C^k -QUASI-ISOMETRY SETS ARE PRE-COMPACT

F. T. FARRELL AND P. ONTANEDA

(Communicated by Chuu-Lian Terng)

ABSTRACT. Let M be a closed smooth manifold. By an argument formally similar to one used in constructing the Levi-Civita connection, it is shown that C^k -quasi-isometry sets in $DIFF^{k+1}(M)$ are C^{k+1} -bounded, where $0 \leq k < \infty$. This implies, using the Arzela-Ascoli theorem, that such sets are pre-compact in $DIFF^k(M)$.

Let M be a closed C^{k+1} -manifold, $1 \leq k < \infty$. Given a C^k -Riemannian metric g on M we know that the space of C^{k+1} -diffeomorphisms that preserve g (i.e. the isometries of (M, g)) with the C^0 -topology is compact [8]. In our study of the space of negatively curved metrics ([3], [4], [5], [6]) we have had to deal with the behavior of diffeomorphisms that “almost” preserve a Riemannian metric “up to order k ”. Even though it has been mentioned sparsely in the literature that “quasi-isometries are pre-compact” (see for instance section 12.55 of Besse’s book [1, p. 253]) we have not found a precise presentation (definitions, statements and proofs) that includes the C^k cases. We believe it is useful to have such a presentation, and in this short paper we aim to give one.

First we introduce some notation and definitions. The space of C^k -Riemannian metrics on M , with the C^k -topology, will be denoted by $\mathcal{MET}^{C^k}(M)$, or simply by $\mathcal{MET}(M)$. The spaces of C^k -self maps and C^k -self-diffeomorphisms of M , with the C^k -topology, will be denoted by $C^k(M, M)$ and $DIFF^k(M, M)$. Recall that $\mathcal{MET}(M)$ is an open set in the normed vector space of all C^k -symmetric two-tensors on M . Hence the notion of a set $A \subset \mathcal{MET}(M)$ being C^k -bounded is well-defined. In fact it is equivalent to saying the set of all matrix entries of the chart representative (with respect to a fixed finite set of charts) of all elements of A is bounded up to order k . Similarly, the notion of $F \subset C^k(M, M)$ being C^k -bounded is well-defined (see item 2 below).

Remarks. (1) A technical point here. The fixed set of charts $\mathcal{A} = \{(U, \phi)\}$ mentioned above has to have the following property. Every chart (U, ϕ) can be extended to a chart (V, ϕ) , with $\bar{U} \subset V$, $\bar{U}$ compact. In what follows we assume that a finite set of charts with this property has been given.

(2) $F \subset C^k(M, M)$ is C^k -bounded if the set of representatives of all $f : U_1 \cap f^{-1}(U_2) \rightarrow U_2$, $f \in F$, $(U_1, \phi_1), (U_2, \phi_2) \in \mathcal{A}$, is bounded up to order k .

(3) The definition of C^k -boundedness (in both cases) does not depend on the choice of the set of charts.

Received by the editors March 4, 2009, and, in revised form, March 6, 2009.

2000 *Mathematics Subject Classification.* Primary 58A05, 58D05, 58D17, 58D19.

Both authors were partially supported by NSF grants.

We say that $A \subset \mathcal{MET}(M)$ has *volume forms away from zero* if the set of positive numbers

$$\left\{ \det(a_{ij}^\alpha)(x), a \in A, \alpha = (U, \phi) \in \mathcal{A}, x \in \phi(U) \right\}$$

is bounded away from zero, where (a_{ij}^α) is the local coordinate expression of $a \in A$ with respect to α . Note that this implies that the set of entries of all $(a_{ij}^\alpha(x))^{-1}$ is bounded, provided A is C^0 -bounded. We say that $A \subset \mathcal{MET}(M)$ is C^k -well-bounded if it is C^k -bounded and has volume forms away from zero.

Let $g, h \in \mathcal{MET}^{C^k}(M)$. A set $F \subset \text{DIFF}^{k+1}(M)$ is a C^k -quasi-isometry set with respect to g, h if the sets $\{f^*h, f \in F\}$ and $\{f_*g, f \in F\}$ are C^k -well-bounded. More generally, a set $F \subset \text{DIFF}^{k+1}(M)$ is a C^k -quasi-isometry set if there are C^k -well-bounded sets $A, B \subset \mathcal{MET}^{C^k}(M)$, such that for every $f \in F$ there is $a \in A$ with $f_*a \in B$. Clearly F is a C^k -quasi-isometry set if it is a C^k -quasi-isometry set with respect to some g and h . It is a consequence of Theorem A below that the first definition is independent of the metrics g and h and is equivalent to the second definition.

Remark. Let F be a C^k -quasi-isometry set. Then any subset of F is a C^k -quasi-isometry set. Also $F^{-1} = \{f^{-1}, f \in F\}$ is a C^k -quasi-isometry set.

Here is our main theorem.

Theorem A. C^k -quasi-isometry sets are pre-compact in $\text{DIFF}^k(M)$, $k \geq 0$.

If F is C^{k+1} -bounded, then it is also C^k -bounded and, since the set of k -derivatives of the elements of $f \in F$ is bounded, these elements are Lipschitz (with the same constant); hence this set is equicontinuous. The Arselà-Ascoli theorem implies then the following:

Lemma. Let $F \subset C^{k+1}(M, M)$ be C^{k+1} -bounded. Then F is pre-compact in $C^k(M, M)$.

Using the lemma above, the proof of Theorem A is reduced to proving:

Theorem B. C^k -quasi-isometry sets are C^{k+1} -bounded.

Remarks. (1) Theorem A implies that the definition of C^k -quasi-isometry set with respect to two metrics g, h does not depend on g, h .

(2) Rigorously, Theorem B implies that C^k -quasi-isometry sets are pre-compact in $C^k(M, M)$. But, using the remark before the statement of Theorem A, it is straightforward to check that they are actually pre-compact in $\text{DIFF}^k(M)$.

The proof of Theorem B implies (and, essentially reduces to) the following interesting fact. Let f be a diffeomorphism between open sets of $\mathbb{R}^n$. Let $Df(x) = \left(\frac{\partial f_i}{\partial x_j}(x) \right)$ be its Jacobian matrix at x . Assume: (i) the entries of $Df^T(x) Df(x)$ are bounded up to order k , and (ii) $\det Df(x)$ is bounded away from zero. Then the entries of $Df(x)$ are bounded up to order k . Note that it is important to assume that the map $x \mapsto Df(x)$ has for images Jacobian matrices; otherwise the result fails: let $x \mapsto C(x) \in O(n)$ be any wild function, but we have that $C^T(x)C(x)$ is constant.

Proof of Theorem B. Let $F \subset \text{DIFF}^{k+1}(M)$ be a C^k -quasi-isometry set. Let A and B be C^k -well-bounded sets such that for every $f \in F$, there is $a \in A$ with

$f_*a \in B$. In what follows the local representatives of $f \in F$, $a \in A$ and $b \in B$ will be denoted by the same letters. Also, for $f \in F$, $Df(x) = \left(\frac{\partial f_i}{\partial x_j}(x) \right)$ denotes the Jacobian matrix of (a local representative of) f . Then, for $f \in F$ there are $a \in A$, $b \in B$ such that:

$$(1) \quad Df(x)^T a(x) Df(x) = b(x)$$

for every $x \in \phi(U)$, $(U, \phi) \in \mathcal{A}$. We now proceed to verify Theorem B by induction on k starting with the special case $k = 0$. Let $c(x)$ be the (unique and smoothly defined) square root of $a(x)$. Then

$$\left(c(x) Df(x) \right)^T \left(c(x) Df(x) \right) = b(x).$$

Since the diagonal entries of this matrix are the norm of the columns of $c(x)Df(x)$ and the entries of $b(x)$ are bounded, we have that the entries of $c(x)Df(x)$ are also bounded. But the entries of $c^{-1}(x)$ are bounded since $\det c(x) = \sqrt{\det a(x)}$ is bounded away from zero. We therefore conclude that the entries of $Df(x)$ are bounded, thus proving the theorem when $k = 0$. Interchanging the sets A and B in the above argument, we obtain also that the $\|Df^{-1}\|$'s are bounded as well.

Proceeding with the induction we assume $k > 0$ and that all derivatives of order $\leq k$ of all f 's are bounded. We prove that all derivatives of order $k + 1$ are also bounded by contradiction. Suppose F is not C^{k+1} -bounded. Then there are sequences (f_m) in F and (x_m) in M such that some $k + 1$ derivative of f_m , at x_m , becomes large as $m \rightarrow \infty$. Since M is compact and (f_m) is pre-compact in $C^0(M, M)$ we can assume that both sequences converge. It follows that there are two charts, $\psi : V' \rightarrow V \subset \mathbb{R}^n$, $\psi' : W' \rightarrow W \subset \mathbb{R}^n$, such that $f_m(V) \subset W$ and some $k + 1$ derivative of $f_m : V \rightarrow W$ becomes large as $m \rightarrow \infty$. (The charts ψ, ψ' are not necessarily in $\mathcal{A}$ but can be extended to charts in $\mathcal{A}$.) We have reduced our problem to $\mathbb{R}^n$.

For a j -multi-index $\alpha = (i_1, \dots, i_j)$, $1 \leq i_\ell \leq n$ (i.e. $\alpha \in \{1, 2, \dots, n\}^j$), we write $\partial x_\alpha = \partial x_{i_1} \cdots \partial x_{i_j}$. If α and β are multi-indices, $\alpha\beta$ denotes the "concatenation" multi-index. Also a 1-multi-index (i) will be denoted just as i . Let α be a k -multi-index. Applying $\frac{\partial^k}{\partial x_\alpha}$ to equation (1) above we get:

$$\left[Df^T a \left(\frac{\partial^k}{\partial x_\alpha}(Df) \right) + \left(\frac{\partial^k}{\partial x_\alpha}(Df) \right)^T a Df \right] + d = \frac{\partial^k}{\partial x_\alpha} b,$$

where d is a sum of products of derivatives of order $\leq k$ of a and f . Hence the expression inside the brackets is bounded. Note that the columns of Df are $f_i = \frac{\partial}{\partial x_i} f$, that is, $Df = [f_1 \cdots f_n]$. To simplify our notation, for a multi-index β we write $f_\beta = \frac{\partial^k}{\partial x_\beta} f$. Using this notation, the equations above tell us that the following set of expressions is uniformly bounded:

$$(2) \quad f_i^T a f_{\alpha j} + f_j^T a f_{\alpha i}$$

for all $1 \leq i, j \leq n$ and every $\alpha \in \{1, \dots, n\}^j$. Note that the second term in (2) is obtained from the first term by interchanging i and j .

Claim. The fact that the expressions in (2) are bounded implies that the following expressions are bounded for every $\beta \in \{1, \dots, n\}^{k+1}$:

$$(Df)^T a f_\beta.$$

Since the Df^{-1} 's and a^{-1} 's are bounded, we conclude that the expressions $f_\beta = \frac{\partial^{k+1}}{\partial x^\beta} f$ are bounded. This contradicts our assumption that “some $k+1$ derivative of $f_m : V \rightarrow W$ becomes large as $m \rightarrow \infty$ ”, and hence proves Theorem B once we verify the claim.

Proof of the claim. To simplify our notation we denote the expression $f_i^T a f_\beta$ as simply $[i, \beta]$. Note that the i -th entry of the column vector $Df^T a f_\beta$ is $[i, \beta]$. Therefore to prove the claim we need to show that $[i, \beta]$ is bounded, for every i and every $(k+1)$ -multi-index β . Note that our argument above proving Theorem B for $k=0$ also proved the claim for $k=0$. Hence we may assume that $k > 0$. Now, with the bracket notation introduced above, the bounded expression in (2) can be rewritten as:

$$(2') \quad [i, \alpha j] + [j, \alpha i]$$

That is, the expression in (2') is bounded for every i, j and k -multi-index α . Since $k+1 \geq 2$, we can write $\beta = \gamma s t$ for some $(k-1)$ -multi-index γ and $1 \leq s, t \leq n$. By cyclically permuting i, s, t and applying the fact that (2') is bounded, we obtain that the following three expressions are bounded:

$$\begin{aligned} [i, \gamma s t] + [t, \gamma s i] \\ [s, \gamma t i] + [i, \gamma t s] \\ [t, \gamma i s] + [s, \gamma i t]. \end{aligned}$$

Notice that only three numbers occur in these three sums, namely:

$$\begin{aligned} A &= [i, \gamma s t] = [i, \gamma t s] \\ B &= [s, \gamma t i] = [s, \gamma i t] \\ C &= [t, \gamma i s] = [t, \gamma s i]. \end{aligned}$$

Therefore the above three bounded terms become:

$$\begin{aligned} A + C \\ A + B \\ B + C. \end{aligned}$$

Consequently all three terms A , B , and C are bounded. This completes the proof of the claim because $A = [i, \beta]$.

Remark. The above proof is formally similar to that given for the Fundamental Lemma of Riemannian Geometry, i.e. for the construction of the Levi-Civita connection (see, for instance, pp. 48-49 of [7]).

REFERENCES

- [1] A. L. Besse, *Einstein Manifolds*. Ergebnisse series, vol. **10**, Springer-Verlag, Berlin, 1987. MR867684 (88f:53087)
- [2] D. G. Ebin, *The manifold of Riemannian metrics*. 1970 Global Analysis, Proc. Sympos. Pure Math., Vol. XV (Berkeley, Calif., 1968), Amer. Math. Soc., Providence, RI, 11-40. MR0267604 (42:2506)
- [3] F.T. Farrell and P. Ontaneda, *The Teichmüller space of pinched negatively curved metrics on a hyperbolic manifold is not contractible*. Ann. of Math. (2) **170** (2009), no. 1, 45–65.
- [4] F.T. Farrell and P. Ontaneda, *On the topology of the space of negatively curved metrics*. Submitted for publication. arXiv:mathDG.0607367

- [5] F.T. Farrell and P. Ontaneda, *Teichmüller spaces and bundles with negatively curved fibers*. Submitted for publication. arXiv:0709.0998
- [6] F.T. Farrell and P. Ontaneda, *On the moduli space of negatively curved metrics of a hyperbolic manifold*. Submitted for publication. arXiv:0805.2635
- [7] J. W. Milnor, *Morse theory*. Annals of Math. Studies, Princeton University Press, Princeton, New Jersey, 1963. MR0163331 (29:634)
- [8] S. B. Myers and N. E. Steenrod, *The group of isometries of a Riemannian manifold*. Ann. of Math. (2) **40** (1939), 400-416. MR1503467

DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK, BINGHAMTON, NEW YORK
13902

DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK, BINGHAMTON, NEW YORK
13902