ON BIMODULES OVER NOETHERIAN PI RINGS

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(Communicated by Martin Lorenz)

ABSTRACT. Let R be a prime Noetherian PI ring, and let I be an ideal in R satisfying $xI \subseteq Ix$ for some x in R . We prove that $xI = Ix$. This is obtained as a corollary of a similar more general result, where I can be taken as any finitely generated torsion-free central R -bimodule.

1. INTRODUCTION

It was proved in [1, Proposition 1.8] that $xR \subseteq Rx$ implies $xR = Rx$ for each prime Noetherian PI ring R . This result has a long history. It was proved by Frölich et al. in [6] and by Vasconcelos in [12] in special cases. It was then conjectured by Montgomery in [10] to hold in general. This was verified by Guralnick et al. in [8] for affine Noetherian prime PI rings. It was then confirmed in [1] and was also extended to a version valid for an invertible ideal.

The following extension to bimodules is our main result.

Theorem A. *Let R be a prime Noetherian PI ring, and let M be a finitely generated torsion-free (from both sides) R -bimodule which is $Z(R)$ -central. Let x in R satisfy $xM \subseteq Mx$. Then $xM = Mx$.*

The next immediate corollary appears to be new.

Corollary B. *Let R be a prime Noetherian PI ring, and let I be an ideal in R satisfying $xI \subseteq Ix$. Then $xI = Ix$.*

This somewhat unexpected “rigidity” result triggers the question of whether $xR = Rx$ also holds in this case. We show at the end of this paper that this is indeed the case if R is, in addition, a maximal order.

Theorem A was partially motivated by our investigation of the Noetherian property of the I -adic ideal completion of a Noetherian PI ring, the results of which will appear elsewhere.

2. PROOFS

Lemma 2.1. *Let R be a prime PI ring, and let M be a finitely generated torsion-free left R -module. Then M is embedded in a finitely generated torsion-free Z -module, where $Z = Z(R)$, the center of R .*

Received by the editors April 18, 2009, and, in revised form, June 23, 2009, and July 19, 2009.
 2000 *Mathematics Subject Classification.* Primary 16P40, 16R20; Secondary 16N20, 16D20.

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Proof. By the torsion-freeness of M we identify M with $R \otimes_R M \subset Q(R) \otimes_R M$. By Formanek (e.g. [11]), we have $R \subset Zw_1 + \dots + Zw_s$, where $w_i \in Q(R)$. Also

$$M = \sum_{i=1}^r Rm_i. \text{ Hence}$$

$$M = \sum_{i=1}^r R \otimes m_i \subset \sum_{i=1}^r \sum_{j=1}^s Zw_j \otimes m_i = \sum_{i,j} Z(w_j \otimes m_i).$$

The latter is a finitely generated Z -submodule of $Q(R) \otimes_R M$ and is Z -torsion-free since $Q(R) \otimes_R M$ is a torsion-free $Q(R)$ -module. $\square$

Corollary 2.2. *Let R be a prime PI ring, and let M be a finitely generated torsion-free left R -module. Then $\text{End}_Z(M)$ is a prime PI ring.*

Proof. $M \subseteq \sum_i Zv_{ij}$, where $v_{ij} = w_j \otimes m_i \in Q(R) \otimes_R M$ as in Lemma 2.1. Let $K \equiv Q(Z)$. Then if $\alpha \in \text{End}_Z(M)$ we define, for each $\delta \in Z$, $\tilde{\alpha}(\delta^{-1}m) \equiv \delta^{-1}\alpha(m)$. This is clearly well defined and $\tilde{\alpha} \in \text{End}_K(K \otimes_Z M)$. So the map $\alpha \rightarrow \tilde{\alpha}$ is an injection, showing that $\text{End}_Z(M)$ is a PI ring. Let $\psi \in \text{End}_K(K \otimes_Z M)$. Find $c \in Z(R)$ such that $cv_{ij} \in R \otimes_R M = M$, so $v_{ij} \in c^{-1}M \subset K \otimes_Z M$. Hence $\psi(v_{ij}) \in K \otimes_Z M$. So find $c_1 \in Z$ such that $c_1\psi(v_{ij}) \in Z \otimes_Z M = M$. Hence $c_1\psi \in \text{End}_Z(M)$, and consequently $\text{End}_Z(M)$ is an order in the matrix ring $\text{End}_K(K \otimes_Z M)$, implying that $\text{End}_Z(M)$ is a prime ring. $\square$

Let S be a central simple finite dimensional K -algebra. Then for each $x \in S$ one defines the reduced norm, which we denote by $\det_S(x)$, as having the properties that $\det_S(x) \in K$ and $\det_S(xy) = \det_S(x)\det_S(y)$. The notation is explained by the fact that if F is a splitting field of S , that is, $S \otimes_K F \cong M_n(F)$, the $n \times n$ matrix ring over F , then $\det_S(x) = \det(x \otimes 1)$, where $\det(\cdot)$ stands for the standard determinant function on $M_n(F)$. For a standard reference see [4, pp. 274 - 275].

Let B be a prime PI ring with center Z . Then $Q(B) = BQ(Z)$ is a central simple finite dimensional $Q(Z)$ -algebra, and by abuse of notation we write $\det_B(x)$ for $\det_{Q(B)}(x)$.

Theorem 2.3. *Let S, R be PI rings, and let ${}_S M_R$ be a bimodule. Let $Z = Z(R)$. Assume that:*

- (1) M_R is a finitely generated and torsion-free R -module,
- (2) ${}_S M$ is a finitely generated and faithful S -module,
- (3) R is a prime Noetherian PI ring, and S is a left Noetherian PI ring.

Then $A \equiv \text{End}_Z(M_Z)$ is prime Noetherian and a finite R_r -module, as well as a finite S_l -module. Moreover S is right Noetherian, $Q(S)$ exists and ${}_S M$ is torsion-free.

Note: Here $R_r = \{x_r | x \in R\}$, where x_r is the right multiplication by x on M , and $S_l = \{y_l | y \in S\}$, where y_l is the left multiplication by y on M .

Proof. By Corollary 2.2, A is a prime PI ring which contains both R_r and S_l . We shall consider the action of A on M on the *right*, implying that M is a right A -module. Also M_R is finitely generated and hence M_{R_r} is finitely generated, which shows that M_A is finitely generated. Also $A \equiv \text{End}_Z(M)$ implies that M_A is faithful. Therefore by the H -condition (e.g. [7, Prop. 8.7]) $A_A \hookrightarrow M_A^n$, for

some n . Since $R_r \subseteq A$ we also have $A_{R_r} \hookrightarrow M_{R_r}^n$, which implies that A is a finite R_r -module, and in particular A is Noetherian (from both sides) and Schelter integral over R_r . Similarly $A_{S_l} \hookrightarrow M_{S_l}^{(n)}$ shows, since S_l is right Noetherian (since $S_l \cong S^0$), that A_{S_l} is a right finitely generated S_l -module, and since S_l is right Noetherian, we get that A is also Schelter integral over S_l . Now by [5], S_l is also left Noetherian and consequently S is also right Noetherian. Now since A is Schelter integral over S_l , then by [2, Theorem 7.1] $\mathcal{C}_{S_l}(\text{Nil}(S_l)) \subseteq \mathcal{C}_A(0)$ and in particular $\mathcal{C}_{S_l}(\text{Nil}(S_l)) \subseteq \mathcal{C}_{S_l}(0)$. This shows that S_l has an Artinian quotient ring and the same holds for S .

We shall next show that M_A is torsion-free. It suffices to show that each $\delta \neq 0 \in Z(A)$ acts regularly on M (since if a in A is regular, then $\det_A(a) \in Q(Z(A))$ is non-zero, and so $\exists c \in Z(A) \neq 0$ such that $c \det_A(a) \in Z(A)$).

We shall first observe that $Z(R_r) \subseteq Z(A)$. Indeed if $\theta \in A$, $z \in Z(R)$, then $m.(\theta \cdot z_r) = \theta(m)z = \theta(mz) = m.(z_r \cdot \theta)$, for each $m \in M$; that is, $\theta z_r = z_r \theta$, for each $\theta \in \text{End}_Z(M) \equiv A$. Therefore if $K \equiv Q(Z(R_r))$, then $Q(R_r) = KR_r \subseteq KA \subseteq Q(Z(A))A = Q(A)$. Now A is integral over R_r , and therefore KA is integral over K and in particular δ is integral over K . So there exist $c_0, \dots, c_n \in Z(R_r)$, $c_n \neq 0$, satisfying $c_0 \delta^n + c_1 \delta^{n-1} + \dots + c_n = 0$. Then $m\delta = 0$ implies $mc_n = 0$, and $m = 0$ by the torsion-freeness of M_R . Finally, if $tm = 0$, for some $t \in \mathcal{C}_S(0)$, then $mt_l = 0$, but $t_l \in \mathcal{C}_{S_l}(0) \subseteq \mathcal{C}_A(0)$, and $m = 0$ by the torsion-freeness of M_A . $\square$

The next result is a surprising corollary.

Theorem 2.4. *Let ${}_S M_R$ be a bimodule which is finitely generated and torsion-free from both sides, where S is a left Noetherian PI and R is a prime Noetherian PI ring. Then there exists a centralizing extension B of both R_r and S_l . Moreover B is finitely generated over both.*

Proof. Let $Z = Z(R)$ and $A = \text{End}_Z(M_Z)$ be as in Theorem 2.3. Then $B \equiv S_l \cdot R_r$ is a centralizing extension of both S_l and R_r . Moreover since by the previous theorem A is a finite S_l -module as well as a finite R_r -module and both are Noetherian, we conclude that B is finite over both. $\square$

Remark 2.5. The previous result suggests that $\text{Spec} R \cong \text{Spec} R_r$ is closely connected to $\text{Spec} S \cong \text{Spec} S^0 \cong \text{Spec} S_l$. This seems to be new even in the finite extension case $R \subset S$.

Corollary 2.6. *Let R be a prime Noetherian PI ring with center $Z(R) \equiv Z$. Then $\text{End}_Z(R)$ is a prime Noetherian PI ring, which is a finite R_r -module as well as a finite R_l -module.*

Remark 2.7. This can be used to give a different proof of [1, Theorem 1.8].

The next result is our main tool in generalizing [1, Theorem 1.8].

Proposition 2.8. *Let R be a prime Noetherian PI ring with center Z , and let M be an R -bimodule which is finitely generated torsion-free (from both sides). Let $\varphi, \psi \in \text{End}_Z M \equiv A$ satisfy:*

- (1) $\det_A \varphi = \det_A \psi \neq 0$,
- (2) $\varphi(M) \subseteq \psi(M)$.

Then $\varphi(M) = \psi(M)$.

Proof. We can clearly extend ψ to $\text{End}_K(MK)$, where $K = Q(Z)$ is the quotient field of Z . Then $\det \psi \neq 0$ implies that ψ has an inverse $\psi^{-1} \in \text{End}_K(MK)$. Hence (2) implies that $\alpha \equiv \psi^{-1} \circ \varphi \in \text{End}_Z M \equiv A$. Moreover,

$$\det_A \alpha = (\det_A \psi^{-1}) \det_A \varphi = (\det_A \psi)^{-1} \det_A \varphi = (\det_A \varphi)^{-1} (\det_A \varphi) = 1.$$

Hence, by the Cayley-Hamilton equation, α has an inverse $\alpha^{-1} \in T(A)$, the trace ring of A . Now by Theorem 2.3 A is a prime Noetherian PI ring and consequently $T(A)$ is a finite centralizing extension of A . Let $(\alpha^{-1})^m + a_1(\alpha^{-1})^{m-1} + \dots + a_m = 0$ be an integral equation of α^{-1} over A , with $a_1, \dots, a_m$ in A . Multiplying this equation by α^{m-1} yields that $\alpha^{-1} \in A$. In particular α is an onto map. Since $\varphi = \psi \circ \alpha$, this shows that $\varphi(M) = \psi(M)$. $\square$

Lemma 2.9. *Let S be a central simple finite dimensional K -algebra. Then $T \equiv \text{End}_K(S)$ is a central simple K -algebra. Moreover, for each $x \in S$ we have $\det_T(x_l) = \det_{S_l}(x_l)^n = \det_S(x)^n = \det_{S_r}(x_r)^n = \det_T(x_r)$, where $n^2 = [S : K]$.*

Proof. This appears for example in [4, pp. 274-275]. $\square$

Lemma 2.10. *Let R be a prime Noetherian PI ring, and let M be a Z -central R -bimodule which is finitely generated and torsion-free from both sides. Let $K \equiv Q(Z)$ be the quotient field of Z , and let $Q(R)$ be the quotient ring of R . Then ${}_{Q(R)}KM$, $KM_{Q(R)}$ are free and $\text{rank}_R M = \text{rank}_R M$.*

Proof. Clearly $KM = MK$ and $Q(R) = KR = RK$. Now KM is a completely reducible left module over $Q(R) \otimes_K Q(R)^\circ \cong \text{End}_K(Q(R)^\circ)$, since the latter is a simple central K -algebra. Consequently $KM \cong \underbrace{V \oplus \dots \oplus V}_{t\text{-times}}$, where V is the

unique irreducible left $Q(R) \otimes_K Q(R)^\circ$ module. Clearly $Q(R)$ is an irreducible $Q(R) \otimes_K Q(R)^\circ$ -module. Therefore $V \cong Q(R)$ implies that ${}_{Q(R)}V \cong_{Q(R)} Q(R)$, showing that KM is a free left $Q(R)$ -module with $\text{rank}_{Q(R)} KM = t$. Similarly using $V_{Q(R)} \cong Q(R)_{Q(R)}$, we show that $\text{rank}_{Q(R)} KM_{Q(R)} = t$ and $KM_{Q(R)}$ is free. $\square$

Lemma 2.11. *Let R be a prime Noetherian PI ring with center Z . Let M be a finitely generated torsion-free Z -central R -bimodule. Then for every $x \in \mathcal{C}_R(0)$, we have $\det_A(x_r) = \det_R^{tn}(x) = \det_A(x_l)$, where $A = \text{End}_Z M$, $t = \text{rank}_R M$ and $n = \text{PI.deg} R$.*

Proof. By Lemma 2.10, $MK = KM$ is a free left $Q(R)$ -module of rank t . Hence $KM = Q(R)u_1 \oplus \dots \oplus Q(R)u_t$, as left $Q(R)$ -modules. Hence $\text{End}_K(MK) \cong M_t(\text{End}_K Q(R))$. Moreover, since $x_l(Q(R)u_i) \subseteq Q(R)u_i$, for each $i = 1, \dots, t$, then the matrix representing $x_l \in \text{End}_K(MK)$ is given by the $t \times t$ matrix:

$$x_l = \begin{pmatrix} x_{l,Q} & 0 & \cdots & 0 \\ 0 & x_{l,Q} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_{l,Q} \end{pmatrix},$$

where $x_{l,Q}$ is the matrix representing left multiplication by x on $Q(R)$.

Consequently $\det_A(x_l) = \det_T^t(x_{l,Q})$, where $T = \text{End}_K(Q(R))$. Let $S = Q(R)$; then by Lemma 2.9 we have $\det_T(x_{l,Q}) = \det_S^n(x)$. Consequently $\det_A(x_l) = \det_S^{tn}(x)$. The other equality follows by a similar argument on the right side using Lemma 2.10. $\square$

Recall that it is proved in [1, Theorem 1.8] that $xR \subseteq Rx$ implies $xR = Rx$, for each prime Noetherian PI ring R . The following extension of this result to R -bimodules is obtained by an entirely different proof.

Theorem 2.12. *Let R be a prime Noetherian PI ring with center Z . Let M be a finitely generated torsion-free (on both sides) R -bimodule which is Z -central. Suppose that $xM \subseteq Mx$, for some $x \in R$. Then $xM = Mx$.*

Proof. We clearly may assume that $x \neq 0$. We shall next show that $x \in \mathcal{C}_R(0)$. Indeed the torsion-freeness of ${}_R M$ implies that ${}_R M$ is faithful and hence $xM \neq 0$. Suppose $xy = 0$ for some non-zero $y \in R$. Then $y \in r - \text{ann}_R xM$, a non-zero two-sided ideal in R , implying the existence of a non-zero $c \in Z \cap (r - \text{ann}_R xM)$. Hence $cxM = xMc = 0$ shows that $cx = 0$ and therefore $x = 0$, which was excluded. Let $\varphi = x_l$, $\psi = x_r$. Then $xM \subseteq Mx$ is equivalent to $\varphi(M) \subseteq \psi(M)$. Also by Lemma 2.11 $\det_A(x_r) = \det_A(x_l) \neq 0$, where $A \equiv \text{End}_Z M$. So all the conditions of Proposition 2.8 are in place and its conclusion $\varphi(M) = \psi(M)$ is the required result. $\square$

Apparently, even the next corollary seems to be new.

Theorem 2.13. *Let R be a prime Noetherian PI ring with $xI \subseteq Ix$, for some two-sided ideal $I \subseteq R$ and $x \in R$. Then $xI = Ix$.*

Question 2.14. Is there a non-normal element $x \in R$ satisfying $xI = Ix$, for some non-zero ideal $I \subseteq R$, where R is a prime Noetherian PI ring?

The next result provides a negative answer to Question 2.14 in the case where R is in addition a maximal order.

Lemma 2.15. *Let R be a prime Noetherian PI maximal order and $xI = Ix$ for some non-zero ideal I in R and $x \in R$. Then x is a normal element in R .*

Proof. If $xy = 0$, then $xIy = 0$, showing, since $x \neq 0$, that $y = 0$. Therefore x is a regular element in R and consequently $x^{-1} \in Q(R)$. We have

$$(xRx^{-1})I = xR(x^{-1}Ix)x^{-1} = xRIx^{-1} = xIx^{-1} = I.$$

Therefore $xRx^{-1} \subseteq O_l(R) \equiv \{t \in Q(R) \mid tI \subseteq I\}$ (see e.g. [9, 3.1.12]). Thus since R is a maximal order, $O_l(R) = R$ (e.g. [9, 5.1.4]). Therefore $xRx^{-1} \subseteq R$, or equivalently $xR \subseteq Rx$. Hence by [1, Proposition 1.8] or Theorem A, $xR = Rx$ and x is a normal element in R . $\square$

A similar argument to the one appearing in Theorem 2.12, using Proposition 2.8, proves the following result.

Theorem 2.16. *Let R, M be as in Theorem 2.12 and x, y be regular elements in R . Suppose that $xMy \subseteq yMx$. Then $xMy = yMx$.*

Question 2.17. It is easily verified that $r - \text{ann}(x) \subseteq r - \text{ann}(y)$ and $l - \text{ann}(y) \subseteq l - \text{ann}(x)$. Thus we merely have to assume in Theorem 2.16 that one of $\{x, y\}$ is regular. Can one avoid it altogether? Even the special case $xRy \subseteq yRx$ appears to be open.

REFERENCES

- [1] A. Braun and C. R. Hajarnavis, *Generator ideals in Noetherian P.I. rings*, J. Algebra 247 (2002), 134–152. MR1873387 (2003a:16033)
- [2] A. Braun and V. Vonessen, *Integrality for P.I. rings*, J. Algebra 151 (1992), 39–79. MR1182014 (94g:16034)
- [3] A. Braun and R. B. Warfield, Jr., *Symmetry and localization in Noetherian prime P.I. rings*, J. Algebra 118 (1988), 322–335. MR969675 (89k:16031)
- [4] P. M. Cohn, *Algebra, III*, Wiley, New York and Chichester, 1991. MR1098018 (92c:00001)
- [5] B. Cortzen and L. W. Small, *Finite extensions of rings*, Proc. Amer. Math. Soc. 103 (1988), 1058–1062. MR954983 (89f:16020)
- [6] A. Frölich, I. Reiner and S. Ullom, *Class groups and Picard groups of orders*, Proc. London Math. Soc. 29 (1974), 405–434. MR0357464 (50:9932)
- [7] R. R. Goodearl and R. B. Warfield, *An Introduction to Noncommutative Noetherian Rings*, London Math. Soc. Student Texts, 16, Cambridge University Press, 1989. MR1020298 (91c:16001)
- [8] R. M. Guralnick, J. C. Robson and L. W. Small, *Normalizing elements in PI rings*, Proc. Amer. Math. Soc. 123 (1995), 1955–1957. MR1301026 (95h:16032)
- [9] J. C. McConnell and J. C. Robson, *Noncommutative Noetherian Rings*, Wiley, New York, 1987. MR934572 (89j:16023)
- [10] S. Montgomery, *A generalized Picard group for prime rings*, in Banach Center Pub. No. 26, pp. 55–63, Polish Acad. Sci., Warsaw, 1990. MR1171225 (93g:16026)
- [11] L. H. Rowen, *Ring Theory*, Academic Press, San Diego, 1988. MR1095047 (94e:16001)
- [12] W. V. Vasconcelos, *On quasi-local regular algebras*, in “Sympos. Math XI”, pp. 11–12, Academic Press, London, 1973. MR0330159 (48:8497)

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