REMOVABLE SETS FOR THE FLUX OF CONTINUOUS VECTOR FIELDS

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ABSTRACT. We show that any closed set E having a σ -finite $(n-1)$ -dimensional Hausdorff measure does *not* support the nonzero distributional divergence of a continuous vector field; in particular it has the property that any C^1 function in $\mathbb{R}^n$ that is harmonic outside it is harmonic in $\mathbb{R}^n$. We also exhibit a compact set E having Hausdorff dimension $n-1$, supporting the nonzero distributional divergence of a continuous vector field yet having the property that any C^1 function that is harmonic outside E is harmonic in $\mathbb{R}^n$.

INTRODUCTION

The problem of removing singularities for the equation $\operatorname{div} v = 0$ can be stated as follows: given a collection $\mathcal{B}$ of locally bounded, measurable vector fields in $\mathbb{R}^n$, give conditions on a compact set E guaranteeing that each vector field $v \in \mathcal{B}$ satisfying $\operatorname{div} v = 0$ outside E (in the distributional sense) satisfies $\operatorname{div} v = 0$ in $\mathbb{R}^n$ (in the distributional sense, too).

This problem has been studied for various classes of functions. When $\mathcal{B}$ is the collection of all (locally) bounded, measurable vector fields, the compact removable sets are *exactly* those having zero $(n-1)$ -dimensional Hausdorff measure (see [2], [8] and [11]). In this short paper, we focus on the case when $\mathcal{B}$ is the collection of all *continuous* vector fields in $\mathbb{R}^n$. Theorem 5.2 (second part) in [11] states that removable sets in this context have Hausdorff dimension at most $n-1$. On the other hand, it is a rather easy observation (see [9, Lemma 8.18] for example) that any compact set E satisfying $\mathcal{H}^{n-1}(E) < \infty$ is removable. In this paper, we extend this property to closed sets having σ -finite $(n-1)$ -dimensional Hausdorff measure (Theorem 12). In particular, each function u that is C^1 in $\mathbb{R}^n$ and harmonic outside a closed set having σ -finite $(n-1)$ -dimensional Hausdorff measure is harmonic in $\mathbb{R}^n$ (apply the removability property to the vector field $v = \nabla u$).

We then show that the dimension condition does *not* characterize removable sets. In particular, we exhibit (Theorem 13) a compact set E that has Hausdorff dimension $n-1$, is not removable, yet satisfies the following condition: any C^1 function $u : \mathbb{R}^n \rightarrow \mathbb{R}$ harmonic outside E is harmonic in $\mathbb{R}^n$.

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1. PRELIMINARIES AND NOTATION

In the sequel, we work in Euclidean space $\mathbb{R}^n$, $n \geq 1$. The Euclidean norm of $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ is denoted $|x|$; for $r > 0$ let $B[x, r] = \{y \in \mathbb{R}^n : |x - y| \leq r\}$ and $Q[x, r] = \{y \in \mathbb{R}^n : |y_i - x_i| \leq r, 1 \leq i \leq n\}$. We write $B[r]$ and $Q[r]$ instead of $B[0, r]$ and $Q[0, r]$, respectively. A *cube* is a set of the form $Q[x, r]$ for $x \in \mathbb{R}^n$ and $r > 0$.

Given $E \subseteq \mathbb{R}^n$, we will write $E \subset\subset U$ to indicate that the closure of E is a compact subset of the open set $U \subseteq \mathbb{R}^n$. We further denote by $\mathcal{L}^n(E)$ or $|E|$ the Lebesgue (outer) measure of E ; in particular we define $\alpha(n) := |B[1]|$. The notation $d(E)$ stands for the *diameter* of E , and for $0 \leq s \leq n$ we let $\mathcal{H}^s(E)$ (resp. $\mathcal{S}^s(E)$) denote the (normalized by $\alpha(s)2^{-s}$) s -dimensional Hausdorff measure (resp. the s -dimensional spherical Hausdorff measure) of E . One has $\mathcal{H}^s(E) \leq \mathcal{S}^s(E) \leq 2^s \mathcal{H}^s(E)$, as is easily checked. We also say that E is *thin* in case $\mathcal{H}^{n-1} \llcorner E$ is σ -finite.

Given $f : X \rightarrow Y$ and $A \subseteq X$, we let $f \upharpoonright A$ denote the *restriction* of f to A . The restriction of an outer measure μ on $\mathbb{R}^n$ to $E \subseteq \mathbb{R}^n$ is the outer measure $\mu \llcorner E$ defined by $\mu \llcorner E(B) := \mu(B \cap E)$.

We denote by $\|f\|_\infty$ and $\|f\|_1$ the L^∞ - and L^1 -norms of $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, respectively. Given $f \in L^1(\mathbb{R}^n)$, we denote by $\|Dg\|$ its total variation. For $0 \leq k \leq \infty$, we give $C^k(\mathbb{R}^n, \mathbb{R}^m)$, $C_c^k(\mathbb{R}^n, \mathbb{R}^m)$ and $C^k(\mathbb{R}^n)$, $C_c^k(\mathbb{R}^n)$ their usual meanings.

2. CHARGES AND FLUXES

The *flux* of any (locally integrable) $v : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the distribution $F_v : C_c^\infty(\mathbb{R}^n) \rightarrow \mathbb{R}$ defined by $F_v(\varphi) = - \int_{\mathbb{R}^n} v(x) \cdot \nabla \varphi(x) dx$. We observe an interesting continuity property of the flux of a *continuous* vector field.

Proposition 1. *Given $v \in C^0(\mathbb{R}^n, \mathbb{R}^n)$, $\varepsilon > 0$, $r > 0$ and $C > 0$, there exists $\delta > 0$ such that $F_v(\varphi) \leq \varepsilon$ provided $\varphi \in C_c^\infty(\mathbb{R}^n)$ satisfies $\{\varphi \neq 0\} \subseteq B[r]$, $\|\nabla \varphi\|_1 \leq C$ and $\|\varphi\|_1 \leq \delta$.*

Proof. Fix $v \in C^0(\mathbb{R}^n, \mathbb{R}^n)$, $\varepsilon > 0$ and choose $w \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$ satisfying $|v(x) - w(x)| \leq \varepsilon/2C$ for each $x \in B[r]$. Let $\theta := 1 + \|\operatorname{div} w\|_\infty$ and say $\delta := \varepsilon/(2\theta)$. For $\varphi \in C_c^\infty(\mathbb{R}^n)$ satisfying $\{\varphi \neq 0\} \subseteq B[r]$, $\|\nabla \varphi\|_1 \leq C$ and $\|\varphi\|_1 \leq \delta$, compute using the integration by parts formula:

$$\begin{aligned} F_v(\varphi) &= \int_{\mathbb{R}^n} [v(x) - w(x)] \cdot \nabla \varphi(x) dx + \int_{\mathbb{R}^n} \varphi(x) \operatorname{div} w(x) dx \\ &\leq \frac{\varepsilon}{2C} \|\nabla \varphi\|_1 + \theta \|\varphi\|_1 \leq \varepsilon. \end{aligned}$$

The proof is complete. $\square$

Remark 2. According to the approximation property of BV functions (see [5, Section 5.2.2]), it is obvious that given $v \in C^0(\mathbb{R}^n, \mathbb{R}^n)$, F_v extends to the linear functional

$$F_v : BV_c(\mathbb{R}^n) \rightarrow \mathbb{R}; g \mapsto - \int_{\mathbb{R}^n} v \cdot d[Dg]$$

on the space $BV_c(\mathbb{R}^n)$ of compactly supported functions having bounded variation in $\mathbb{R}^n$. Furthermore for any $\varepsilon > 0$, $r > 0$ and $C > 0$ there exists $\delta > 0$ such that $F(g) \leq \varepsilon$ provided $g \in BV_c(\mathbb{R}^n)$ satisfies $\{g \neq 0\} \subseteq B[r]$, $\|Dg\| \leq C$ and $\|g\|_1 \leq \delta$.

Given $r > 0$ and $C > 0$, let

$$BV_{r,C} := \{g \in BV_c(\mathbb{R}^n) : \text{supp } g \subseteq B[r], \|Dg\| \leq C\}$$

and denote by $\mathfrak{B}$ the collection of all absorbing, balanced convex sets $\mathbf{U} \subseteq BV_c(\mathbb{R}^n)$ having the property that for all $r > 0$ and $C > 0$, the set $\mathbf{U} \cap BV_{r,C}$ is a relative L^1 -neighborhood of g in $BV_{r,C}$. According to [3, Theorem 3.3], for example, $\mathfrak{B}$ is a neighborhood basis at 0 for a locally convex topology $\mathfrak{T}$ on $BV_c(\mathbb{R}^n)$ which is characterized by the following two properties:

- (A) $\mathfrak{T}$ is weaker than the L^1 topology on $BV_{r,C}$ for all $r > 0, C > 0$;
- (B) for any locally convex space X , a linear map $f : BV_c(\mathbb{R}^n) \rightarrow X$ is $\mathfrak{T}$ -continuous whenever the restriction $f \upharpoonright BV_{r,C}$ is L^1 -continuous on $BV_{r,C}$ for all $r > 0, C > 0$.

Given $v \in C^0(\mathbb{R}^n, \mathbb{R}^n)$, condition (B) together with Remark 2 implies in particular that F_v is $\mathfrak{T}$ -continuous on $BV_c(\mathbb{R}^n)$.

Definition 3. A $\mathfrak{T}$ -continuous linear functional F on $BV_c(\mathbb{R}^n)$ is called a *charge*.

The following theorem, due to T. De Pauw and W.F. Pfeffer [4], is very deep and identifies fluxes and charges.

Theorem 4 (De Pauw-Pfeffer). *Given a charge F , there exists a vector field $v \in C^0(\mathbb{R}^n, \mathbb{R}^n)$ for which $F = F_v$.*

On the other hand, N.C. Phuc and M. Torres [11] give a sufficient condition for a Radon measure to define a charge.

Theorem 5. *Assume that μ is a Radon measure in $\mathbb{R}^n$ and that for each compact set $K \subseteq \mathbb{R}^n$ one has*

$$\lim_{\delta \rightarrow 0} \sup_{0 < r < \delta} \sup_{x \in K} \frac{\mu(B[x, r])}{r^{n-1}} = 0.$$

Then the linear functional

$$C_c^\infty(\mathbb{R}^n) \rightarrow \mathbb{R}, \varphi \mapsto \int_{\mathbb{R}^n} \varphi d\mu$$

extends in a unique way to a charge.

3. REMOVING THIN SETS

The following lemma will be useful in the sequel.

Lemma 6. *Assume that $E \subseteq \mathbb{R}^n$ is bounded and satisfies $\mathcal{H}^{n-1}(E) < \infty$. For all $\varepsilon > 0$ and $\eta > 0$, there exists an open set U containing E and satisfying the following conditions:*

- (A) $|U| \leq \varepsilon$,
- (B) $\|D\chi_U\| \leq \eta + \beta(n)\mathcal{H}^{n-1}(E)$,

where $\beta(n) := 2^{n-1}$.

Proof. First, observe that the dominated convergence theorem yields $|B(E, r)| \leq \varepsilon$ in case $r > 0$ is sufficiently small, say $r \leq \delta$. Continue by choosing a countable family $(B_k)_{k \geq 1}$ of open balls with diameter less than δ such that $U := \bigcup_{k=1}^\infty B_k \supseteq E$ and $2^{1-n} \alpha(n-1) \sum_{k=1}^\infty d(B_k)^{n-1} \leq \mathcal{S}^{n-1}(E) + \eta$. Without loss of generality, we

can assume that $B_k \cap E \neq \emptyset$ for each $k \geq 1$; in particular we have $U \subseteq B(E, \delta)$ and hence $|U| \leq \varepsilon$. On the other hand,

$$\|D\chi_U\| \leq \sum_{k=1}^{\infty} \|D\chi_{B_k}\| = 2^{1-n} \alpha(n-1) \sum_{k=1}^{\infty} d(B_k)^{n-1} \leq 2^{n-1} \mathcal{H}^{n-1}(E) + \eta,$$

and the proof is complete. $\square$

Lemma 7. *If $E \subseteq \mathbb{R}^n$ is closed and thin, then the collection of all $g \in BV_c(\mathbb{R}^n)$ that vanishes in a neighborhood of E is $\mathfrak{T}$ -dense in $BV_c(\mathbb{R}^n)$.*

Proof. First, write E as an increasing union $\bigcup_{k=1}^{\infty} E_k$ where $\mathcal{H}^{n-1}(E_k) < \infty$ for each $k \geq 1$. Replacing if necessary E_k by $E_k \cap B(k)$ we may assume that E_k is bounded for each $k \geq 1$. We further define for each $k \geq 1$ a constant $C_k := \|g\|_{\infty} [1 + \sum_{j=1}^k \mathcal{H}^{n-1}(E_j)] + \|Dg\|$ and choose $r > 0$ so that $\text{supp } g \subseteq B[r]$.

Now fix $\mathbf{U}$ a $\mathfrak{T}$ -neighborhood of 0 in $BV_c(\mathbb{R}^n)$. According to the definition of $\mathfrak{T}$, there exists $\delta_k > 0$ such that any $h \in BV_c(\mathbb{R}^n)$ satisfying $\text{supp } h \subset B[r]$, $\|Dh\| \leq C_k$ and $\|h\|_1 \leq \delta_k$ belongs to $2^{-k}\mathbf{U}$.

For each $k \geq 1$, choose according to Lemma 6 an open set $U_k \supseteq E_k \cap \text{supp } g$ satisfying $\|g\|_{\infty} |U_k| \leq \delta_k$ and $\|D\chi_{U_k}\| \leq 2^{-k} + \beta(n) \mathcal{H}^{n-1}(E_k)$. Observe that by compactness, there exists $l \geq 1$ such that $E \cap \text{supp}(g) \subseteq \bigcup_{k=1}^l U_k$.

Define $g_1 := g\chi_{U_1}$, and for $k \geq 2$, let $g_k := g\chi_{U_k \setminus \bigcup_{j=1}^{k-1} U_j}$. Fix $k \geq 1$. We obviously have $\text{supp } g_k \subseteq B[r]$ and $\|g_k\|_1 \leq \delta_k$, while

$$\|Dg_k\| \leq \|g\|_{\infty} \sum_{j=1}^k \|D\chi_{U_j}\| + \|Dg\| \leq \|g\|_{\infty} \left[1 + \beta(n) \sum_{j=1}^k \mathcal{H}^{n-1}(E_j) \right] + \|Dg\| \leq C_k,$$

where we used [10, Proposition 1.8.4] for example to estimate $\|D\chi_{U_k \setminus \bigcup_{j=1}^{k-1} U_j}\|$. Consequently we get $g_k \in 2^{-k}\mathbf{U}$.

We finish the argument by observing that $g' := \sum_{k=1}^l g_k$ equals g on a neighborhood of $E \cap \text{supp } g$ (and hence equals g on a neighborhood of E) and belongs to $\sum_{k=1}^l 2^{-k}\mathbf{U} \subseteq \mathbf{U}$. $\square$

Observation 8. It is not sufficient that E be thin for the conclusion of Lemma 7 to hold, as the following example shows.

Example 9. Assume that $n = 1$ and let E denote the set of rational numbers (which is thin but not closed). If $g \in BV_c(\mathbb{R})$ vanishes on an open set $U \supset E$, then choose according to [1, Theorem 3.27] a function $\tilde{g} = g$ a.e. whose pointwise variation equals $\|Dg\|$. We claim that $\{\tilde{g} \neq 0\}$ is countable, for otherwise given any $k \geq 1$, one would be able to find a finite set of $B_k \subset \{\tilde{g} \neq 0\}$ such that $\sum_{x \in E_k} \tilde{g}(x) \geq k$. This would imply that the pointwise variation of $\tilde{g}$ is infinite because $\{\tilde{g} = 0\}$ is dense in $\mathbb{R}$. Consequently $g = 0$ a.e.

Definition 10. The *support* of a charge F , denoted $\text{supp } F$, is the complementary set of the largest open set $U \subseteq \mathbb{R}^n$ having the following property: $F(g) = 0$ provided $g \in BV_c(\mathbb{R}^n)$ satisfies $\{g \neq 0\} \subset\subset U$.

Observation 11. Given a charge F , the support of F coincides with the support (defined in the usual way) of the distribution $F \upharpoonright C_c^{\infty}(\mathbb{R}^n)$ uniquely determining F .

An immediate corollary of Lemma 7 is the following.

Theorem 12. *If F is a charge and if $\text{supp } F$ is thin, then $F \equiv 0$.*

4. A NONREMOVABLE SET WITH DIMENSION $n - 1$

Given a cube Q , we denote by $\sigma(Q)$ the length of its edges. For $0 < \lambda < 1/2$, we let $\mathcal{E}(Q, \lambda)$ stand for the collection of 2^n cubes contained in Q whose edges have length $\lambda\sigma(Q)$, arranged in such a way that each cube of $\mathcal{E}(Q, \lambda)$ has a common vertex with Q .

Fix a sequence $\lambda = (\lambda_k)_{k \geq 1} \subseteq (0, 1/2)$. Write $E_0 := [0, 1]^n$ and $\mathcal{E}_0 = \{E_0\}$. For each $k \geq 1$, define inductively

$$\mathcal{E}_k(\lambda_1, \dots, \lambda_k) := \bigcup \{ \mathcal{E}(Q, \lambda_k) : Q \in \mathcal{E}_{k-1}(\lambda_1, \dots, \lambda_{k-1}) \}$$

and $E_k(\lambda_1, \dots, \lambda_k) := \bigcup \mathcal{E}_k(\lambda_1, \dots, \lambda_k)$ (or briefly $\mathcal{E}_k$ and E_k when the underlying sequence is clear). In particular, each cube in $\mathcal{E}_k$ has edges of length $\sigma_k := \lambda_1 \dots \lambda_k$. Finally we let $E(\lambda) = \bigcap_{k=1}^{\infty} E_k$.

Associated to λ , we choose $h : [0, \infty) \rightarrow [0, \infty)$ as a nondecreasing function satisfying $h(0) = 0$ and $h(\sigma_k) = 2^{-kn}$ for each $k \geq 1$. Recall that the Hausdorff h -measure of $E(\lambda)$ is positive and finite (see [7, Section 4.9]).

Theorem 13. *There is a compact set $E \subseteq \mathbb{R}^n$ having Hausdorff dimension $n - 1$ and satisfying the following properties:*

- (A) *there exists $v \in C^0(\mathbb{R}^n, \mathbb{R}^n)$ and a nonzero Radon measure μ supported in E such that $\operatorname{div} v = \mu$ (in the distribution sense);*
- (B) *if $u \in C^1(\mathbb{R}^n)$ is harmonic outside E , then u is harmonic in $\mathbb{R}^n$.*

In particular, $\mathcal{H}^{n-1} \llcorner E$ is not σ -finite.

Proof. For $k \geq 3$, let $\ell_k = 2^{-\frac{n}{n-1} \log k / \log(k-1)}$ and assume that $\lambda_k = \ell_k$ for k sufficiently large. Then:

- (1) $\mathcal{H}^{n-1} \llcorner E(\lambda)$ is not σ -finite,
- (2) $\dim_{\mathcal{H}}(E(\lambda)) = n - 1$;

the proofs of both facts are standard.

Given $k \geq 1$, we associate to E_k a Radon measure $\mu_k := |E_k|^{-1} \mathcal{L}^n \llcorner E_k$ supported in E_k . Observe that $\mu_k(\mathbb{R}^n) = 1$ for each k . Hence, [7, Theorem 1.23] ensures that there exists a subsequence of (μ_k) weakly*-converging to a Radon measure μ supported in $E(\lambda)$.

We now turn to showing that

$$(1) \quad \lim_{\delta \rightarrow 0} \sup_{0 < r < \delta} \sup_{x \in \mathbb{R}^n} \frac{\mu(B[x, r])}{r^{n-1}} = 0.$$

Observe that it suffices to show (1) with $\mathbb{R}^n$ replaced by $E(\lambda)$ and $B[x, r]$ replaced by $Q[x, r]$.

Claim 14. Fix $x \in E(\lambda)$ and k sufficiently large for $\lambda_k = \ell_k$ to hold. If $\sigma_{k+1} \leq r \leq \sigma_k$, then

$$\frac{\mu(Q[x, r])}{r^{n-1}} \leq 2^{-kn}.$$

Proof of the claim. Observe that $Q[x, r]$ is contained in 2^n cubes $K_1, \dots, K_{2^n}$ having x as a common vertex and edges of length σ_k . In particular, we get for $1 \leq i \leq 2^n$ and some arbitrary but fixed $Q_* \in \mathcal{E}_{k-1}$:

$$|E_l \cap K_i| \leq |E_l \cap Q| \quad \text{for any } Q \in \mathcal{E}(Q_*, \lambda_k),$$

and consequently

$$\mu_l(Q[x, r]) \leq 2^{-ln} \sigma_l^{-n} |E_l \cap Q_*| = 2^{-ln} 2^{(k+1)n}.$$

This yields

$$\frac{\mu_l(Q[x, r])}{r^{n-1}} \leq \frac{2^{(k-l+1)n}}{\sigma_{k+1}^{n-1}} \leq 2^{-ln} \leq 2^{-kn};$$

the conclusion follows for $\mu(Q[x, r]) \leq \overline{\lim}_l \mu_l(Q[x, r])$. $\square$

Claim 14 together with Theorems 4 and 5 yields a vector field $v \in C^0(\mathbb{R}^n, \mathbb{R}^n)$ such that $\operatorname{div} v = \mu$ in the distribution sense. Moreover, the equality $\mu(\mathbb{R}^n) = 1$ ensures that μ is nonzero. This proves property (A) of Theorem 13.

The proof of property (B) uses a recent result by A. Ruiz de Villa and X. Tolsa [12]. Observe that for k_0 sufficiently large

$$\tau := \sum_{k=k_0}^{\infty} \frac{1}{(2^{kn} \sigma_k^{n-1})^2} = \sum_{k=k_0}^{\infty} \left(\frac{1}{\log k} \right)^{2(n-1)} = \infty.$$

According to [12, Theorem 6], we get

$$\kappa^c(E(\lambda)) := \sup\{|\langle \Delta u, 1 \rangle| : u \in C^1(\mathbb{R}^n), \operatorname{supp}(\Delta u) \subseteq E, \|\nabla u\|_{\infty} \leq 1\} = \tau^{-1/2} = 0.$$

It follows from [6, Proposition 2.2] that the above equality is equivalent to property (B). The proof is complete. $\square$

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