LANTERN RELATIONS AND RATIONAL BLOWDOWNS

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ABSTRACT. We discuss a connection between the lantern relation in mapping class groups and the rational blowing down process for 4-manifolds. More precisely, if we change a positive relator in Dehn twist generators of the mapping class group by using a lantern relation, the corresponding Lefschetz fibration changes into its rational blowdown along a copy of the configuration C_2 . We exhibit examples of such rational blowdowns of Lefschetz fibrations whose blowup is homeomorphic but not diffeomorphic to the original fibration.

1. INTRODUCTION

Lefschetz fibrations relate the topology of symplectic 4-manifolds to the combinatorics on positive relators in Dehn twist generators of mapping class groups. Fuller introduced a substitution technique for constructing positive relators to obtain an example of non-holomorphic Lefschetz fibrations of genus three [15], [14]. Many constructions of Lefschetz fibrations as positive relators can be interpreted as generalizations of his construction (cf. [5]), while what such substitutions mean geometrically has been less investigated.

In this paper we study a particular substitution, the lantern substitution (or the $L^{\pm 1}$ -substitution for short), for positive relators of mapping class groups. The corresponding surgical operation on Lefschetz fibrations turns out to be the rational blowing down process, which was discovered by Fintushel and Stern [6], along with a copy of the configuration C_2 (i.e., a -4 -framed unknot in Kirby diagrams). Applying a theorem of Usher [19], we give examples of such rational blowdowns of Lefschetz fibrations whose blowup is homeomorphic but not diffeomorphic to the original fibration.

In Section 2 we review the lantern relation in mapping class groups and define the lantern substitution for positive relators. We discuss a relation between lantern relations and rational blowdowns in Section 3 and state the main theorem in Section 4. We then exhibit some examples in Section 5 and end by observing other relations in Section 6.

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2. LANTERN RELATIONS AND SUBSTITUTIONS

Let Σ_g be a closed oriented surface of genus g (≥ 2) and $\mathcal{M}_g$ the mapping class group of Σ_g . We denote by $\mathcal{F}$ the free group generated by all isotopy classes $\mathcal{S}$ of simple closed curves on Σ_g . There is a natural epimorphism $\varpi : \mathcal{F} \rightarrow \mathcal{M}_g$ which sends (the isotopy class of) a simple closed curve a on Σ_g to the right-handed Dehn twist t_a along a . We set $\mathcal{R} := \text{Ker } \varpi$ and call each element of $\mathcal{R}$ a *relator* in the generators $\mathcal{S}$ of $\mathcal{M}_g$. A word in the generators $\mathcal{S}$ is called *positive* if it includes no negative exponents. We put $W(c) := t_{a_r}^{\varepsilon_r} \cdots t_{a_1}^{\varepsilon_1}(c) \in \mathcal{S}$ for $c \in \mathcal{S}$ and $W = a_r^{\varepsilon_r} \cdots a_1^{\varepsilon_1} \in \mathcal{F}$ ($a_1, \dots, a_r \in \mathcal{S}, \varepsilon_1, \dots, \varepsilon_r \in \{\pm 1\}$), and put $WV := W(c_1) \cdots W(c_s) \in \mathcal{F}$ for $V = c_1 \cdots c_s \in \mathcal{F}$ ($c_1, \dots, c_s \in \mathcal{S}$).

We begin with a precise definition of the lantern relation [2], [9].

Definition 2.1. Let a and b be simple closed curves on Σ_g with geometric intersection number 2 and algebraic intersection number 0. We orient a and b locally on a neighborhood of each intersection point $p \in a \cap b$ such that the intersection number $(a \cdot b)_p$ at p is $+1$. Resolving all intersection points according to the local orientations, we obtain a new simple closed curve c . A regular neighborhood of $a \cup b$, which can be chosen to include c , is a genus 0 subsurface Σ of Σ_g with four boundary components. We denote simple closed curves parallel to four boundary components of Σ by d_1, d_2, d_3 , and d_4 . The relation

$$t_{d_1} t_{d_2} t_{d_3} t_{d_4} = t_a t_b t_c$$

is called the *lantern relation*. We put $L := L(a, b) = abcd_4^{-1}d_3^{-1}d_2^{-1}d_1^{-1} \in \mathcal{R}$.

Let $\varrho \in \mathcal{R}$ ($\varrho \neq 1$) be a positive relator of $\mathcal{M}_g$. Let a, b, c, d_1, d_2, d_3 , and d_4 be curves as in Definition 2.1. Suppose that ϱ includes $d_1 d_2 d_3 d_4$ as a subword: $\varrho = U \cdot d_1 d_2 d_3 d_4 \cdot V$ ($U, V \in \mathcal{F}$). Since ϱ and $U \cdot L \cdot U^{-1}$ are both relators of $\mathcal{M}_g$, the positive word

$$\begin{aligned} \varrho' &= U \cdot abc \cdot V (= U \cdot abcd_4^{-1}d_3^{-1}d_2^{-1}d_1^{-1} \cdot d_1 d_2 d_3 d_4 \cdot V \\ &= U \cdot L \cdot U^{-1} \cdot U \cdot d_1 d_2 d_3 d_4 \cdot V = U \cdot L \cdot U^{-1} \cdot \varrho) \end{aligned}$$

is also a relator of $\mathcal{M}_g$. The length of the word ϱ' is equal to that of ϱ minus one.

Definition 2.2. We say that ϱ' is obtained by applying an *L-substitution* to ϱ . Conversely, ϱ is said to be obtained by applying an *L⁻¹-substitution* to ϱ' . We also call these two kinds of operations *lantern substitutions* (cf. [5]).

We next recall a definition of Lefschetz fibrations (cf. [13], [8]).

Definition 2.3. Let M be a closed oriented smooth 4-manifold. A smooth map $f : M \rightarrow S^2$ is called a *Lefschetz fibration* of genus g if it satisfies the following conditions:

- (i) f has finitely many critical values $b_1, \dots, b_n \in S^2$ and f is a smooth fiber bundle over $S^2 - \{b_1, \dots, b_n\}$ with fiber Σ_g ;
- (ii) for each i ($i = 1, \dots, n$), there exists a unique critical point p_i in the *singular fiber* $f^{-1}(b_i)$ such that f is locally written as $f(z_1, z_2) = z_1^2 + z_2^2$ with respect to some

local complex coordinates around p_i and b_i which are compatible with orientations of M and S^2 ;

(iii) no fiber contains a -1 -sphere.

Remark 2.4. A more general definition can be found in Chapter 8 of [8]. We treat also Lefschetz fibrations with boundary in the proof of Theorem 3.1.

Suppose that $g \geq 2$. According to theorems of Kas and Matsumoto, there exists a one-to-one correspondence between the isomorphism classes of Lefschetz fibrations and the equivalence classes of positive relators modulo simultaneous conjugations

$$c_1 \cdots c_n \sim_W W(c_1) \cdots W(c_n),$$

and elementary transformations

$$\begin{aligned} c_1 \cdots c_i \cdot c_{i+1} \cdots c_n &\sim c_1 \cdots c_{i+1} \cdot c_{i+1}^{-1}(c_i) \cdots c_n, \\ c_1 \cdots c_i \cdot c_{i+1} \cdots c_n &\sim c_1 \cdots c_i(c_{i+1}) \cdot c_i \cdots c_n, \end{aligned}$$

where $c_1 \cdots c_n \in \mathcal{R}$ is a positive relator in the generator $\mathcal{S}$ and $W \in \mathcal{F}$. This correspondence is described by using the holonomy (or monodromy) homomorphism induced by the classifying map of f restricted on $S^2 - \{b_1, \dots, b_n\}$ (cf. [8], [13], and [3]). We denote (the isomorphism class of) a Lefschetz fibration associated to a positive relator $\varrho \in \mathcal{R}$ by $M_\varrho \rightarrow S^2$.

Let $\varrho, \varrho' \in \mathcal{R}$ be positive relators of $\mathcal{M}_g$ and let $M_\varrho, M_{\varrho'}$ be the corresponding Lefschetz fibrations over S^2 , respectively. Suppose that the relator ϱ' is obtained by applying an L -substitution to the relator ϱ . The Euler characteristic and the signature of a Lefschetz fibration $M_{\varrho'} \rightarrow S^2$ with monodromy ϱ' are computed as follows:

$$e(M_{\varrho'}) = e(M_\varrho) - 1, \quad \sigma(M_{\varrho'}) = \sigma(M_\varrho) + 1$$

([5], Theorem 4.3 and Proposition 3.12). We investigate relations between M_ϱ and $M_{\varrho'}$ and several properties of them in the subsequent sections.

3. RATIONAL BLOWDOWNS VIA LANTERN RELATIONS

Let $\varrho, \varrho' \in \mathcal{R}$ be positive relators of $\mathcal{M}_g$ and let $M_\varrho, M_{\varrho'}$ be the corresponding Lefschetz fibrations over S^2 , respectively.

Theorem 3.1. *If ϱ' is obtained by applying an L -substitution to ϱ , then the 4-manifold $M_{\varrho'}$ is a rational blowdown of M_ϱ along a configuration $C_2 \subset M_\varrho$.*

Proof. We take a subsurface Σ of Σ_g and curves a, b, c, d_1, d_2, d_3 , and d_4 on Σ as in Definition 2.1. Let N, N' be Lefschetz fibrations over D^2 with fiber Σ corresponding to the positive words $d_1 d_2 d_3 d_4, abc$, respectively. $\square$

Drawing a Kirby diagram of N , sliding the central -1 -framed unknot over the other three -1 -framed unknots, and canceling three 1-handle/2-handle pairs, we obtain a -4 -framed unknot (Figure 1). Thus N is diffeomorphic to the total space of a D^2 -bundle over S^2 with Euler number -4 , which is denoted by C_2 in [6] (see also [8], Section 8.5).

Drawing a Kirby diagram of N' and sliding and canceling handles as in Figure 2, we obtain a pair of a dotted circle and a $+1$ -framed unknot with linking number $+2$. This means that N' is diffeomorphic to a rational 4-ball with boundary $L(4, 1)$, which is denoted by B_2 in [6] (see also [8], Section 8.5).

FIGURE 1

FIGURE 2

From construction, N (resp. N') can be considered to be a submanifold of M_ϱ (resp. $M_{\varrho'}$). It is also easily seen that $M_\varrho - \text{int } N$ and $M_{\varrho'} - \text{int } N'$ are diffeomorphic to each other. Hence we have

$$M_{\varrho'} \approx N' \cup_{\partial N} (M_\varrho - \text{int } N) \approx B_2 \cup_{L(4,1)} (M_\varrho - \text{int } C_2).$$

This completes the proof of Theorem 3.1. $\square$

4. SMOOTH STRUCTURES

Let $\varrho, \varrho' \in \mathcal{R}$ be positive relators of $\mathcal{M}_g$ and let $M_\varrho, M_{\varrho'}$ be the corresponding Lefschetz fibrations over S^2 , respectively. Suppose that ϱ' is obtained by applying k times L -substitutions ($k \geq 1$), elementary transformations, and simultaneous conjugations to ϱ . Suppose also that $e(M_\varrho) + \sigma(M_\varrho) \geq 2$. We choose a positive relator $\varsigma \in \mathcal{R}$ ($\varsigma \neq 1$) such that $M_\varsigma - \nu F$ is simply-connected and either the word ς includes at least one separating curve as a factor, or $\sigma(M_\varrho) + \sigma(M_\varsigma)$ is not divisible by 16. Here νF is an open fibered neighborhood of a general fiber F of M_ς . Taking a fiber sum of M_ϱ (resp. $M_{\varrho'}$) with M_ς , we obtain a new Lefschetz fibration $M_1 := M_\varrho \#_F M_\varsigma$ (resp. $M_2 := M_{\varrho'} \#_F M_\varsigma$) with monodromy $\varrho \cdot_W \varsigma$ (resp. $\varrho' \cdot_{W'} \varsigma$) for some $W \in \mathcal{F}$ (resp. $W' \in \mathcal{F}$). It is obvious that $\varrho' \cdot_{W'} \varsigma$ is obtained by applying k times L -substitutions, elementary transformations, and simultaneous conjugations to $\varrho \cdot_W \varsigma$.

Theorem 4.1. *The 4-manifold M_1 is homeomorphic but not diffeomorphic to a k times blowup $M_2 \# k \overline{\mathbb{CP}}^2$ of M_2 . Moreover, both of these 4-manifolds do not dissolve.*

Proof. Let $j : F_i \hookrightarrow M_i$ be the inclusion map from a general fiber F_i into the total space M_i ($i = 1, 2$). The induced homomorphism $j_\# : \pi_1(F_i) \rightarrow \pi_1(M_i)$ is surjective and the kernel of $j_\#$ includes the normal subgroup N of $\pi_1(M_i)$ generated by the vanishing cycles of M_i (cf. [1], Lemma 3.2). Since $M_\varsigma - \nu F$ is simply-connected and $j_\#$ is the composition of homomorphisms $\pi_1(F_i) \rightarrow \pi_1(M_\varsigma - \nu F) \rightarrow \pi_1(M_i)$, the group $\pi_1(M_i)$ must be trivial ($i = 1, 2$).

M_1 is a non-spin 4-manifold because either it has a component of a separating singular fiber which represents a homology class of square -1 , or $\sigma(M_1)$ is not divisible by 16. It is easily seen from the observation above that $e(M_2) = e(M_1) - k$ and $\sigma(M_2) = \sigma(M_1) + k$. By virtue of Freedman's classification theorem, both M_1 and $M_2 \# k\overline{\mathbb{CP}}^2$ are homeomorphic to $\#b_2^+(M_1)\mathbb{CP}^2 \# b_2^-(M_1)\overline{\mathbb{CP}}^2$ because they are simply-connected, non-spin, and have the same Euler characteristic and the same signature.

M_1 is a fiber sum $M_\varrho \#_F M_\varsigma$ of non-trivial Lefschetz fibrations M_ϱ and M_ς . By Gompf's theorem ([8], Theorem 10.2.18), M_1 admits a symplectic structure with symplectic fibers. It follows from a theorem of Usher [19] that M_1 is a minimal symplectic 4-manifold. Since $b_2^+(M_1) = b_2^+(M_\varrho) - b_1(M_\varrho) + b_2^+(M_\varsigma) + 2g - 1 > 1$, M_1 does not contain any smooth -1 -sphere as a consequence of Seiberg-Witten theory ([17], [18]; cf. [8], Remark 10.2.4(a)). On the other hand, $M_2 \# k\overline{\mathbb{CP}}^2$ has a natural smooth -1 -sphere. Hence M_1 and $M_2 \# k\overline{\mathbb{CP}}^2$ cannot be diffeomorphic.

Because M_1 and $M_2 \# k\overline{\mathbb{CP}}^2$ admit symplectic structures and $b_2^+(M_1) > 1$, these manifolds cannot be diffeomorphic to $\#b_2^+(M_1)\mathbb{CP}^2 \# b_2^-(M_1)\overline{\mathbb{CP}}^2$ ([16], [10]; cf. [8], Theorem 10.1.14). $\square$

Remark 4.2. We do not use any explicit property of rational blowdowns to prove Theorem 4.1. The proof above is rather similar to that of Theorem 4.8 of [3]. It is likely that M_ϱ is homeomorphic but not diffeomorphic to $M_{\varrho'} \# k\overline{\mathbb{CP}}^2$ (without taking fiber sums with M_ς) in a general setting. On the other hand, a certain rational blowdown along C_2 happens to be diffeomorphic to an honest blowdown of the original 4-manifold: $E(1)_2 (\approx E(1) \approx \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2)$ is a rational blowdown of $E(1) \# \overline{\mathbb{CP}}^2 (\approx \mathbb{CP}^2 \# 10\overline{\mathbb{CP}}^2)$ along C_2 ([6], Proposition 3.2; cf. [8], Theorem 8.5.9 and Theorem 8.3.11).

5. EXAMPLES

We apply theorems in previous sections to explicit examples.

Example 5.1. Suppose that $g \geq 4$. Choose and fix $2g + 1$ simple closed curves $c_1, c_2, \dots, c_{2g+1}$ on Σ_g such that c_i and c_{i+1} intersect transversely at one point for $i = 1, 2, \dots, 2g$ and $c_i \cap c_j = \emptyset$ if $|i - j| > 1$. Let $d_{h,1}$ and $d_{h,2}$ be simple closed curves parallel to the boundary of a regular neighborhood of $c_1 \cup c_2 \cup \dots \cup c_{2h+1}$ for $h = 1, \dots, g - 2$. We consider positive words

$$\begin{aligned} F_{h-1}^{\text{even}} &:= d_{h-1,1} d_{h-1,2} (c_{2h} c_{2h-1} \cdots c_2 c_1) (c_{2h+1} c_{2h} \cdots c_3 c_2) \\ &\quad \cdots (c_{2g} c_{2g-1} \cdots c_{2g-2h+2} c_{2g-2h+1}) (c_1 c_2 \cdots c_{2g})^{4g-2h+2}, \\ \bar{F}_{g-h-1}^{\text{even}} &:= (c_2 c_3 \cdots c_{2g+1})^{2g+2h+2} (c_{2g-2h+1} \cdots c_3 c_2) (c_{2g-2h+2} \cdots c_4 c_3) \\ &\quad \cdots (c_{2g+1} \cdots c_{2h+3} c_{2h+2}) d_{h,1} d_{h,2} \end{aligned}$$

for $h = 2, \dots, g - 2$. It was shown in Section 4 of [5] that both F_{h-1}^{even} and $\bar{F}_{g-h-1}^{\text{even}}$ are positive relators of $\mathcal{M}_g$. Since $d_{h-1,1}, d_{h-1,2}, d_{h,1}$, and $d_{h,2}$ bound a genus 0 subsurface Σ of Σ_g with four boundary components, we can take simple closed curves a_h, b_h , and c_h on Σ so that

$$L = L(a_h, b_h) := a_h b_h c_h d_{h-1,1}^{-1} d_{h-1,2}^{-1} d_{h,1}^{-1} d_{h,2}^{-1}$$

is the relator corresponding to a lantern relation. Then we apply an L -substitution to $\bar{F}_{g-h-1}^{\text{even}} F_{h-1}^{\text{even}}$ to obtain a positive relator V_h .

Suppose that $g \geq 2$ and g is even. Let d be the boundary curve of a regular neighborhood of $c_1 \cup \cdots \cup c_g$. It was shown in Section 4 of [3] that the positive word

$$Q := (c_1 c_2 \cdots c_{2g+1})^{g+1} \cdot d \cdot w_1(c_{g+1}) \cdot w_2(c_{g+2}) \cdots w_{g+1}(c_{2g+1}) \\ (W_i := (c_{i+g-1} \cdots c_{i+1} c_i)^{-1} \quad (i = 1, \dots, g+1))$$

is a positive relator of $\mathcal{M}_g$. Suppose that $g \geq 3$ and g is odd. Let d be the boundary curve of a regular neighborhood of $c_1 \cup \cdots \cup c_{g-1}$. It was shown in Section 4 of [3] that the positive word

$$Q := (c_1 c_2 \cdots c_{2g+1})^{g+2} \cdot d \cdot (c_{g-1} \cdots c_2 c_1)^2 \cdot w_1(c_g) \cdot w_2(c_{g+1}) \cdots w_{g+2}(c_{2g+1}) \\ (W_i := (c_{i+g-2} \cdots c_{i+1} c_i)^{-1} \quad (i = 1, \dots, g+2))$$

is a positive relator of $\mathcal{M}_g$.

For every $g (\geq 4)$ and $h = 2, \dots, g-2$, we set $\varrho := \bar{F}_{g-h-1}^{\text{even}} F_{h-1}^{\text{even}}$, $\varrho' := V_h$, and $\varsigma := Q$. Since ϱ' is obtained by applying an L -substitution to ϱ , it turns out from Theorem 3.1 that $M_{\varrho'}$ is a rational blowdown of M_{ϱ} along a copy of C_2 . The Euler characteristic and the signature of $M_{\varrho'}$ are equal to $12g^2 + 6g + 8gh - 8h^2 + 7$ and $-6g^2 - 8g - 4gh + 4h^2 - 3$, respectively. $M_{\varsigma} - \nu F$ is simply-connected and ς includes one separating curve. The Euler characteristic and the signature of M_{ς} are $2g^2 + 7$ and $-(g^2 + 2g + 3)$ for even g , and $2g^2 + 4g + 7$ and $-(g+2)^2$ for odd g , respectively. We set $M_1 := M_{\varrho} \#_F M_{\varsigma}$ and $M_2 := M_{\varrho'} \#_F M_{\varsigma}$. It follows from Theorem 4.1 that M_1 is homeomorphic but not diffeomorphic to $M_2 \# \overline{\mathbb{CP}}^2$ and both of these do not dissolve. If we use Q^n ($n \geq 2$) instead of Q , we obtain infinitely many pairs of homeomorphic but non-diffeomorphic 4-manifolds for a fixed $g (\geq 4)$.

Example 5.2. The second example is a Lefschetz fibration of genus 3 given by an order-three element in $\mathcal{M}_3$ as defined in §3.2 of [4]. Consider the relations

$$(t_{c_1} t_{c_2})^6 = t_{d_1} = t_{\delta_1}, \quad (t_{c_4} t_{e_1} t_{c_4} t_{f_1})^3 = t_{d_2} t_{\delta_2} = t_{\delta_1} t_{\delta_2}, \quad \text{and} \quad (t_{c_7} t_{c_6})^6 = t_{d_3} = t_{\delta_2}$$

on tori with one and two components, Figure 3, which can also be seen as subsurfaces of a closed surface of genus three, Figure 4. The order-three element on this surface is obtained by writing

$$(t_{c_1} t_{c_2})^6 t_{\delta_1}^{-1} (t_{c_4} t_{e_1} t_{c_4} t_{f_1})^{-3} t_{\delta_1} t_{\delta_2} (t_{c_7} t_{c_6})^6 t_{\delta_2}^{-1} \\ = \left((t_{c_1} t_{c_2})^2 (t_{c_4} t_{e_1} t_{c_4} t_{f_1})^{-1} (t_{c_7} t_{c_6})^2 \right)^3 = 1$$

after gluing together the bounded tori with alternating orientations. We need to eliminate the negative exponent in this expression in order to obtain a positive relator. We do this in a few steps. First, we write $(t_{c_4} t_{e_1} t_{c_4} t_{f_1})^3 = t_{\delta_1} t_{\delta_2}$ as $(t_{c_4} t_{e_1} t_{c_4} t_{f_1})^{-1} = t_{\delta_1}^{-1} (t_{c_4} t_{e_1} t_{c_4} t_{f_1})^2 t_{\delta_2}^{-1}$. Next, we replace $t_{\delta_1}^{-1}$ and $t_{\delta_2}^{-1}$ using lantern relations $t_{\delta_1} t_{x_1} t_{c_3} = t_{c_1} t_{c_1} t_{e_1} t_{f_1}$ and $t_{\delta_2} t_{x_2} t_{c_5} = t_{e_1} t_{f_1} t_{c_7} t_{c_7}$. Solving these relations for $t_{\delta_1}^{-1}$ and $t_{\delta_2}^{-1}$ we obtain $t_{\delta_1}^{-1} = t_{x_1} t_{c_3} t_{c_1}^{-1} t_{c_1}^{-1} t_{e_1}^{-1} t_{f_1}^{-1}$ and $t_{\delta_2}^{-1} = t_{x_2} t_{c_5} t_{e_1}^{-1} t_{f_1}^{-1} t_{c_7}^{-1} t_{c_7}^{-1}$. Finally, substituting these into the order-three element above and simplifying the

FIGURE 5

FIGURE 6

as follows:

$$\begin{aligned}
 \varrho &= (c_1 c_2 x_1 c_3 r c_8 c_8 c_4 x_2 c_5 c_6 c_7)^3 = (c_1 c_2 x_1 c_3 \cdot f_1^{-1}(c_4) \cdot c_8 c_8 c_4 x_2 c_5 c_6 c_7)^3 \\
 &\sim f_1^{-1}(c_1 c_2 x_1 c_3 c_4 c_8 c_8 \cdot f_1(c_4) \cdot x_2 c_5 c_6 c_7)^3 \sim (c_1 c_2 x_1 c_3 c_4 c_8 c_8 \cdot f_1(c_4) \cdot x_2 c_5 c_6 c_7)^3 \\
 &\sim (c_1(c_2) \cdot c_1 x_1 c_3 c_4 c_8 c_8 \cdot f_1(c_4) \cdot x_2 c_5 c_6 c_7)^3 \\
 &\sim (c_1(c_2) \cdot x_1 c_3 c_4 c_8 c_8 \cdot f_1(c_4) \cdot x_2 c_5 c_6 c_7 \cdot c_1)^3 \\
 &\sim (c_1 \cdot c_1(c_2) \cdot x_1 c_3 c_4 c_8 c_8 \cdot f_1(c_4) \cdot x_2 c_5 c_6 c_7)^3 \\
 &\sim (c_1 \cdot c_1(c_2) \cdot x_1 c_3 c_4 c_8 c_8 \cdot c_5 \cdot c_5^{-1} f_1(c_4) \cdot c_5^{-1}(x_2) \cdot c_6 c_7)^3 \\
 &\sim (c_1 \cdot c_1(c_2) \cdot x_1 c_3 c_4 c_5 c_8 c_8 \cdot f_1 c_5^{-1}(c_4) \cdot \bar{x}_2 \cdot c_6 c_7)^3 \quad (\bar{x}_2 := c_5^{-1}(x_2)) \\
 &\sim (c_1 \cdot c_1(c_2) \cdot x_1 c_3 c_5 \cdot c_5^{-1}(c_4) \cdot c_8 c_8 \cdot f_1 c_5^{-1}(c_4) \cdot \bar{x}_2 c_7 \cdot c_7^{-1}(c_6))^3 \\
 &\sim (c_1 \cdot c_1(c_2) \cdot x_1 c_3 c_5 \cdot s_2 \cdot c_8 c_8 \cdot f_1(s_2) \cdot \bar{x}_2 c_7 \cdot r_3)^3 \quad (s_2 := c_5^{-1}(c_4), r_3 := c_7^{-1}(c_6)) \\
 &\sim (c_1 \cdot c_1(c_2) \cdot x_1 c_3 c_5 c_7 s_2 c_8 c_8 \cdot f_1(s_2) \cdot \bar{x}_2 r_3)^3 \\
 &\sim (c_1^2(c_2) \cdot c_1 \cdot x_1 c_3 c_5 c_7 s_2 c_8 c_8 \cdot f_1(s_2) \cdot \bar{x}_2 r_3)^3 \\
 &\sim (\bar{y}_1 x_1 \cdot c_1 c_3 c_5 c_7 \cdot s_2 c_8 c_8 \cdot f_1(s_2) \cdot \bar{x}_2 r_3)^3 =: \tau \quad (\bar{y}_1 := c_1^2(c_2)).
 \end{aligned}$$

We apply elementary transformations to ϱ' as follows:

$$\begin{aligned}
 \varrho' &= (\bar{y}_1 x_1 t v s_2 c_8 f_1 c_8 s_2 \bar{x}_2 r_3)^3 \sim (\bar{y}_1 x_1 t v s_2 c_8 \cdot f_1(c_8 s_2 \bar{x}_2 r_3) \cdot f_1)^3 \\
 &\sim (f_1 \cdot \bar{y}_1 x_1 t v s_2 c_8 \cdot f_1(c_8 s_2 \bar{x}_2 r_3))^3 \sim (f_1(\bar{y}_1 x_1) \cdot f_1 t v \cdot s_2 c_8 \cdot f_1(c_8 s_2 \bar{x}_2 r_3))^3 \\
 &= (\bar{y}_1 x_1 \cdot f_1 t v \cdot s_2 c_8 c_8 \cdot f_1(s_2) \cdot \bar{x}_2 r_3)^3 =: \tau'.
 \end{aligned}$$

Thus τ' is obtained by applying three times L -substitutions to τ by virtue of the lantern relation $f_1 t v = c_1 c_3 c_5 c_7$, and $M_{\varrho'} = X_{3,3}$ turns out to be a rational blowdown of $M_{\varrho} = X_3$ along three copies of C_2 from Theorem 3.1.

We set $\varsigma := (c_1 c_2 c_3 c_4 c_5 c_6 c_7^2 c_6 c_5 c_4 c_3 c_2 c_1)^2 \in \mathcal{R}$ and put $M_1 := M_{\varrho} \#_F M_{\varsigma}$ and $M_2 := M_{\varrho'} \#_F M_{\varsigma}$. Both of M_1 and $M_2 \# 3\overline{\mathbb{CP}}^2$ are simply-connected and have the Euler characteristic 56 and signature -36 . Hence Theorem 4.1 tells us that M_1 , $M_2 \# 3\overline{\mathbb{CP}}^2$, and $\#9\mathbb{CP}^2 \# 45\overline{\mathbb{CP}}^2$ are homeomorphic but mutually non-diffeomorphic.

We next exhibit an example of lantern substitution for genus 2 fibrations and pose a problem about it.

Example 5.3. Let ϱ (resp. ϱ') be a positive relator of $\mathcal{M}_2$ given as follows (see Figure 7, Figure 8, Figure 9, and Figure 4 of [4]):

$$\begin{aligned}\varrho &:= (c_5 c_4 c_3 c_2 c_1^2 c_2 c_3 c_4 c_5)^2, \\ \varrho' &:= {}_{c_3 c_4}(\delta) {}_{c_3 c_4}^{-1}(x) \cdot \bar{k} \bar{h} c_5 \cdot {}_{c_1}^{-4}(c_2) {}_{c_1}^{-1}(c_2) {}_{c_2^{-1} c_3}(c_4) \cdot k \cdot {}_{c_2^{-1} c_3}^{-1}(h) {}_{c_2^{-1} c_3}^{-1}(c_4) {}_{c_1^2}(c_2) \cdot \bar{k} \bar{h} c_5 c_4.\end{aligned}$$

Let M_ϱ (resp. $M_{\varrho'}$) be the corresponding Lefschetz fibration of genus 2 over S^2 . It is well known that M_ϱ is diffeomorphic to $\mathbb{CP}^2 \# 13 \overline{\mathbb{CP}}^2$ (cf. [8]). ϱ' is obtained by applying elementary transformations and four times L -substitutions to ϱ as follows:

$$\begin{aligned}\varrho &= (c_5 c_4 c_3 c_2 c_1^2 c_2 c_3 c_4 c_5)^2 \\ &\sim c_5 c_4 c_3 c_2 c_1 \cdot c_1 c_2 \cdot {}_{c_3}(c_4) \cdot c_3 c_5 \cdot c_5 c_3 \cdot {}_{c_3}^{-1}(c_4) \cdot c_2 c_1 \cdot c_1 c_2 c_3 c_4 c_5 \\ &\xrightarrow{L} c_5 c_4 c_3 c_2 c_1 \cdot c_1 c_2 \cdot {}_{c_3}(c_4) \cdot c_1 k h \cdot {}_{c_3}^{-1}(c_4) \cdot c_2 c_1 \cdot c_1 c_2 c_3 c_4 c_5 \\ &\sim c_5 c_4 c_3 \cdot c_1^2 \cdot {}_{c_1}^{-2}(c_2) \cdot c_2 \cdot {}_{c_3}(c_4) \cdot c_1 k h \cdot {}_{c_3}^{-1}(c_4) \cdot c_2 c_1 \cdot c_1 c_2 c_3 c_4 c_5 \\ &\sim c_5^2 c_1^2 \cdot c_4 c_3 \cdot {}_{c_1}^{-2}(c_2) \cdot c_2 \cdot {}_{c_3}(c_4) \cdot c_1 k h \cdot {}_{c_3}^{-1}(c_4) \cdot c_2 c_1 \cdot c_1 c_2 c_3 c_4 \\ &\xrightarrow{L} c_3 \delta x \cdot c_4 c_3 \cdot {}_{c_1}^{-2}(c_2) \cdot c_2 \cdot {}_{c_3}(c_4) \cdot c_1 k h \cdot {}_{c_3}^{-1}(c_4) \cdot c_2 c_1 \cdot c_1 c_2 c_3 c_4 \\ &\sim \delta x c_4 c_3 \cdot {}_{c_1}^{-2}(c_2) \cdot c_2 \cdot {}_{c_3}(c_4) \cdot c_1 k h \cdot {}_{c_3}^{-1}(c_4) \cdot c_2 c_1 \cdot c_1 c_2 c_3 c_4 c_3 \\ &\sim \delta x c_4 c_3 \cdot {}_{c_1}^{-2}(c_2) \cdot c_2 \cdot {}_{c_3}(c_4) \cdot c_1 k h \cdot {}_{c_3}^{-1}(c_4) \cdot c_2 \cdot c_1^2(c_2) \cdot c_1^2 c_3^2 \cdot {}_{c_3}^{-1}(c_4) \\ &\xrightarrow{L} \delta x c_4 c_3 \cdot {}_{c_1}^{-2}(c_2) \cdot c_2 \cdot {}_{c_3}(c_4) \cdot c_1 k h \cdot {}_{c_3}^{-1}(c_4) \cdot c_2 \cdot c_1^2(c_2) \cdot \bar{k} \bar{h} c_5 \cdot {}_{c_3}^{-1}(c_4) \\ &\sim {}_{c_3}^{-1}(c_4) \cdot \delta x c_4 c_3 \cdot {}_{c_1}^{-2}(c_2) \cdot c_2 c_1 \cdot {}_{c_3}(c_4) \cdot k h \cdot {}_{c_3}^{-1}(c_4) \cdot c_2 \cdot c_1^2(c_2) \cdot \bar{k} \bar{h} c_5 \\ &\sim {}_{c_3}^{-1}(c_4) \cdot \delta x c_4 c_3 \cdot {}_{c_1}^{-2}(c_2) \cdot c_2 c_1 c_2 \cdot {}_{c_2^{-1} c_3}^{-1}(c_4) \cdot k \cdot {}_{c_2^{-1} c_3}^{-1}(h) {}_{c_2^{-1} c_3}^{-1}(c_4) {}_{c_1^2}(c_2) \cdot \bar{k} \bar{h} c_5 \\ &\sim {}_{c_3}^{-1}(c_4) \cdot \delta x c_4 c_3 \cdot {}_{c_1}^{-2}(c_2) \cdot c_1 c_2 c_1 \cdot {}_{c_2^{-1} c_3}^{-1}(c_4) \cdot k \cdot {}_{c_2^{-1} c_3}^{-1}(h) {}_{c_2^{-1} c_3}^{-1}(c_4) {}_{c_1^2}(c_2) \cdot \bar{k} \bar{h} c_5 \\ &\sim {}_{c_3}^{-1}(c_4) \cdot \delta x c_4 c_3 c_1^2 \cdot {}_{c_1}^{-4}(c_2) {}_{c_1}^{-1}(c_2) {}_{c_2^{-1} c_3}(c_4) \cdot k \cdot {}_{c_2^{-1} c_3}^{-1}(h) {}_{c_2^{-1} c_3}^{-1}(c_4) {}_{c_1^2}(c_2) \cdot \bar{k} \bar{h} c_5 \\ &\sim {}_{c_3}^{-1} c_4 c_3 (\delta) {}_{c_3}^{-1} c_4 c_3 (x) {}_{c_3}^{-1}(c_4) \cdot c_4 c_3 c_1^2 \cdot {}_{c_1}^{-4}(c_2) {}_{c_1}^{-1}(c_2) {}_{c_2^{-1} c_3}(c_4) \cdot k \\ &\quad \cdot {}_{c_2^{-1} c_3}^{-1}(h) {}_{c_2^{-1} c_3}^{-1}(c_4) {}_{c_1^2}(c_2) \cdot \bar{k} \bar{h} c_5 \\ &\sim {}_{c_3}^{-1} c_4 c_3 (\delta) {}_{c_3}^{-1} c_4 c_3 (x) \cdot c_4 c_3^2 c_1^2 \cdot {}_{c_1}^{-4}(c_2) {}_{c_1}^{-1}(c_2) {}_{c_2^{-1} c_3}(c_4) \cdot k \\ &\quad \cdot {}_{c_2^{-1} c_3}^{-1}(h) {}_{c_2^{-1} c_3}^{-1}(c_4) {}_{c_1^2}(c_2) \cdot \bar{k} \bar{h} c_5 \\ &\sim c_4 \cdot {}_{c_3}(\delta) {}_{c_3 c_4}^{-1}(x) \cdot c_3^2 c_1^2 \cdot {}_{c_1}^{-4}(c_2) {}_{c_1}^{-1}(c_2) {}_{c_2^{-1} c_3}(c_4) \cdot k \cdot {}_{c_2^{-1} c_3}^{-1}(h) {}_{c_2^{-1} c_3}^{-1}(c_4) {}_{c_1^2}(c_2) \cdot \bar{k} \bar{h} c_5 \\ &\sim {}_{c_3}(\delta) {}_{c_3 c_4}^{-1}(x) \cdot c_3^2 c_1^2 \cdot {}_{c_1}^{-4}(c_2) {}_{c_1}^{-1}(c_2) {}_{c_2^{-1} c_3}(c_4) \cdot k \cdot {}_{c_2^{-1} c_3}^{-1}(h) {}_{c_2^{-1} c_3}^{-1}(c_4) {}_{c_1^2}(c_2) \cdot \bar{k} \bar{h} c_5 c_4 \\ &\xrightarrow{L} {}_{c_3}(\delta) {}_{c_3 c_4}^{-1}(x) \cdot \bar{k} \bar{h} c_5 \cdot {}_{c_1}^{-4}(c_2) {}_{c_1}^{-1}(c_2) {}_{c_2^{-1} c_3}(c_4) \cdot k \cdot {}_{c_2^{-1} c_3}^{-1}(h) {}_{c_2^{-1} c_3}^{-1}(c_4) {}_{c_1^2}(c_2) \cdot \bar{k} \bar{h} c_5 c_4 \\ &= \varrho',\end{aligned}$$

where the symbol $\xrightarrow{L}$ stands for an L -substitution. Thus $M_{\varrho'}$ turns out to be a four times rational blowdown of $M_\varrho \approx \mathbb{CP}^2 \# 13 \overline{\mathbb{CP}}^2$ along copies of C_2 from Theorem 3.1.

We set $\varsigma := \varrho \in \mathcal{R}$ and put $M_1 := M_\varrho \#_F M_\varsigma$ and $M_2 := M_{\varrho'} \#_F M_\varsigma$. Both M_1 and $M_2 \# 4 \overline{\mathbb{CP}}^2$ are simply-connected and have the Euler characteristic 36 and

FIGURE 7

FIGURE 8

FIGURE 9

signature -24 . Hence Theorem 4.1 tells us that M_1 , $M_2 \# 4\overline{\mathbb{CP}}^2$, and $\#5\mathbb{CP}^2 \# 29\overline{\mathbb{CP}}^2$ are homeomorphic but mutually non-diffeomorphic.

We denote the manifold $M_{\theta'}$ of Example 5.3 by E . Since E is simply-connected and has the Euler characteristic 12 and signature -8 , E is homeomorphic to $E(1) = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$ from Freedman's classification theorem.

Problem 5.4. Does E decompose into a non-trivial fiber sum of other Lefschetz fibrations? Is E isomorphic to a fiber sum of two copies of Matsumoto's fibration?

Y. Sato told the first author that E is not diffeomorphic to $E(1)$ because $E(1)$ does not admit a Lefschetz fibration of genus 2. If E decomposes into a non-trivial fiber sum, then it is minimal by virtue of Usher's theorem [19]. Matsumoto's fibration (Example B of [13]) is a Lefschetz fibration of genus 2 with 6 non-separating and 2 separating singular fibers, and total space diffeomorphic to $S^2 \times T^2 \# 4\overline{\mathbb{CP}}^2$. It is easy to see that an appropriately twisted fiber sum of two copies of Matsumoto's fibration is homeomorphic but not diffeomorphic to $E(1)$. Another possible way to examine the manifold E would be to compute the Seiberg-Witten invariants of E by the rational blowdown formula of Fintushel and Stern [6].

6. OTHER RELATIONS

We finally observe effects of substitutions for other relations. Luo [11] improved Gervais' infinite presentation [7] of $\mathcal{M}_g$ to show that the set $\mathcal{R}$ of relators is normally generated by all commutativity, all braid, all 2-chain, and all lantern relators. We briefly review the definitions of the commutativity, braid and 2-chain relations.

Let a, b be disjoint essential simple closed curves on Σ_g . The relation

$$t_a t_b = t_b t_a$$

in $\mathcal{M}_g$ is called a *commutativity relation*. A regular neighborhood Σ of $a \cup b$ is the disjoint union of two annuli.

Let a, b be simple closed curves on Σ_g which intersect transversely at one point. The relation

$$t_a t_b t_a = t_b t_a t_b$$

in $\mathcal{M}_g$ is called a *braid relation*. A regular neighborhood Σ of $a \cup b$ is a torus with one boundary component. Let c be a simple closed curve parallel to the boundary of Σ . The relation

$$(t_a t_b)^6 = t_c$$

in $\mathcal{M}_g$ is called a *chain relation of length 2*, or a *2-chain relation* for short.

Both sides of each relation above correspond to Lefschetz fibrations over D^2 with fiber Σ . It is not difficult to draw Kirby diagrams of those manifolds and find out what they are (cf. [8], Chapter 8). We actually obtain the manifolds in Table 1.

TABLE 1

relation	manifold for LHS	manifold for RHS	common boundary
commutativity	$D^4 \amalg D^4$	$D^4 \amalg D^4$	$S^3 \amalg S^3$
braid	$X(S^2, -2)$	$X(S^2, -2)$	$\mathbb{RP}^3$
2-chain	$M_c(2, 3, 6)$	$X(T^2, -1)$	$\Sigma(2, 3, 6)$
lantern	C_2	B_2	$L(4, 1)$

The symbol $X(B, e)$ stands for the total space of a D^2 -bundle over B with Euler number e . The Milnor fiber $M_c(2, 3, 6)$ and the Brieskorn manifold $\Sigma(2, 3, 6)$ are defined by

$$\begin{aligned} M_c(2, 3, 6) &:= \{(x, y, z) \in \mathbb{C}^3 \mid x^2 + y^3 + z^6 = \varepsilon\} \cap D^6, \\ \Sigma(2, 3, 6) &:= \{(x, y, z) \in \mathbb{C}^3 \mid x^2 + y^3 + z^6 = 0\} \cap S^5 \end{aligned}$$

(see [8], Figure 8.13, for the Kirby diagram). Substitutions for commutativity and braid relations do not change the original manifold (cf. [20], Figure 34, and [3], Appendix A), whereas those for 2-chain and lantern relations do.

It might be interesting to extend Table 1 to that for various other relations, such as chain relations of length n (≥ 3), the star relation, and Matsumoto's relations [12]. No relation seems to be known to correspond to a rational blowing down process along C_p for $p \geq 3$.

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