A NOTE ON CR MAPPINGS OF POSITIVE CODIMENSION

JEAN-CHARLES SUNYÉ

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ABSTRACT. We prove the following Artin type approximation theorem for smooth CR mappings: given $M \subset \mathbb{C}^N$ a connected real-analytic CR submanifold that is minimal at some point, $M' \subset \mathbb{C}^{N'}$ a real-analytic subset, and $H: M \rightarrow M'$ a C^∞ -smooth CR mapping, there exists a dense open subset $\mathcal{O} \subset M$ such that for any $q \in \mathcal{O}$ and any positive integer k there exists a germ at q of a real-analytic CR mapping $H^k: (M, q) \rightarrow M'$ whose k -jet at q agrees with that of H up to order k .

1. INTRODUCTION

Given germs of real-analytic submanifolds M and M' embedded in complex spaces, a fundamental question is to decide whether the formal equivalence of M and M' implies their biholomorphic equivalence. While this need not be in general the case in view of a well known example due to Moser-Webster [12] (see also [7, 8]), recent results due to Baouendi, Mir, Rothschild and Zaitsev [5, 3] provide a partial positive answer when the submanifolds are furthermore assumed to be CR. In [5, 3], the positive solution is obtained by approximating in the Krull topology a given formal holomorphic equivalence by a convergent one, following the spirit of Artin's approximation theorem [1]. In this paper, we prove the following Artin type approximation theorem for arbitrary smooth CR mappings of any positive codimension.

Theorem 1.1. *Let $M \subset \mathbb{C}^N$ be a connected real-analytic CR submanifold that is minimal at some point, $M' \subset \mathbb{C}^{N'}$ be a real-analytic subset, and $H: M \rightarrow M'$ be a C^∞ -smooth CR mapping. Then there exists a dense open subset $\mathcal{O} \subset M$ such that for any $q \in \mathcal{O}$ and any positive integer k there exists a germ at q of a real-analytic CR mapping $H^k: (M, q) \rightarrow M'$ whose k -jet at q agrees with that of H up to order k .*

Here minimality is meant in the sense of Tumanov (see Section 2 for the precise definition). To the author's knowledge, Theorem 1.1 is the first result of its kind for mappings of positive codimension between arbitrary real-analytic submanifolds. When the target is a real-algebraic set instead of a real-analytic set, then Theorem 1.1 follows from the work of Meylan, Mir and Zaitsev [10]. Observe that

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Theorem 1.1 is also new even in the case $N = N'$ since there is no rank assumption on the mapping under consideration (compare with [5, 3, 13]). On the other hand, we do not know whether one may choose in Theorem 1.1 the dense open subset $\mathcal{O} \subset M$ to be a Zariski open subset independent of the mapping H . Note that when M' is real-algebraic, such a choice is possible and follows from the main result of [10]. For more details related to Artin type approximation in CR geometry, we refer the reader to the survey paper [11].

In this paper we shall give a rather elementary and self-contained proof of Theorem 1.1. For this, we will use several main steps of [6] for which we will provide simplified proofs of the results needed for this paper.

We will organize the paper as follows. Section 2 contains some basic definitions and technical lemmas used in Section 3. In Section 3, we give some elementary properties of a complex-analytic set invariantly attached to a graph of a smooth CR-mapping. The last section is devoted to the proof of Theorem 1.1.

2. PRELIMINARIES

In this section we first recall some basic definitions and prove a lemma used in Section 3. For basic background on CR analysis, we refer the reader to [2]. Let $M \subset \mathbb{C}^N$ be a real-analytic generic submanifold of codimension d . Let us recall that M is said to be minimal at $p \in M$ if there is no germ of a real submanifold $S \subset M$ through p such that the complex tangent space of M at q is tangent to S at every $q \in S$ and $\dim_{\mathbb{R}} S < \dim_{\mathbb{R}} M$ (see [2]).

Following [6], for a $\mathcal{C}^\infty$ -smooth CR mapping $f: M \rightarrow \mathbb{C}^{N'}$ and for $p \in M$, we denote $\mathcal{T}_p(f)$ as the germ of the smallest complex analytic set in $\mathbb{C}^{N+N'}$ containing the germ of the graph of f at $(p, f(p))$. The integer $\dim \mathcal{T}_p(f) - N$ will be called the *degree of partial analyticity* of f at p and denoted by $\deg_p f$. We may observe that, if M is minimal at p , the degree of partial analyticity of f at p is non-negative (see Remark 3.2).

We will need the following well known reflection principle (see e.g. [4, 9]).

Proposition 2.1 (Reflection principle). *Let M be a real-analytic generic submanifold in $\mathbb{C}^N$ minimal at some point $p \in M$. Assume that $u: M \rightarrow \mathbb{C}$ is a continuous CR function near p in M and that $\Theta: \left(\mathbb{C}^{2N+1}, \left(p, \bar{p}, \overline{u(p)}\right)\right) \rightarrow \mathbb{C}$ is the germ at $\left(p, \bar{p}, \overline{u(p)}\right)$ of a holomorphic function such that the function $F(z) = \Theta\left(z, \bar{z}, \overline{u(z)}\right)$ is CR near p in M . Then there exists a unique holomorphic extension of F near p in $\mathbb{C}^N$.*

In what follows, we say that a $\mathcal{C}^\infty$ -smooth mapping $h: \Omega \rightarrow \mathbb{C}^l$, with Ω being a real manifold, is not identically zero near a point p_0 if the germ of h at p_0 is non-zero, i.e. if we find points p as close as we want to p_0 such that $h(p) \neq 0$.

We will also use the notion of wedge. For $p \in M$, we consider an open neighborhood U of p in $\mathbb{C}^N$ and a local defining real-analytic function $\rho: U \rightarrow \mathbb{R}^d$ of M near p . If Γ is an open convex cone in $\mathbb{R}^d$ with vertex at the origin, an open set $\mathcal{W}$ of the form $\{z \in U, \rho(z, \bar{z}) \in \Gamma\}$ is called a *wedge of edge M* in the direction Γ centered at p .

The following result is a lemma from [6] for which we provide a more elementary proof.

Lemma 2.2. *Let $M \subset \mathbb{C}^N$ be a real-analytic generic submanifold, minimal at $p \in M$, let $F: (M, p) \rightarrow \mathbb{C}^s$ and $u: (M, p) \rightarrow \mathbb{C}^t$ be two germs of $\mathcal{C}^\infty$ -smooth CR mappings and let $\psi: \left(\mathbb{C}^{2N+t+s}, \left(p, \bar{p}, \overline{u(p)}, F(p)\right)\right) \rightarrow \mathbb{C}$ be a germ at $\left(p, \bar{p}, \overline{u(p)}, F(p)\right)$ of a holomorphic function. Assume that $\psi\left(z, \bar{z}, \overline{u(z)}, F(z)\right) \equiv 0$ for $z \in M$ near p and that the function $(z, w) \mapsto \psi\left(z, \bar{z}, \overline{u(z)}, w\right)$ is not identically zero near $(p, F(p))$ in $M \times \mathbb{C}^s$. Then there exists $q \in M$ as close as we want to p such that $\deg_q F < s$.*

Proof. This result will be proved by induction on the integer s . First, we consider the case where $s = 1$.

Let A be the set of points p_1 near p in M such that the holomorphic function $\mathbb{C} \ni w \mapsto \psi\left(p_1, \bar{p}_1, \overline{u(p_1)}, w\right)$ is not identically zero near $F(p_1)$ in $\mathbb{C}$. We may find $p_1 \in A$ as close as we want to p . Indeed, since $\psi\left(z, \bar{z}, \overline{u(z)}, w\right)$ is not identically zero near $(p, F(p))$ in $M \times \mathbb{C}$, there exists (p_1, p'_1) as close as we want to $(p, F(p))$ such that $\psi\left(p_1, \bar{p}_1, \overline{u(p_1)}, p'_1\right) \neq 0$. So, the holomorphy of $\mathbb{C} \ni w \mapsto \psi\left(p_1, \bar{p}_1, \overline{u(p_1)}, w\right)$ implies it cannot be identically zero near $F(p_1)$ in $\mathbb{C}$. Moreover, for $z \in A$ fixed, since the holomorphic function $\mathbb{C} \ni w \mapsto \psi\left(z, \bar{z}, \overline{u(z)}, w\right)$ doesn't vanish identically near $F(z)$ in $\mathbb{C}$, there exists a unique positive integer k_z such that $\frac{\partial^{k_z} \psi}{\partial w^{k_z}}\left(z, \bar{z}, \overline{u(z)}, F(z)\right) \neq 0$ and, for any integer $k < k_z$, $\frac{\partial^k \psi}{\partial w^k}\left(z, \bar{z}, \overline{u(z)}, F(z)\right) = 0$. Now we fix a sufficiently small open neighborhood V of p in M and consider the integer

$$K = \min\{k_z, z \in A \cap V\}.$$

We may pick $p_1 \in A \cap V$ such that $k_{p_1} = K$, and we have

$$(2.1) \quad \frac{\partial^{K-1} \psi}{\partial w^{K-1}}\left(p_1, \bar{p}_1, \overline{u(p_1)}, F(p_1)\right) = 0,$$

$$(2.2) \quad \frac{\partial^K \psi}{\partial w^K}\left(p_1, \bar{p}_1, \overline{u(p_1)}, F(p_1)\right) \neq 0.$$

Since ψ is holomorphic near $\left(p, \bar{p}, \overline{u(p)}, F(p)\right)$ in $\mathbb{C}^{2N+t+1}$ by the implicit function theorem, there exists a germ at $\left(p_1, \bar{p}_1, \overline{u(p_1)}\right)$ of a holomorphic function

$$\Theta: \left(\mathbb{C}^{2N+t}, \left(p_1, \bar{p}_1, \overline{u(p_1)}\right)\right) \rightarrow \mathbb{C}$$

such that the zeros of $\frac{\partial^{K-1} \psi}{\partial w^{K-1}}$ near $\left(p_1, \bar{p}_1, \overline{u(p_1)}, F(p_1)\right)$ in $\mathbb{C}^{2N+t+1}$ are given by the equation $w = \Theta(z, \bar{z}, \nu)$. On the other hand, we may observe that the function $\frac{\partial^{K-1} \psi}{\partial w^{K-1}}\left(z, \bar{z}, \overline{u(z)}, F(z)\right)$ is identically zero near p_1 in M . Suppose, in order to reach a contradiction, that it is false. In this case, we may find p_2 as close as we want to p_1 in M such that $\frac{\partial^{K-1} \psi}{\partial w^{K-1}}\left(p_2, \bar{p}_2, \overline{u(p_2)}, F(p_2)\right) \neq 0$, so there is a point $p_2 \in A \cap V$ with $k_{p_2} < K$. This is a contradiction in view of the definition of K . So $\frac{\partial^{K-1} \psi}{\partial w^{K-1}}\left(z, \bar{z}, \overline{u(z)}, F(z)\right) \equiv 0$ for $z \in M$ near p_1 , and, from the remark on the zeros of $\frac{\partial^{K-1} \psi}{\partial w^{K-1}}$, we obtain that $F(z) = \Theta\left(z, \bar{z}, \overline{u(z)}\right)$ near p_1 in M . But if p_1 is close enough to p , we may assume that M is minimal at p_1 (since M is real-analytic and minimal at p), and consequently we may apply Proposition 2.1 to obtain the

existence of a holomorphic extension $\tilde{F}$ of F near p_1 in $\mathbb{C}^N$. Thus the graph of F is contained, near p_1 , in the graph of $\tilde{F}$, which is a complex analytic set of dimension N . Consequently, the dimension of $\mathcal{T}_{p_1}(F)$ is less than or equal to N . So we proved that, for an arbitrary small neighborhood V of p in M , there exists $p_1 \in V$ such that $\deg_{p_1} F \leq 0$. This finishes the proof of the lemma for $s = 1$.

Now, we assume that the lemma holds for $s - 1$, and for any $t \in \mathbb{N}$, any germs of CR mappings $F: (M, p) \rightarrow \mathbb{C}^{s-1}$, $u: (M, p) \rightarrow \mathbb{C}^t$ and any germ at $(p, \bar{p}, \overline{u(p)}, F(p))$ of a holomorphic function $\psi: (\mathbb{C}^{2N+t+s-1}, (p, \bar{p}, \overline{u(p)}, F(p))) \rightarrow \mathbb{C}$. Our aim is to prove the same result for s . We write $w = (w', w_s) \in \mathbb{C}^{s-1} \times \mathbb{C}$ and $F = (F', F_s) \in \mathbb{C}^{s-1} \times \mathbb{C}$.

First, we consider the case where $\psi(z, \bar{z}, \overline{u(z)}, F'(z), w_s) \equiv 0$ for $(z, w_s) \in M \times \mathbb{C}$ near $(p, F_s(p))$. Taking the Taylor series of ψ in w_s at $F_s(p)$,

$$(2.3) \quad \psi(z, \zeta, \nu, w', w_s) = \sum_{k \in \mathbb{N}} \psi_k(z, \zeta, \nu, w') (w_s - F_s(p))^k,$$

we obtain that, for any $k \in \mathbb{N}$, $\psi_k(z, \bar{z}, \overline{u(z)}, F'(z)) \equiv 0$ for $z \in M$ near p and that there exists k_0 such that $\psi_{k_0}(z, \bar{z}, \overline{u(z)}, w')$ doesn't vanish identically near $(p, F'(p))$ in $M \times \mathbb{C}^{s-1}$. So, by the induction hypothesis, there exists $q \in M$ as close as we want to p such that $\deg_q F' < s - 1$, which implies $\deg_q F < s$. This completes the proof for this case.

To finish the proof, we have to consider the case where $\psi(z, \bar{z}, \overline{u(z)}, F'(z), w_s)$ doesn't vanish identically near $(p, F_s(p))$ in $M \times \mathbb{C}$. By the same method as in the case $s = 1$, we show that, for point p_1 as close as we want to p where M is minimal, there exists a germ at $(p_1, \bar{p}_1, \overline{u(p_1)}, F'(p_1))$ of a holomorphic function $\Theta: (\mathbb{C}^{2N+t+s-1}, (p_1, \bar{p}_1, \overline{u(p_1)}, F'(p_1))) \rightarrow \mathbb{C}$ such that

$$(2.4) \quad F_s(z) = \Theta(z, \bar{z}, \overline{u(z)}, F'(z))$$

near p_1 in M . Let $\Theta(z, \zeta, \nu, w') = \sum_{\alpha \in \mathbb{N}^{s-1}} \Theta_\alpha(z, \zeta, \nu) (w' - F'(p_1))^\alpha$ be the Taylor series of Θ in w' at $F'(p_1)$.

If every $\Theta_\alpha(z, \bar{z}, \overline{u(z)})$ is CR near p_1 in M , then by Proposition 2.1 (recall that M is minimal at p_1) the function $M \times \mathbb{C}^{s-1} \ni (z, w') \mapsto \Theta(z, \bar{z}, \overline{u(z)}, w') \in \mathbb{C}$ can be holomorphically extended near $(p_1, F'(p_1))$ in $\mathbb{C}^{N+s-1}$. We denote the extension by $\tilde{\Theta}$. The graph of F is contained in the graph of $\tilde{\Theta}$, which is a complex submanifold of $\mathbb{C}^{N+s}$ of dimension $N + s - 1$. This is equivalent to saying that the degree of partial analyticity of F at p_1 is smaller than $s - 1$.

If there is a multi-index $\alpha \in \mathbb{N}^{s-1}$ such that the mapping $\Theta_\alpha(z, \bar{z}, \overline{u(z)})$ is not CR, then there exists a vector field $\bar{L} = \sum_{j=1}^N a_j(z, \bar{z}) \frac{\partial}{\partial \bar{z}_j}$ near p in $\mathbb{C}^N$, where $a_1, \dots, a_N$ are real-analytic functions near p , such that $\bar{L}|_M$ is a CR vector field and

$$\bar{L}(\Theta(z, \bar{z}, \overline{u(z)}, w')) \neq 0$$

near $(p_1, F'(p_1))$ in $M \times \mathbb{C}^{s-1}$. Using the chain rule, we may observe that there exists a holomorphic function Ψ_1 near $\left(p_1, \overline{p_1}, \bar{L}\left(\overline{u(z)}\right)\Big|_{z=p_1}, F'(p_1)\right)$ in $\mathbb{C}^{2N+t+s-1}$ such that

$$(2.5) \quad \bar{L}\left(\Theta\left(z, \bar{z}, \overline{u(z)}, w'\right)\right) = \Psi_1\left(z, \bar{z}, \bar{L}\left(\overline{u(z)}\right), w'\right)$$

near $(p_1, F'(p_1))$ in $M \times \mathbb{C}^{s-1}$. Since M is minimal at p , Tumanov's extension theorem (see [2]) implies that there exists a holomorphic extension $\tilde{u}$ of u in a wedge $\mathcal{W}$ of edge M centered at p . We may assume that the mapping $\tilde{u}$ is $\mathcal{C}^\infty$ -smooth on $\mathcal{W}$ up to the edge M . Moreover, for any $z \in \mathcal{W} \cup (M \cap V)$, where V is a sufficiently small neighborhood of p in $\mathbb{C}^N$,

$$\bar{L}\left(\overline{\tilde{u}(z)}\right) = \sum_{j=1}^N a_j(z, \bar{z}) \frac{\partial \tilde{u}}{\partial z_j}(z).$$

Now, for any $j \in \{1, \dots, N\}$, $\frac{\partial \tilde{u}}{\partial z_j}$ is holomorphic in $\mathcal{W}$ and $\mathcal{C}^\infty$ up to the edge M . Thus, the mapping U whose components are the restrictions to M near p of the derivatives of $\tilde{u}$ is a CR mapping near p in M . Consequently, from the identity (2.5), and since $\bar{L}(u(z)) = \bar{L}(\tilde{u}(z))$, for $z \in M$ close enough to p , we may find a germ at $\left(p_1, \overline{p_1}, \overline{U(p_1)}, F'(p_1)\right)$ of a holomorphic function

$$\Psi: \left(\mathbb{C}^{2N+t+Nt+s-1}, \left(p_1, \overline{p_1}, \overline{U(p_1)}, F'(p_1)\right)\right) \rightarrow \mathbb{C}$$

such that

$$\bar{L}\left(\Theta\left(z, \bar{z}, \overline{u(z)}, w'\right)\right) = \Psi\left(z, \bar{z}, \overline{U(z)}, F'(z)\right)$$

near $(p_1, F'(p_1))$ in $M \times \mathbb{C}^{s-1}$. Moreover, we know that $\bar{L}\left(\Theta(z, \bar{z}, \overline{u(z)}, F'(z))\right) \equiv \bar{L}(F_s(z)) \equiv 0$ near p_1 in M , since F is CR; i.e. $\Psi\left(z, \bar{z}, \overline{U(z)}, F'(z)\right) \equiv 0$ near p_1 in M . So, since we saw that $\Psi\left(z, \bar{z}, \overline{U(z)}, w'\right) \not\equiv 0$ near $(p_1, F'(p_1))$ in $M \times \mathbb{C}^{s-1}$, the induction assumption implies that the degree of partial analyticity of F' is strictly smaller than $s-1$ for points in M as close as we want to p_1 and therefore as close to p . This finishes the proof of Lemma 2.2. $\square$

As in [6], one gets from Lemma 2.2 the following result.

Lemma 2.3. *Let $M \subset \mathbb{C}^N$ be a real-analytic generic submanifold, minimal at $p \in M$, let $F: (M, p) \rightarrow \mathbb{C}^s$ be a germ of a CR mapping and let*

$$\psi: \left(\mathbb{C}^{2N+2s}, \left(p, \bar{p}, \overline{F(p)}, F(p)\right)\right) \rightarrow \mathbb{C}$$

be a germ at $\left(p, \bar{p}, \overline{F(p)}, F(p)\right)$ of a holomorphic function. Assume that

$$\psi\left(z, \bar{z}, \overline{F(z)}, F(z)\right) \equiv 0$$

for $z \in M$ near p and that the function $(z, v, w) \mapsto \psi(z, \bar{z}, v, w)$ is not identically zero near $\left(p, \overline{F(p)}, F(p)\right)$ in $M \times \mathbb{C}^{2s}$. Then, there exists $q \in M$ as close as we want to p such that $\deg_q F < s$.

Proof. First, we assume that $\psi(z, \bar{z}, \overline{F(z)}, w)$ does not vanish identically near $(p, F(p))$ in $M \times \mathbb{C}^s$. Since $\psi(z, \bar{z}, \overline{F(z)}, F(z)) \equiv 0$ for $z \in M$ near p , we may apply Lemma 2.2 and deduce that the degree of partial analyticity of F at q is strictly smaller than s for q arbitrarily close to p .

Now we treat the case where $\psi(z, \bar{z}, \overline{F(z)}, w) \equiv 0$ for $(z, w) \in M \times \mathbb{C}^s$ near $(p, F(p))$. For this we consider the Taylor series of ψ in w at $F(p)$:

$$(2.6) \quad \psi(z, \zeta, v, w) = \sum_{\alpha \in \mathbb{N}^s} \psi_\alpha(z, \zeta, v) (w - F(p))^\alpha.$$

The assumption implies that, for any $\alpha \in \mathbb{N}^s$, $\psi_\alpha(z, \bar{z}, \overline{F(z)}) \equiv 0$ near p in M . However, by assumption, there is a multi-index α_0 such that $\psi_{\alpha_0}(z, \bar{z}, v) \not\equiv 0$ near $(p, \overline{F(p)})$ in $M \times \mathbb{C}^s$, and Lemma 2.2 gives the desired result. $\square$

3. PROPERTIES OF $\mathcal{T}_p(H)$

In this section, we fix a real-analytic generic submanifold $M \subset \mathbb{C}^N$ and a $\mathcal{C}^\infty$ -smooth CR mapping $H: M \rightarrow M'$ on M with values in a real-analytic set $M' \subset \mathbb{C}^{N'}$. We shall give some properties of the degree of partial analyticity of H and of the complex analytic set $\mathcal{T}_p(H)$ for $p \in M$. All the results of this section can be found in [6], but since the proofs we shall give are rather elementary compared to [6], we include them in this paper for completeness.

The following lemma is a direct consequence of the boundary uniqueness theorem.

Lemma 3.1. *Let M , M' and H be as above. If H admits a holomorphic extension $\tilde{H}$ on a wedge of edge M centered at p , then the graph of $\tilde{H}$ near $(p, H(p))$ is contained in $\mathcal{T}_p(H)$.*

Remark 3.2. If M is minimal at $p \in M$, Tumanov's extension theorem implies that, in the above setting, there is a unique extension $\tilde{H}$ of H holomorphic on a wedge of edge M centered at p . Thus $\mathcal{T}_p(H)$ contains the graph of $\tilde{H}$ near $(p, H(p))$. This implies that the dimension of $\mathcal{T}_p(H)$ at p is greater than N , i.e. that the degree of partial analyticity of H at p is non-negative.

The two following lemmas describe the regular points of the complex analytic set $\mathcal{T}_p(H)$.

Lemma 3.3. *In the above setting, assume that the degree of partial analyticity of H is constant near some point p in M . Then there exists an open neighborhood U_p^1 of p in M such that the set $\Sigma_p^1 \subset M \cap U_p^1$ of points $q \in U_p^1$ for which $\mathcal{T}_p(H)$ is not regular at $(q, H(q))$ is a closed set with empty interior.*

Proof. We may find an open neighborhood U_p^1 of p in M on which $\mathcal{T}_p(H)$ is a complex analytic set and on which the degree of partial analyticity of H is constant equal to s . Let $\Sigma_p^1 \subset M \cap U_p^1$ be the set of points $q \in U_p^1$ for which $\mathcal{T}_p(H)$ is not regular at $(q, H(q))$. In view of the classical definition of regular points of a complex analytic set, Σ_p^1 is a closed subset of $M \cap U_p^1$. To prove that its interior is empty, assume by contradiction that we may find an open subset V of M contained in Σ_p^1 . Thus, for any $q \in V$, the graph of H near $(q, H(q))$ is contained in the set of

the singular points of $\mathcal{T}_p(H)$, which is a complex analytic set of dimension strictly smaller than $N + s$. So, the dimension of $\mathcal{T}_q(H)$ is also strictly smaller than $N + s$ for any $q \in V$. This is impossible, since the degree of partial analyticity of H is constant equal to s on V . $\square$

Lemma 3.4. *In the above setting, assume that M is minimal at $p \in M$, that the degree of partial analyticity of H is constant equal to s near p in M , and write $t = N' - s$. Then there are an open neighborhood $U_p^2 \subset U_p^1$ of p in M (with U_p^1 given by Lemma 3.3) and a closed set with empty interior $\Sigma_p^2 \subset M \cap U_p^2$ such that, for any $q \in U_p^2 \setminus \Sigma_p^2$, there are holomorphic coordinates $(u', v') \in \mathbb{C}^s \times \mathbb{C}^t$ near $H(q)$ for which $H = (F, G) \in \mathbb{C}^s \times \mathbb{C}^t$, and a germ at $(q, F(q))$ of a holomorphic mapping $T_q: (\mathbb{C}^{N+s}, (q, F(q))) \rightarrow \mathbb{C}^t$ such that $\mathcal{T}_p(H)$ is given near $(q, H(q))$ by the equation $v' = T_q(z, u')$. In particular $\mathcal{T}_p(H)$ is regular at $(q, H(q))$.*

Proof. Since M is minimal at p and H is CR on M , Tumanov's extension theorem implies that there exists a holomorphic extension $\tilde{H}$ of H in a wedge $\mathcal{W}$ of edge M centered at p which is C^∞ -smooth up to the edge. By Lemma 3.1 the graph of $\tilde{H}$ is contained in $\mathcal{T}_p(H)$ near $(p, H(p))$. It means that we may choose an open neighborhood Δ of p in $\mathbb{C}^N$ such that $(z, \tilde{H}(z)) \in \mathcal{T}_p(H)$ for any $z \in \Delta \cap \mathcal{W}$.

On the other hand, if U_p^1 is the open neighborhood of p given by Lemma 3.3, we define $U_p^2 = U_p^1 \cap \Delta$ and $\Sigma_p^2 = \Sigma_p^1 \cap U_p^2$. Now, for a fixed point $q \in U_p^2 \setminus \Sigma_p^2$, $\mathcal{T}_p(H)$ is regular at $(q, H(q))$. Moreover, since the degree of partial analyticity of H is constant equal to s on U_p^1 , the dimension of $\mathcal{T}_p(H)$ at $(q, H(q))$ is $N + s$. So, there exist an open neighborhood $\mathcal{U} \subset U_p^1$ of q in $\mathbb{C}^N$, an open neighborhood $\mathcal{V}$ of $H(q)$ in $\mathbb{C}^{N'}$ and a holomorphic mapping $f: \mathcal{U} \times \mathcal{V} \rightarrow \mathbb{C}^t$ of rank t at $(q, H(q))$ such that

$$(3.1) \quad \mathcal{T}_p(H) = \{(z, z') \in \mathcal{U} \times \mathcal{V}, f(z, z') = 0\}.$$

Thus, for $z \in \mathcal{U}' = \mathcal{U} \cap \mathcal{W} \cap \tilde{H}^{-1}(\mathcal{V})$, we have

$$(3.2) \quad f(z, \tilde{H}(z)) = 0.$$

So $\frac{\partial f}{\partial z}(z, \tilde{H}(z)) + \frac{\partial f}{\partial z'}(z, \tilde{H}(z)) \cdot \frac{\partial \tilde{H}}{\partial z}(z) \equiv 0$ in $\mathcal{U}'$. Since the mapping $\tilde{H}$ is C^∞ -smooth up to the edge M , we have the following identity:

$$(3.3) \quad \frac{\partial f}{\partial z}(z, \tilde{H}(z)) + \frac{\partial f}{\partial z'}(z, \tilde{H}(z)) \cdot \frac{\partial \tilde{H}}{\partial z}(z) \equiv 0$$

on $M \cap \mathcal{U}'$. But f is of rank t at $(q, H(q))$ and the identity (3.3) shows that the columns of $\frac{\partial f}{\partial z}(q, H(q))$ are a linear combination of the columns of $\frac{\partial f}{\partial z'}(q, H(q))$. Consequently the rank of $\frac{\partial f}{\partial z'}(q, H(q))$ is t . So, by the implicit function theorem there exist holomorphic coordinates $(u', v') \in \mathbb{C}^s \times \mathbb{C}^t$ for which $H = (F, G) \in \mathbb{C}^s \times \mathbb{C}^t$ and a holomorphic mapping T_q near $(q, F(q))$ in $\mathbb{C}^{N+s}$ such that the zeros of f are given by points of the form $(z, u', T_q(z, u')) \in \mathbb{C}^N \times \mathbb{C}^s \times \mathbb{C}^t$. $\square$

For the next result, we will denote $\pi: \mathbb{C}^N \times \mathbb{C}^{N'} \rightarrow \mathbb{C}^N$ and $\pi': \mathbb{C}^N \times \mathbb{C}^{N'} \rightarrow \mathbb{C}^{N'}$ as the canonical projections.

Lemma 3.5. *In the above setting, assume that M is minimal at $p \in M$ and that the degree of partial analyticity of H is constant equal to s near p in M . Then there are an open neighborhood $U_p^3 \subset U_p^2$ of p in M (with U_p^2 given by Lemma 3.4) and*

a closed set with empty interior $\Sigma_p^3 \subset M \cap U_p^3$ such that, for any $q \in U_p^3 \setminus \Sigma_p^3$, there is a neighborhood Ω_q of $(q, H(q))$ in $\mathbb{C}^{N+N'}$ satisfying

$$\pi'(\mathcal{T}_p(H)|_{M \times \mathbb{C}^{N'} \cap \Omega_q}) \subset M'.$$

Proof. Let $\rho': \mathbb{C}^{N'} \rightarrow \mathbb{R}^{d'}$ be a local defining real-analytic function of M' near $H(p)$. Since $H(M) \subset M'$, we have the mapping identity

$$(3.4) \quad \rho'(H(z), \overline{H(z)}) \equiv 0$$

for $z \in M$ near p .

We consider U_p^2 and Σ_p^2 respectively as the open neighborhood of p in M and the closed set with empty interior given by Lemma 3.4. Let $U_p^3 \subset U_p^2$ be a sufficiently small connected neighborhood of p in M . Since M is real-analytic and minimal at p , the set Σ_1 of points in U_p^3 where M is not minimal is a closed set with empty interior. Thus, $\Sigma_p^3 = (\Sigma_p^2 \cap U_p^3) \cup \Sigma_1$ is a closed set with empty interior. Moreover, for any $q \in U_p^3 \setminus \Sigma_p^3$, M is minimal at q , and we may find holomorphic coordinates $(u', v') \in \mathbb{C}^s \times \mathbb{C}^t$ near $H(q)$ for which $H = (F, G) \in \mathbb{C}^s \times \mathbb{C}^t$, and a germ at $(q, F(q))$ of a holomorphic mapping $T_q: (\mathbb{C}^{N+s}, (q, F(q))) \rightarrow \mathbb{C}^t$ such that $\mathcal{T}_p(H)$ is given near $(q, H(q))$ by the equation $v' = T_q(z, u')$. If U_p^3 is chosen small enough we may assume that the graph of H is contained near $q \in U_p^3 \setminus \Sigma_p^3$ in $\mathcal{T}_p(H)$; i.e. we have the mapping identity

$$(3.5) \quad G(z) = T_q(z, F(z))$$

for $z \in M$ near q . We may also assume that the mapping defined by

$$(3.6) \quad \Psi(z, \zeta, \nu', u') = \rho'(u', T_q(z, u'), \nu', \overline{T_q(\bar{\zeta}, \bar{\nu'})})$$

is holomorphic near $(q, \bar{q}, \overline{F(q)}, F(q))$ in $\mathbb{C}^{2N+2s}$ for any $q \in U_p^3 \setminus \Sigma_p^3$.

From (3.5) and (3.4), we obtain

$$(3.7) \quad \rho'(F(z), T_q(z, F(z)), \overline{F(z)}, \overline{T_q(z, F(z))}) \equiv 0$$

for $z \in M$ near q . Thus, by (3.7), we have

$$(3.8) \quad \Psi(z, \bar{z}, \overline{F(z)}, F(z)) \equiv 0$$

for $z \in M$ near q . Since the degree of analyticity of H is constant equal to s on U_p^3 , from (3.5), we deduce that the degree of partial analyticity of F is constant equal to s on U_p^3 . So, the identity (3.8) and Lemma 2.3 (recall that M is minimal at q) imply that

$$(3.9) \quad \Psi(z, \bar{z}, \nu', u') \equiv 0$$

for $(z, \nu', u') \in M \times \mathbb{C}^{2s}$ near $(q, \overline{F(q)}, F(q))$. This identity is equivalent to

$$(3.10) \quad \rho'(u', T_q(z, u'), \nu', \overline{T_q(z, \bar{\nu'})}) \equiv 0$$

for $(z, \nu', u') \in M \times \mathbb{C}^{2s}$ near $(q, \overline{F(q)}, F(q))$. So taking $\nu' = \bar{u}'$ in (3.10), we obtain that $(u', T_q(z, u')) \in M'$ as soon as z is close enough to q in M and u' is close enough to $F(q)$ in $\mathbb{C}^s$. This finishes the proof of the lemma. $\square$

We conclude this section with a lemma which makes use of the upper semi-continuity of the partial analyticity degree; we leave the details to the reader.

Lemma 3.6. *In the above setting, there exists a closed set with empty interior $\Sigma_2 \subset M$ such that the degree of partial analyticity of H is constant on each connected component of $M \setminus \Sigma_2$.*

4. PROOF OF THEOREM 1.1

In this section, we keep the assumptions and the notation of Section 3; i.e we consider a real-analytic generic submanifold M in $\mathbb{C}^N$ and a $\mathcal{C}^\infty$ -smooth CR mapping $H: M \rightarrow M'$ where $M' \subset \mathbb{C}^{N'}$ is a real-analytic set. Theorem 1.1 will be a consequence of the following result.

Proposition 4.1. *In the above setting, assume that M is minimal at $p \in M$ and that the degree of partial analyticity of H is constant equal to s near p in M . Let U_p^3 and $\Sigma_p^3 \subset M \cap U_p^3$ be, respectively, the open neighborhood of p in M and the closed set with empty interior given by Lemma 3.5. Then for any $q \in U_p^3 \setminus \Sigma_p^3$ and for any positive integer k there exists a germ at q of a real-analytic CR mapping $H^k: (M, q) \rightarrow M'$ whose k -jet at q agrees with that of H up to order k .*

Proof. For any $q \in U_p^3 \setminus \Sigma_p^3$, we may choose holomorphic coordinates $z' = (u', v') \in \mathbb{C}^s \times \mathbb{C}^t$ and a germ at $(q, F(q))$ of a holomorphic mapping $T_q: (\mathbb{C}^{N+s}, (q, F(q))) \rightarrow \mathbb{C}^t$ such that $\mathcal{T}_p(H)$ is given by the equation $v' = T_q(z, u')$ near $(q, H(q))$. We may also find a neighborhood Ω_q of $(q, H(q))$ in $\mathbb{C}^{N+N'}$ such that $\pi'(\mathcal{T}_p(H)|_{M \times \mathbb{C}^{N'}} \cap \Omega_q) \subset M'$.

We fix a point $q \in U_p^3 \setminus \Sigma_p^3$ and a positive integer k . Since the graph of H is contained in $\mathcal{T}_p(H)$ near $(q, H(q))$, we have the following mapping identity:

$$(4.1) \quad G(z) = T_q(z, F(z))$$

near q in M . Let F^k be the k -th order Taylor polynomial of F at q (that is holomorphic since H is CR; see [2]). We define the holomorphic mapping G^k near q in $\mathbb{C}^N$ by setting

$$(4.2) \quad G^k(z) = T_q(z, F^k(z)).$$

Thus $H^k = (F^k, G^k)$ is a holomorphic mapping near q in $\mathbb{C}^N$ with values in $\mathbb{C}^{N'}$. Moreover, by definition of H^k and from (4.1), the k first derivatives of H^k at q coincide with that of H . So to complete the proof of Proposition 4.1, we have to show that H^k sends M into M' near q . From the definition of H^k and the local defining equation of $\mathcal{T}_p(H)$ near $(q, H(q))$ in $\mathbb{C}^{N+N'}$, we obtain that there exists an open neighborhood $\widetilde{\Omega}_q$ of $(q, H(q))$ in $\mathbb{C}^{N+N'}$ such that

$$\mathcal{G}_{H^k} \cap \left(M \times \mathbb{C}^{N'} \right) \cap \widetilde{\Omega}_q \subset \mathcal{T}_p(H)|_{M \times \mathbb{C}^{N'}} \cap \Omega_q,$$

where $\mathcal{G}_{H^k}$ is the graph of the mapping H^k . Since $\pi'(\mathcal{T}_p(H)|_{M \times \mathbb{C}^{N'}} \cap \Omega_q) \subset M'$ by Lemma 3.5, the conclusion of the proposition follows. $\square$

Now, we are able to prove our main result, Theorem 1.1.

Proof of Theorem 1.1. We first note that Theorem 1.1 holds in the case where $N = 1$ or $N' = 1$. We may therefore assume that $N, N' \geq 2$.

We first treat the case where M is generic. Since M is real-analytic and connected, there exists a real-analytic subvariety Σ_1 of M such that M is minimal at each point $p \in M \setminus \Sigma_1$. Moreover, from Lemma 3.6, there exists a closed set with empty interior $\Sigma_2 \subset M$ such that the degree of partial analyticity of H is constant

on each connected components of $M \setminus \Sigma_2$. Thus, $\Sigma = \Sigma_1 \cup \Sigma_2$ is a closed subset of M with empty interior. Fix a point $p \in M \setminus \Sigma$; by Proposition 4.1, we may find an open subset U_p^3 of p in M and a closed set with empty interior $\Sigma_p^3 \subset M \cap U_p^3$ such that, for every $q \in U_p^3 \setminus \Sigma_p^3$, the conclusion of Theorem 1.1 holds at q . Thus, the set $\mathcal{O} = \bigcup_{p \in M \setminus \Sigma} (U_p^3 \setminus \Sigma_p^3)$ does the job, and this finishes the proof of Theorem 1.1 for the generic case.

If M is not generic, for any $p \in M \setminus \Sigma_1$ (where Σ_1 again denotes the set of nonminimal points of M) we may assume, thanks to a local holomorphic change near p , that $M = \widetilde{M}_p \times \{0\} \subset \mathbb{C}_{z_1}^{N-r_1} \times \mathbb{C}_{z_2}^{r_1}$, where r_1 is a non-negative integer and $\widetilde{M}_p$ is a connected real-analytic generic submanifold which is minimal (see [2]). From the generic case treated above, there exists a dense open subset $\widetilde{\mathcal{O}}_p \subset \widetilde{M}_p$ such that, for any non-negative integer k and any $\tilde{q} \in \widetilde{\mathcal{O}}_p$, there exists a germ at $\tilde{q}$ of a real-analytic CR mapping $H_1^k: (\widetilde{M}_p, q) \rightarrow M'$ whose k -jet at $\tilde{q}$ agrees with that of $\widetilde{M}_p \ni z_1 \mapsto H(z_1, 0)$. Since $\bigcup_{p \in M \setminus \Sigma_1} (\widetilde{\mathcal{O}}_p \times \{0\})$ is a dense open subset of M , the proof of Theorem 1.1 is complete. $\square$

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LABORATOIRE DE MATHMATIQUES RAPHAL SALEM, CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE, UNIVERSIT DE ROUEN, AVENUE DE L’UNIVERSIT, B.P. 12, 76801 SAINT ETIENNE DU ROUVRAY, FRANCE

E-mail address: jean-charles.sunye@etu.univ-rouen.fr