THE LAPLACE TRANSFORM OF THE PSI FUNCTION

ATUL DIXIT

(Communicated by Peter A. Clarkson)

ABSTRACT. An expression for the Laplace transform of the psi function

$$L(a) := \int_0^\infty e^{-at} \psi(t+1) dt$$

is derived using two different methods. It is then applied to evaluate the definite integral

$$M(a) = \frac{4}{\pi} \int_0^\infty \frac{x^2 dx}{x^2 + \ln^2(2e^{-a} \cos x)},$$

for $a > \ln 2$ and to resolve a conjecture posed by Olivier Oloa.

1. INTRODUCTION

Let $\psi(x)$ denote the logarithmic derivative of the gamma function $\Gamma(x)$, i.e.,

$$(1.1) \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

The psi function has been studied extensively and still continues to receive attention from many mathematicians. Many of its properties are listed in [6, pp. 952–955]. Surprisingly, an explicit formula for the Laplace transform of the psi function, i.e.,

$$(1.2) \quad L(a) := \int_0^\infty e^{-at} \psi(t+1) dt,$$

is absent from the literature. Recently in [5], the nature of the Laplace transform was studied by demonstrating the relationship between $L(a)$ and the Glasser-Manna-Oloa integral

$$(1.3) \quad M(a) := \frac{4}{\pi} \int_0^\infty \frac{x^2 dx}{x^2 + \ln^2(2e^{-a} \cos x)},$$

namely, that, for $a > \ln 2$,

$$(1.4) \quad M(a) = L(a) + \frac{\gamma}{a},$$

where γ is the Euler's constant. In [1], T. Amdeberhan, O. Espinosa and V. H. Moll obtained certain analytic expressions for $M(a)$ in the complementary range $0 < a \leq \ln 2$. Our goal here is to derive an explicit expression of $L(a)$ for $a > 0$ in terms of elementary functions and a certain simple infinite series which cannot be evaluated

Received by the editors October 28, 2008.

2000 *Mathematics Subject Classification*. Primary 33B15.

Key words and phrases. Laplace transform, gamma function, psi function.

©2009 American Mathematical Society
 Reverts to public domain 28 years from publication

in terms of elementary functions. We give two different proofs, the second of which is shorter. However, it makes use of a formula (equation 2.31) in [1], viz., equation (3.1) in our paper, which requires considerable work for its derivation. Also, our first proof is self-contained and elementary. We then use our expression for the Laplace transform of the psi function to evaluate $M(a)$ for $a > \ln 2$ and thereby resolve a conjecture posed by Olivier Oloa in [8]. Our main theorem can be stated as follows:

Theorem 1.1. *Let $a > 0$. If $\psi(x)$ is defined as in (1.1), then*

$$(1.5) \quad \begin{aligned} L(a) = \int_0^\infty e^{-at} \psi(t+1) dt &= \left(\frac{1}{e^a - 1} - \frac{1}{a} + 1 \right) \ln \left(\frac{2\pi}{a} \right) + 2a \sum_{n=1}^\infty \frac{\ln n}{a^2 + 4n^2\pi^2} \\ &\quad + \frac{1}{4} \left(\psi \left(\frac{ia}{2\pi} \right) + \psi \left(-\frac{ia}{2\pi} \right) \right) - \frac{\gamma + \ln a}{a}. \end{aligned}$$

2. LAPLACE TRANSFORM OF THE PSI FUNCTION: FIRST PROOF

First, from [2, p. 259, formula 6.3.18] for $|\arg z| < \pi$, as $z \rightarrow \infty$,

$$(2.1) \quad \psi(z) \sim \ln z - \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} - \frac{1}{252z^6} + \cdots$$

We now state a lemma¹ which will be subsequently used in the proof. The integral involved in the lemma exists because of (2.1). This integral gives the motivation for decomposing $L(a)$ into the two integrals mentioned in equation (2.9) below. This integral evaluation is implicit in the work of A.P. Guinand on a certain transformation formula involving the psi function (see the footnote on the last page in [7]), but he neither proves it nor explicitly states it in [7].

Lemma 2.1.

$$(2.2) \quad \int_0^\infty \left(\psi(t+1) - \frac{1}{2(t+1)} - \ln t \right) dt = \frac{1}{2} \ln 2\pi.$$

Proof. Let I denote the integral on the left-hand side. Then,

$$(2.3) \quad \begin{aligned} I &= \int_0^\infty \frac{d}{dt} \left(\ln \frac{e^t \Gamma(t+1)}{t^t \sqrt{t+1}} \right) dt \\ &= \lim_{t \rightarrow \infty} \ln \frac{e^t \Gamma(t+1)}{t^t \sqrt{t+1}} - \lim_{t \rightarrow 0} \ln \frac{e^t \Gamma(t+1)}{t^t \sqrt{t+1}} \\ &= \ln \left(\lim_{t \rightarrow \infty} \frac{e^t \Gamma(t+1)}{t^t \sqrt{t+1}} \right) - \ln \left(\lim_{t \rightarrow 0} e^t \Gamma(t+1) \right) - \lim_{t \rightarrow 0} t \ln t - \lim_{t \rightarrow 0} \frac{1}{2} \ln(t+1) \\ &= \ln \lim_{t \rightarrow \infty} \frac{e^t \Gamma(t+1)}{t^t \sqrt{t+1}}. \end{aligned}$$

Stirling's formula [6, p. 945, formula 8.327] tells us that

$$(2.4) \quad \Gamma(z) \sim z^{z-1/2} e^{-z} \sqrt{2\pi},$$

as $|z| \rightarrow \infty$ and $|\arg z| \leq \pi - \delta$, where $0 < \delta < \pi$. Hence, employing (2.4), we find that

$$(2.5) \quad \frac{e^t \Gamma(t+1)}{t^t \sqrt{t+1}} \sim \left(1 + \frac{1}{t} \right)^t \frac{\sqrt{2\pi}}{e},$$

¹The author is indebted to M. L. Glasser for the proof of this lemma. The author's original proof is significantly longer than that given by Glasser.

so that

$$(2.6) \quad \lim_{t \rightarrow \infty} \frac{e^t \Gamma(t+1)}{t^t \sqrt{t+1}} = \sqrt{2\pi}.$$

Thus from (2.3) and (2.6), we conclude that

$$(2.7) \quad I = \frac{1}{2} \ln 2\pi. \quad \square$$

Now we begin with the proof of Theorem 1.1. Without mentioning it explicitly, throughout this paper we make use of the fact that

$$(2.8) \quad \psi(x+1) = \psi(x) + \frac{1}{x}.$$

Using (2.1), we can decompose the Laplace transform of the psi function into two integrals,

$$(2.9) \quad \begin{aligned} \int_0^\infty e^{-at} \psi(t+1) dt &= \int_0^\infty e^{-at} \left(\psi(t+1) - \frac{1}{2(t+1)} - \ln t \right) dt \\ &\quad + \int_0^\infty e^{-at} \left(\frac{1}{2(t+1)} + \ln t \right) dt \\ &= I_1 + I_2, \text{ say,} \end{aligned}$$

where (2.1) ensures that I_1 exists. The integral I_2 is easy to evaluate. Using the definition of the incomplete gamma function

$$(2.10) \quad \Gamma(0, a) = \int_a^\infty \frac{e^{-t}}{t} dt,$$

and the fact [6, p. 602, formula 4.331, no. 1] that

$$(2.11) \quad \int_0^\infty e^{-ax} \ln x dx = -\frac{\gamma + \ln a}{a}, \quad \text{for } a > 0,$$

we obtain

$$(2.12) \quad I_2 = \frac{e^a}{2} \Gamma(0, a) - \frac{\gamma + \ln a}{a}.$$

Now let $I_1 = I_1(a)$. Differentiating under the integral sign, we have

$$(2.13) \quad \begin{aligned} I_1'(a) &= \int_0^\infty -te^{-at} \left(\psi(t+1) - \frac{1}{2(t+1)} - \ln t \right) dt \\ &= \int_0^\infty e^{-at} \left(t \ln t + \frac{t}{2(t+1)} - t\psi(t+1) \right) dt \\ &= \int_0^\infty e^{-at} \left(t \ln t + \frac{1}{2} - \frac{1}{2(t+1)} - t \left(\psi(t) + \frac{1}{t} \right) \right) dt \\ &= - \int_0^\infty \frac{e^{-at}}{2(t+1)} dt - \int_0^\infty te^{-at} \left(\psi(t) + \frac{1}{2t} - \ln t \right) dt \\ &= -\frac{e^a}{2} \Gamma(0, a) - I_3, \text{ say.} \end{aligned}$$

Using a well-known formula [9, p. 248] in a slightly simplified form, we have for $\operatorname{Re} z > 0$,

$$(2.14) \quad \psi(z) + \frac{1}{2z} - \ln z = - \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) e^{-tz} dt.$$

Thus,

$$\begin{aligned}
 I_3 &= - \int_0^\infty t e^{-at} \int_0^\infty \left(\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2} \right) e^{-xt} dx dt \\
 &= - \int_0^\infty \left(\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2} \right) \int_0^\infty t e^{-(x+a)t} dt dx \\
 (2.15) \quad &= - \int_0^\infty \left(\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2} \right) \frac{dx}{(x+a)^2}.
 \end{aligned}$$

The inversion of the order of integration above is easily justifiable. Next from [3, p. 191], we have for $x \neq 0$,

$$(2.16) \quad \sum_{n=1}^{\infty} \frac{1}{x^2 + 4n^2\pi^2} = \frac{1}{2x} \left(\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2} \right).$$

Hence,

$$\begin{aligned}
 I_3 &= - \int_0^\infty \frac{2x dx}{(x+a)^2} \sum_{n=1}^{\infty} \frac{1}{x^2 + 4n^2\pi^2} \\
 &= - \frac{1}{a} \sum_{n=1}^{\infty} \int_0^\infty \frac{2ax}{(x+a)^2(x^2 + 4n^2\pi^2)} dx \\
 &= - \frac{1}{a} \sum_{n=1}^{\infty} \int_0^\infty \frac{(x^2 + 2ax + a^2) - (x^2 + a^2)}{(x+a)^2(x^2 + 4n^2\pi^2)} dx \\
 (2.17) \quad &= - \frac{1}{a} \sum_{n=1}^{\infty} \left(\int_0^\infty \frac{dx}{x^2 + 4n^2\pi^2} - \int_0^\infty \frac{x^2 + a^2}{(x+a)^2(x^2 + 4n^2\pi^2)} dx \right),
 \end{aligned}$$

where the inversion of integration and summation can be easily seen via absolute convergence. Using Mathematica, we find that

$$\begin{aligned}
 &\int_0^\infty \frac{x^2 + a^2}{(x+a)^2(x^2 + 4n^2\pi^2)} dx \\
 &= \frac{a^4 - 8a^2n^2\pi^2 + 32an^3\pi^2 + 16n^4\pi^4 + 8a^3n(1 + \ln(2)) + 32an^3\pi^2 \ln(a/2n\pi) + 8a^3n \ln(n\pi/a)}{4n(a^2 + 4n^2\pi^2)^2} \\
 &= \frac{(a^2 + 4n^2\pi^2)^2 - 16a^2n^2\pi^2 + 8a^3n(1 + \ln(2\pi n/a)) + 32an^3\pi^2(1 - \ln(2\pi n/a))}{4n(a^2 + 4n^2\pi^2)^2}.
 \end{aligned}$$

(2.18)

Hence from (2.17) and (2.18), we deduce after some simplification that

$$\begin{aligned}
 I_3 &= \frac{1}{4a} \sum_{n=1}^{\infty} \frac{32an^3\pi^2(1 - \ln(2\pi n/a)) + 8a^3n(1 + \ln(2\pi n/a)) - 16a^2n^2\pi^2}{n(a^2 + 4n^2\pi^2)^2} \\
 (2.19) \quad &= S_1 + S_2 + S_3 + S_4 - S_5,
 \end{aligned}$$

where

$$(2.20) \quad S_1 = 8\pi^2 \sum_{n=1}^{\infty} \frac{n^2}{(a^2 + 4n^2\pi^2)^2},$$

$$(2.21) \quad S_2 = \frac{\ln(2\pi/a)}{4a} \sum_{n=1}^{\infty} \frac{8a^3n - 32an^3\pi^2}{n(a^2 + 4n^2\pi^2)^2},$$

$$(2.22) \quad S_3 = \frac{1}{4a} \sum_{n=1}^{\infty} \frac{(8a^3n - 32an^3\pi^2) \ln n}{n(a^2 + 4n^2\pi^2)^2},$$

$$(2.23) \quad S_4 = 2a^2 \sum_{n=1}^{\infty} \frac{1}{(a^2 + 4n^2\pi^2)^2},$$

and

$$(2.24) \quad S_5 = 4a\pi^2 \sum_{n=1}^{\infty} \frac{n}{(a^2 + 4n^2\pi^2)^2}.$$

Decomposing S_1 into partial fractions,

$$\begin{aligned} S_1 &= 2 \sum_{n=1}^{\infty} \frac{1}{a^2 + 4n^2\pi^2} - 2a^2 \sum_{n=1}^{\infty} \frac{1}{(a^2 + 4n^2\pi^2)^2} \\ &= 2 \sum_{n=1}^{\infty} \frac{1}{a^2 + 4n^2\pi^2} + a \sum_{n=1}^{\infty} \frac{d}{da} \left(\frac{1}{a^2 + 4n^2\pi^2} \right) \\ (2.25) \quad &= 2 \sum_{n=1}^{\infty} \frac{1}{a^2 + 4n^2\pi^2} + a \frac{d}{da} \left(\sum_{n=1}^{\infty} \frac{1}{a^2 + 4n^2\pi^2} \right), \end{aligned}$$

and then using (2.16), we find after some simplification that

$$(2.26) \quad S_1 = \frac{2 \coth(a/2) - a/\sinh^2(a/2)}{8a}.$$

Similarly, breaking S_2 into partial fractions and then employing (2.16) and (2.26), we find upon simplification that

$$(2.27) \quad S_2 = \frac{\ln(2\pi/a)}{4a} \left(\frac{a}{\sinh^2(a/2)} - \frac{4}{a} \right).$$

The sum S_4 is similarly evaluated as

$$(2.28) \quad S_4 = \frac{4 + a^2 + a \sinh(a) - 4 \cosh(a)}{8a^2 \sinh^2(a/2)}.$$

Hence from (2.26), (2.27) and (2.28) and using the identities

$$\begin{aligned} \sinh(2x) &= 2 \sinh(x) \cosh(x), \\ \cosh(2x) &= 1 + 2 \sinh^2(x), \end{aligned}$$

we find upon simplification that

$$(2.29) \quad S_1 + S_2 + S_4 = \frac{-4 + 2a \coth(a/2) + (-4 + a^2/\sinh^2(a/2)) \ln(2\pi/a)}{4a^2}.$$

Thus, from (2.19), (2.22), (2.24) and (2.29),

$$\begin{aligned}
 I_3 &= \frac{-4 + 2a \coth(a/2) + (-4 + a^2/\sinh^2(a/2)) \ln(2\pi/a)}{4a^2} \\
 &\quad + \sum_{n=1}^{\infty} \frac{2(a^2 + 4n^2\pi^2) \ln n - 4an\pi^2 - 16n^2\pi^2 \ln n}{(a^2 + 4n^2\pi^2)^2} \\
 &= \frac{-4 + 2a \coth(a/2) + (-4 + a^2/\sinh^2(a/2)) \ln(2\pi/a)}{4a^2} \\
 (2.30) \quad &+ 2 \sum_{n=1}^{\infty} \frac{\ln n}{a^2 + 4n^2\pi^2} - 4\pi^2 \sum_{n=1}^{\infty} \frac{an}{(a^2 + 4n^2\pi^2)^2} - 16\pi^2 \sum_{n=1}^{\infty} \frac{n^2 \ln n}{(a^2 + 4n^2\pi^2)^2}.
 \end{aligned}$$

Thus from (2.13) and (2.30), we have

$$\begin{aligned}
 I_1'(a) &= -\frac{e^a}{2} \Gamma(0, a) - \frac{-4 + 2a \coth(a/2) + (-4 + a^2/\sinh^2(a/2)) \ln(2\pi/a)}{4a^2} \\
 (2.31) \quad &- 2 \sum_{n=1}^{\infty} \frac{\ln n}{a^2 + 4n^2\pi^2} + 4\pi^2 \sum_{n=1}^{\infty} \frac{an}{(a^2 + 4n^2\pi^2)^2} + 16\pi^2 \sum_{n=1}^{\infty} \frac{n^2 \ln n}{(a^2 + 4n^2\pi^2)^2}.
 \end{aligned}$$

Now using the fact [6, p. 953, formula 8.363, no. 8]

$$(2.32) \quad \psi'(x) = \sum_{n=0}^{\infty} \frac{1}{(x+n)^2}$$

to integrate the second to last series in (2.31), we deduce that

$$\begin{aligned}
 I_1(a) &= -\frac{e^a}{2} \Gamma(0, a) - \frac{1}{2} \ln(a) + \frac{(-2 + a \coth(a/2)) \ln(2\pi/a)}{2a} \\
 &\quad - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\ln n \tan^{-1}(a/2n\pi)}{n} + \frac{4\pi^2}{16\pi^2} \left(\psi \left(1 + \frac{ia}{2\pi} \right) + \psi \left(1 - \frac{ia}{2\pi} \right) \right) \\
 (2.33) \quad &+ 16\pi^2 \sum_{n=1}^{\infty} \frac{n^2 \ln n \left(\tan^{-1} \left(\frac{a}{2n\pi} \right) + \frac{2an\pi}{a^2 + 4n^2\pi^2} \right)}{16n^3\pi^3} + c,
 \end{aligned}$$

where c is a constant of integration. Thus after simplification, we obtain

$$\begin{aligned}
 I_1(a) &= -\frac{e^a}{2} \Gamma(0, a) - \frac{1}{2} \ln(a) + \frac{(-2 + a \coth(a/2)) \ln(2\pi/a)}{2a} \\
 (2.34) \quad &+ \frac{1}{4} \left(\psi \left(1 + \frac{ia}{2\pi} \right) + \psi \left(1 - \frac{ia}{2\pi} \right) \right) + 2a \sum_{n=1}^{\infty} \frac{\ln n}{a^2 + 4n^2\pi^2} + c.
 \end{aligned}$$

Let $a \rightarrow 0$ in (2.34). Taking the limit under the integral sign on the left-hand side of (2.34), using Lebesgue's dominated convergence theorem, and then using Lemma 2.1 and the fact that $\psi(1) = -\gamma$,

$$(2.35) \quad \frac{1}{2} \ln 2\pi = -\frac{1}{2} \lim_{a \rightarrow 0} (e^a \Gamma(0, a) + \ln(a)) + \lim_{a \rightarrow 0} \left(\frac{(-2 + a \coth(a/2)) \ln(2\pi/a)}{2a} \right) - \frac{\gamma}{2} + c.$$

From [6, p. 951, formula 8.359, no. 1], we find that

$$(2.36) \quad \Gamma(0, x) = -\text{Ei}(-x).$$

Also, from [6, p. 935, formula 8.214, no. 1], for $x < 0$,

$$(2.37) \quad \text{Ei}(x) = \gamma + \ln(-x) + \sum_{k=1}^{\infty} \frac{x^k}{k \cdot k!}.$$

From (2.36) and (2.37), we readily get the asymptotic expansion for the incomplete gamma function as

$$(2.38) \quad \Gamma(0, x) = -\gamma - \ln x + x - \frac{x^2}{4} + \frac{x^3}{18} - \cdots,$$

as $x \rightarrow 0^+$. Hence,

$$(2.39) \quad \lim_{a \rightarrow 0} (e^a \Gamma(0, a) + \ln(a)) = -\gamma.$$

Next, using the series expansion of $\coth x$ [6, p. 42, formula 1.411, no. 8],

$$(2.40) \quad \coth x = \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} - \cdots, \quad \text{for } 0 < x^2 < \pi^2,$$

we deduce that

$$(2.41) \quad \begin{aligned} & \lim_{a \rightarrow 0} \frac{(-2 + a \coth(a/2)) \ln(2\pi/a)}{2a} \\ &= \lim_{a \rightarrow 0} \left[\frac{1}{2a} \left(-2 + a \left(\frac{2}{a} + \frac{a}{6} - \frac{a^3}{360} + \cdots \right) \right) \ln \left(\frac{2\pi}{a} \right) \right] \\ &= \lim_{a \rightarrow 0} \left[\left(\frac{a}{12} - \frac{a^3}{720} + \cdots \right) \ln 2\pi - \left(\frac{a}{12} - \frac{a^3}{720} + \cdots \right) \ln a \right] \\ &= 0. \end{aligned}$$

Thus from (2.35), (2.39) and (2.41), we obtain

$$(2.42) \quad c = \frac{1}{2} \ln 2\pi.$$

Now substituting (2.42) in (2.34), we have

$$(2.43) \quad \begin{aligned} I_1(a) &= -\frac{e^a}{2} \Gamma(0, a) - \frac{1}{2} \ln(a) + \frac{(-2 + a \coth(a/2)) \ln(2\pi/a)}{2a} + 2a \sum_{n=1}^{\infty} \frac{\ln n}{a^2 + 4n^2\pi^2} \\ &+ \frac{1}{4} \left(\psi \left(1 + \frac{ia}{2\pi} \right) + \psi \left(1 - \frac{ia}{2\pi} \right) \right) + \frac{1}{2} \ln 2\pi. \end{aligned}$$

From (2.9), (2.12) and (2.43), we have

$$(2.44) \quad \begin{aligned} \int_0^{\infty} e^{-at} \psi(t+1) dt &= \frac{(-2 + a \coth(a/2)) \ln(2\pi/a)}{2a} + 2a \sum_{n=1}^{\infty} \frac{\ln n}{a^2 + 4n^2\pi^2} \\ &+ \frac{1}{4} \left(\psi \left(1 + \frac{ia}{2\pi} \right) + \psi \left(1 - \frac{ia}{2\pi} \right) \right) + \frac{1}{2} \ln \left(\frac{2\pi}{a} \right) \\ &- \frac{\gamma + \ln a}{a}. \end{aligned}$$

Finally (2.44) simplifies to (1.5). This completes the proof of Theorem 1.1.

3. SECOND PROOF

In [1, Cor. 2.2], Amdeberhan, Espinosa and Moll showed that

$$(3.1) \quad \int_0^\infty e^{-at} \ln \Gamma(t) dt = -\frac{\gamma + \ln a}{ace^a} + \frac{A(a-c)}{a^2c} - \frac{1}{2a} \Lambda\left(\frac{a}{2\pi}\right) + 2 \sum_{j=1}^\infty \frac{\ln j}{a^2 + 4\pi^2 j^2},$$

where $a > 0$, $c = 1 - e^{-a}$, $A = \ln 2\pi + \gamma$ and

$$(3.2) \quad \Lambda(z) = \lim_{n \rightarrow \infty} \left(\sum_{j=1}^n \frac{j}{j^2 + z^2} - \ln n \right),$$

a generalization of Euler's constant. Now

$$(3.3) \quad \int_0^\infty e^{-at} \ln \Gamma(t+1) dt = \int_0^\infty e^{-at} \ln \Gamma(t) dt + \int_0^\infty e^{-at} \ln t dt$$

and

$$(3.4) \quad \begin{aligned} \int_0^\infty e^{-at} \psi(t+1) dt &= \int_0^\infty \frac{d}{dt} (e^{-at} \ln \Gamma(t+1)) dt + a \int_0^\infty e^{-at} \ln \Gamma(t+1) dt \\ &= \lim_{t \rightarrow \infty} e^{-at} \ln \Gamma(t+1) - \lim_{t \rightarrow 0} e^{-at} \ln \Gamma(t+1) + a \int_0^\infty e^{-at} \ln \Gamma(t+1) dt. \end{aligned}$$

Now using (2.1), we find that

$$(3.5) \quad \begin{aligned} \lim_{t \rightarrow \infty} e^{-at} \ln \Gamma(t+1) &= \lim_{t \rightarrow \infty} \frac{\psi(t+1)}{ae^{at}} \\ &= \frac{1}{a} \lim_{t \rightarrow \infty} \frac{1}{e^{at}} \left(\ln(t+1) - \frac{1}{2(t+1)} - \frac{1}{12(t+1)^2} + \dots \right) \\ &= \frac{1}{a} \lim_{t \rightarrow \infty} \frac{\ln(t+1)}{e^{at}} \\ &= \frac{1}{a^2} \lim_{t \rightarrow \infty} \frac{1}{e^{at}(t+1)} \\ &= 0. \end{aligned}$$

Also,

$$(3.6) \quad \lim_{t \rightarrow 0} e^{-at} \ln \Gamma(t+1) = 0.$$

Thus using (3.4), (3.5), (3.6), (3.3), (3.1) and (2.11), we find that

$$(3.7) \quad \begin{aligned} \int_0^\infty e^{-at} \psi(t+1) dt &= a \int_0^\infty e^{-at} \ln \Gamma(t) dt + a \int_0^\infty e^{-at} \ln t dt \\ &= a \left(-\frac{\gamma + \ln a}{ace^a} + \frac{A(a-c)}{a^2c} - \frac{1}{2a} \Lambda\left(\frac{a}{2\pi}\right) + 2 \sum_{j=1}^\infty \frac{\ln j}{a^2 + 4\pi^2 j^2} \right) - (\gamma + \ln a) \\ &= -\frac{\gamma + \ln a}{e^a - 1} + \frac{A(a-c)}{ac} - \frac{1}{2} \Lambda\left(\frac{a}{2\pi}\right) + 2a \sum_{j=1}^\infty \frac{\ln j}{a^2 + 4\pi^2 j^2} - (\gamma + \ln a). \end{aligned}$$

Now we show that

$$(3.8) \quad \Lambda\left(\frac{a}{2\pi}\right) = \lim_{n \rightarrow \infty} \left(\sum_{j=1}^n \frac{4\pi^2 j}{a^2 + 4\pi^2 j^2} - \ln n \right) = \frac{-1}{2} \left(\psi\left(\frac{ia}{2\pi}\right) + \psi\left(-\frac{ia}{2\pi}\right) \right).$$

A series representation for the ψ function [6, p. 952, formula 8.632, no. 1] gives

$$(3.9) \quad \psi(x) = -\gamma - \sum_{j=0}^{\infty} \left(\frac{1}{x+j} - \frac{1}{j+1} \right).$$

Thus, using (3.9), we find that

(3.10)

$$\begin{aligned} \psi\left(\frac{ia}{2\pi}\right) + \psi\left(-\frac{ia}{2\pi}\right) &= -2\gamma - \sum_{j=0}^{\infty} \left(\frac{1}{j+ia/2\pi} - \frac{1}{j+1} \right) \\ &\quad - \sum_{j=0}^{\infty} \left(\frac{1}{j-ia/2\pi} - \frac{1}{j+1} \right) \\ &= -2\gamma - 2 \sum_{j=0}^{\infty} \left(\frac{j}{j^2 + a^2/4\pi^2} - \frac{1}{j+1} \right) \\ &= 2 - 2 \left(\sum_{j=1}^{\infty} \frac{j}{j^2 + a^2/4\pi^2} - \frac{1}{j+1} + \gamma \right) \\ &= 2 - 2 \lim_{n \rightarrow \infty} \left(\sum_{j=1}^n \left(\frac{1}{j} - \frac{1}{j+1} \right) + \sum_{j=1}^n \frac{j}{j^2 + a^2/4\pi^2} - \ln n \right) \\ &= 2 - 2 \lim_{n \rightarrow \infty} \left(1 + 4\pi^2 \sum_{j=1}^n \frac{j}{a^2 + 4\pi^2 j^2} - \ln n \right) \\ (3.11) \quad &= -2 \lim_{n \rightarrow \infty} \left(\sum_{j=1}^n \frac{4\pi^2 j}{a^2 + 4\pi^2 j^2} - \ln n \right), \end{aligned}$$

where in the antepenultimate line, we have made use of the limit definition of γ ,

$$(3.12) \quad \gamma = \lim_{n \rightarrow \infty} \left(\sum_{j=1}^n \frac{1}{j} - \ln n \right).$$

This proves (3.8). Finally, using (3.8), we see that after a routine simplification, (3.7) is equivalent to (1.5). This finishes the second proof.

4. OLOA'S CONJECTURE

In [8], Olivier Oloa conjectured the following: Let $\alpha > 0$ and

$$(4.1) \quad I(\alpha) = \int_0^{\pi/2} \frac{\theta^2}{\theta^2 + \ln^2(2\alpha \cos \theta)} d\theta.$$

If $\alpha < 1/2$, then

$$(4.2) \quad \frac{4}{\pi} I(\alpha) = -\frac{\gamma}{\ln \alpha} - \frac{\gamma}{1-\alpha} - \frac{1}{1-\alpha} \ln(-\ln \alpha) - \frac{\ln \alpha}{1-\alpha} \int_0^1 \alpha^t \ln \Gamma(t) dt,$$

and if $\alpha \geq 1/2$, then

$$(4.3) \quad \frac{4}{\pi} I(\alpha) = -\frac{\gamma}{\ln \alpha} - \frac{\gamma}{1-\alpha} - \frac{1}{1-\alpha} \ln \left(\frac{\alpha \ln \alpha}{\alpha-1} \right) + \frac{\ln \alpha}{\alpha-1} \int_0^1 \alpha^t \ln \Gamma(t) dt.$$

The second part of the conjecture, namely (4.3), was proved in [1]. Here, using the expression for the Laplace transform of the psi function that we have obtained, we are able to prove (4.2), namely the first part of the conjecture as well. First, we explicitly evaluate $M(a)$ when $a > \ln 2$, which is then subsequently used to prove (4.2). To that end, from (1.4) and (1.5), we readily obtain

$$(4.4) \quad \begin{aligned} M(a) = & \left(\frac{1}{e^a - 1} - \frac{1}{a} + 1 \right) \ln \left(\frac{2\pi}{a} \right) + 2a \sum_{n=1}^{\infty} \frac{\ln n}{a^2 + 4n^2\pi^2} \\ & + \frac{1}{4} \left(\psi \left(\frac{ia}{2\pi} \right) + \psi \left(-\frac{ia}{2\pi} \right) \right) - \frac{\ln a}{a}. \end{aligned}$$

In [1], the following evaluation for $M(a)$ was given for $0 < a \leq \ln 2$:

$$(4.5) \quad M(a) = \frac{\gamma}{a} + \frac{a + \ln(1 - e^{-a}) - \gamma - \ln a}{1 - e^{-a}} + \frac{a}{1 - e^{-a}} \int_0^1 e^{-at} \ln \Gamma(t) dt.$$

Also Lemma 2.3 in [1] states that for $a > 0$, $c = 1 - e^{-a}$ and $A = \ln 2\pi + \gamma$, we have

$$(4.6) \quad \int_0^1 e^{-at} \ln \Gamma(t) dt = \frac{A(a-c)}{a^2} - \frac{c}{2a} \Lambda \left(\frac{a}{2\pi} \right) + 2c \sum_{j=1}^{\infty} \frac{\ln j}{a^2 + 4\pi^2 j^2},$$

where a , c and A are defined as in (3.1) and $\Lambda(z)$ is defined in (3.2).

Thus from (4.5) and (4.6), $M(a)$ can be evaluated for $0 < a \leq \ln 2$, which combined with (4.4) implies that the integral $M(a)$ can be evaluated for all positive values of a .

Next we proceed to prove (4.2). First of all, letting $\alpha = e^{-a}$ in (4.1), we see that (4.2) is equivalent to showing

$$(4.7) \quad M(a) = \frac{\gamma}{a} - \frac{\gamma + \ln a}{1 - e^{-a}} + \frac{a}{1 - e^{-a}} \int_0^1 e^{-at} \ln \Gamma(t) dt.$$

From (4.4) and (4.7), it suffices to show that

$$(4.8) \quad \begin{aligned} & \frac{1}{4} \left(\psi \left(\frac{ia}{2\pi} \right) + \psi \left(-\frac{ia}{2\pi} \right) \right) + 2a \sum_{n=1}^{\infty} \frac{\ln n}{a^2 + 4n^2\pi^2} + \left(\frac{1}{e^a - 1} - \frac{1}{a} + 1 \right) \ln 2\pi \\ & = \frac{\gamma}{a} - \frac{\gamma}{1 - e^{-a}} + \frac{a}{1 - e^{-a}} \int_0^1 e^{-at} \ln \Gamma(t) dt. \end{aligned}$$

Simplifying (4.8) leads to

$$(4.9) \quad \begin{aligned} \int_0^1 e^{-at} \ln \Gamma(t) dt = & \frac{1 - e^{-a}}{a} \left[\frac{1}{4} \left(\psi \left(\frac{ia}{2\pi} \right) + \psi \left(-\frac{ia}{2\pi} \right) \right) + 2a \sum_{n=1}^{\infty} \frac{\ln n}{a^2 + 4n^2\pi^2} \right. \\ & \left. - \frac{\gamma}{a} + \frac{\gamma}{1 - e^{-a}} + \frac{a}{1 - e^{-a}} + \left(\frac{1}{e^a - 1} - \frac{1}{a} + 1 \right) \ln 2\pi \right]. \end{aligned}$$

Finally using (3.8), after a routine simplification, we find that (4.9) is equivalent to (4.6). This finishes the proof of (4.2).

ACKNOWLEDGMENTS

The author wishes to express his gratitude to Professor Bruce C. Berndt for careful reading of the earlier version of this manuscript and for his helpful comments. The author would also like to thank M. Lawrence Glasser and Alexander Williams for their help.

REFERENCES

- [1] T. Amdeberhan, O. Espinosa and V.H. Moll, *The Laplace transform of the digamma function: An integral due to Glasser, Manna and Oloa*, Proc. Amer. Math. Soc., **136** (2008) 3211–3221. MR2407086 (2009d:33005)
- [2] M. Abramowitz and I.A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*. National Bureau of Standards Applied Mathematics Series, 55, U.S. Government Printing Office, Washington, D.C., 1964. MR0167642 (29:4914)
- [3] J. B Conway, *Functions of One Complex Variable*, 2nd ed., Springer-Verlag, New York-Berlin, 1978. MR503901 (80c:30003)
- [4] O. Espinosa and V.H. Moll, *On some integrals involving the Hurwitz zeta function. Part I*, The Ramanujan Journal, **6** (2002) 159–188. MR1908196 (2003f:11127)
- [5] M.L. Glasser and D. Manna, *On the Laplace transform of the psi function*, “Tapas in Experimental Mathematics” (T. Amdeberhan and V. Moll, eds.), Contemporary Mathematics, vol. 457, Amer. Math. Soc., Providence, RI, 2008, pp. 205–214. MR2427673
- [6] I.S. Gradshteyn and I.M. Ryzhik, eds., *Table of Integrals, Series, and Products*, 5th ed., Academic Press, Boston, MA, 1994. MR1243179 (94g:00008)
- [7] A.P. Guinand, *Some formulae for the Riemann zeta-function*, J. London Math. Soc., **22** (1947) 14–18. MR0022928 (9:279g)
- [8] O. Oloa, *Some Euler-type integrals and a new rational series for Euler’s constant*, “Tapas in Experimental Mathematics” (T. Amdeberhan and V. Moll, eds.), Contemporary Mathematics, vol. 457, Amer. Math. Soc., Providence, RI, 2008, 253–264. MR2427676
- [9] E.T. Whittaker and G.N. Watson, *A Course of Modern Analysis*, 4th ed., Cambridge Univ. Press, London, 1927. MR0178117 (31:2375)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, 1409 WEST GREEN STREET,
URBANA, ILLINOIS 61801

E-mail address: aadixit2@illinois.edu