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DOMINANCE OF A RATIONAL MAP TO THE COBLE QUARTIC

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ABSTRACT. We show the dominance of the restriction map from a moduli space of stable sheaves on the projective plane to the Coble sixfold quartic. With the dominance and the interpretation of a stable sheaf on the plane in terms of hyperplane arrangements, we expect these tools to reveal the geometry of the Coble quartic.

1. INTRODUCTION

Let C be a smooth non-hyperelliptic curve of genus 3 over complex numbers. Then C is embedded into $\mathbb{P}_2 \simeq \mathbb{P}H^0(K_C)^*$ by canonical embedding as a plane quartic curve. The moduli space $SU_C(2,K_C)$ of semistable vector bundles of rank 2 with canonical determinant over C is known to be a hypersurface in $\mathbb{P}_7$, called the 'Coble quartic', [3], [13]. Let $\mathcal{W}^r$ be the closure of the following set

$$\{E \in SU_C(2,K_C) \mid h^0(C,E) \geq r+1\}. \tag{1}$$

Then we have the following inclusions [14] on the Brill-Noether loci,

$$SU_C(2,K_C) \supset \mathcal{W} \supset \mathcal{W}^1 \supset \mathcal{W}^2 \supset \mathcal{W}^3 = \emptyset, \tag{2}$$

where $\mathcal{W} = \mathcal{W}^0$. Many properties on the geometry of these Brill-Noether loci have been discovered in [14].

Let $\overline{M}(c_1,c_2)$ be the moduli space of stable sheaves of rank 2 with the Chern classes (c_1,c_2) on the projective plane. The dimension of this space is known to be $4c_2-3$ if $c_1=0$ [2] and $4c_2-4$ if $c_1=-1$ [9]. Then there exists a rational map [8]

$$\Phi_k : \overline{M}(1,k) \dashrightarrow SU_C(2,K_C)\ ,\ 1 \leq k \leq 4, \tag{3}$$

defined by sending E to $E|_C$. It is shown in [8] that Φ_k is a dominant map to $\mathcal{W}^2, \mathcal{W}^1$ and $\mathcal{W}$, for $k=1,2,3$, respectively. In this article, we give a proof of the dominance of the rational map Φ_4. This is equivalent to the dominance of the rational map from $\overline{M}(3,6)$ to $SU_C(2,3K_C)$ by twisting. For a general bundle $E \in SU_C(2,3K_C)$, we embed C with $\mathbb{P}_2$ into a Grassmannian $Gr(5,2)$ and take the pull-back of the universal quotient bundle of $Gr(5,2)$ to $\mathbb{P}_2$. This bundle is shown to be stable and have the Chern classes (3,6).

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As a quick consequence, we can obtain the old result that $SU_C(2, K_C)$ is unirational since $\overline{M}(1,4)$ is rational. The unirationality implies the rationally connectedness. We see how we can obtain a rational curve through two general points of the Coble quartic in terms of hyperplane arrangements.

The restriction of vector bundles on $\mathbb{P}_2$ to plane curves was also studied in [7], where the author investigated the restriction of the tangent bundle of $\mathbb{P}_2$ to plane curves and gave the conditions for a vector bundle E on a plane curve to be a pull-back of the tangent bundle of $\mathbb{P}_2$, twisted by $\mathcal{O}_{\mathbb{P}_1}(-1)$.

For the background on vector bundles, we suggest [12] as a good reference.

2. Embedding plane quartics in Grassmannians

Let E be a semistable vector bundle of rank 2 with the determinant $3K_C$ over C, i.e. $E \in SU_C(2, 3K_C)$. By the following lemma, we can obtain a morphism

$$\varphi : C \to Gr(H^0(E), 2)$$

sending $p \in C$ to the 2-dimensional quotient space E_p of $H^0(E)$.

Lemma 2.1. *$H^1(C, E) = 0$ and E is globally generated.*

Proof. $H^1(E) \simeq H^0(E^* \otimes K_C) \neq 0$ implies the existence of a nonzero homomorphism $E \to \mathcal{O}_C(K_C)$ which contradicts the semistability of E. Now, by the same argument, we have $H^1(E(-p)) = 0$ for all $p \in C$. From the long exact sequence of the sequence

$$0 \to E(-p) \to E \to E_p \to 0,$$

we obtain the surjective evaluation map $H^0(E) \to E_p$, which implies the global generation of E. $\square$

In fact, the morphism φ fits in the diagram

$$\begin{tikzcd} C \arrow[r, "\varphi"] \arrow[d, hook, "|3K_C|"'] & Gr(H^0(E), 2) \arrow[d, hook, "\vartheta"] \\ \mathbb{P}H^0(3K_C)^* \arrow[r, dashed, "\mathbb{P}\lambda^*"] & \mathbb{P}(\bigwedge^2 H^0(E)^*) \end{tikzcd}$$

where ϑ is the Plucker embedding and $\mathbb{P}\lambda^*$ comes from the dual of the homomorphism

$$\lambda : \bigwedge^2 H^0(E) \to H^0(\bigwedge^2 E) \simeq H^0(3K_C).$$

By the following lemma, $\mathbb{P}\lambda^*$ is an embedding and so is φ for general E.

Lemma 2.2. *The homomorphism λ is surjective for general $E \in SU_C(2, 3K_C)$.*

Proof. If E is stable, then by the Nagata-Severi theorem [11], we have the exact sequence for $E(-K_C)$,

$$0 \to \mathcal{O}(D) \to E(-K_C) \to \mathcal{O}(K_C - D) \to 0,$$

where D is a divisor of degree 1. For general E, we have $H^0(E(-K_C)) = 0$, i.e. we can assume that $H^0(\mathcal{O}(D)) = 0$; i.e. D is non-effective.

Let $L = \mathcal{O}(K_C + D)$ and $F = \mathcal{O}(2K_C - D)$. Then we have

$$0 \to L \to E \to F \to 0.$$

Note that $h^0(L) = 3$, $h^0(F) = 5$ and $h^1(L) = h^1(F) = 0$ and, from the long exact sequence of the above sequence, we have

$$H^0(E) \simeq H^0(L) \oplus H^0(F),$$

and hence it is enough to show the surjectivity of the map

$$H^0(L) \otimes H^0(F) \to H^0(L \otimes F) \simeq H^0(3K_C).$$

For every $p \in C$, $h^0(L(-p)) = 2 + h^1(L(-p)) = 2 + h^0(p - D) = 2$ since D is not effective. Hence, we can have a map from C to $Gr(2, H^0(L))$ sending p to $H^0(L(-p))$. Since $Gr(2, H^0(L)) \simeq \mathbb{P}_2$, we can choose $W \in Gr(2, H^0(L))$ which is not the same as $H^0(L(-p))$ for any $p \in C$. Then by the choice of W, it does not have base locus on C. Now consider the map

$$W \otimes H^0(F) \to H^0(3K_C).$$

By the Base-Point-Free Pencil Trick [1], the kernel of this map is isomorphic to $H^0(C, F \otimes L^{-1})$, and this is isomorphic to $H^0(K_C - 2D)$. Note that $h^0(K_C - 2D) = h^0(2D)$ by the Riemann-Roch theorem. If $h^0(2D) = 0$, then $W \otimes H^0(F)$ is isomorphic to $H^0(3K_C)$ by the counting of the dimensions. Hence, it is enough to show that $H^0(2D) = 0$ for general E.

Assume that $h^0(2D) > 0$, and then $\mathcal{O}(2D)$ is an element of the theta divisor in $\mathrm{Pic}^2(C)$. The map

$$\mathrm{Pic}^1(C) \to \mathrm{Pic}^2(C),$$

defined by $D \mapsto 2D$, is a finite surjective map of degree 64. Hence the subvariety of $\mathrm{Pic}^1(C)$ whose elements are D such that $h^0(D) = 0$ and $h^0(2D) > 0$ is of 2 dimensions. For these divisors D, the extensions of $\mathcal{O}(K_C - D)$ by $\mathcal{O}(D)$ are parametrized by $\mathbb{P}_3$, which means that the vector bundles that do not satisfy $h^0(2D) = 0$ are of at most 5 dimensions. Hence $h^0(2D) = 0$ in general. $\square$

Now, for the 5-dimensional subspace $V \subset H^0(E)$, we have the following diagram:

$$\begin{tikzcd} C \arrow[r, dashed, "\varphi_V"] \arrow[d, hook, "|3K_C|"'] & Gr(V,2) \arrow[d, hook, "\vartheta"] \\ \mathbb{P}H^0(3K_C)^* \arrow[r, dashed, "\mathbb{P}\lambda^*"] & \mathbb{P}(\bigwedge^2 V^*) \end{tikzcd} \tag{4}$$

Consider a natural map

$$\begin{tikzcd} \mathbb{P}(\wedge^2 \mathcal{E}) \arrow[r, "f"] \arrow[d] & \mathbb{P}(\wedge^2 V_7) \\ Gr(5, V_7), & \end{tikzcd} \tag{5}$$

where $\mathcal{E}$ is the universal subbundle, V_7 is a 7-dimensional vector space and $Gr(5, V_7)$ is the Grassmannian of 5-dimensional subspaces of V_7. Over $[V_5] \in Gr(5, V_7)$, the fibre $\wedge^2 V_5$ is linearly embedded into $\wedge^2 V_7$.

Lemma 2.3. *The image of f is the secant variety of $Gr(2, V_7) \subset \mathbb{P}(\wedge^2 V_7)$, and its dimension is equal to* 17.

Proof. Let $[x] \in \mathrm{Im}(f)$; i.e. there exists a V_5 such that $x \in \wedge^2 V_5$. Consider $G = Gr(2, V_5) \subset \mathbb{P}(\wedge^2 V_5)$. Since the secant variety of G is $\mathbb{P}(\wedge^2 V_5)$, we can express x by

$$(v \wedge w) \text{ or } (v_1 \wedge v_2 + v_3 \wedge v_4),$$

which proves that $\mathrm{Im}(f)$ is contained in the secant variety of $Gr(2, V_7)$.

Now we show the inclusion $\mathrm{Sec}(Gr(2, V_7)) \hookrightarrow \mathrm{Im}(f)$. Assume that x is a general point in the secant variety. This means that

$$x = v_1 \wedge v_2 + v_3 \wedge v_4,$$

where $U = \langle v_1, v_2, v_3, v_4 \rangle$ is a 4-dimensional space. For any $V_5 \supset U$, we have $x \in \wedge^2 V_5$. This shows that

$$\mathrm{Sec}(Gr(2, V_7)) = \mathrm{Im}(f),$$

since both sides are closed subvarieties of $\mathbb{P}(\wedge^2 V_7)$. Also the set of such V_5 is 2-dimensional and $\dim f^{-1}([x]) = 2$. Hence the dimension of $\mathrm{Im}(f)$ is 17, since $\dim(\mathbb{P}(\wedge^2 \mathcal{E})) = 19$. □

Remark 2.4. $Gr(2, V_7)$ is a Scorza variety of defect $\delta = 4$ [16]. So, it is known that $\dim \mathrm{Sec}(Gr(2, V_7)) = 17$.

Lemma 2.5. *For general $E \in SU_C(2, 3K_C)$ and general 5-dimensional vector subspace $V \subset H^0(E)$, the restriction of λ to $\wedge^2 V$,*

$$\lambda : \bigwedge^2 V \to H^0(3K_C),$$

is an isomorphism.

Proof. In the proof of (2.2), let

$$V_7 := W \oplus V_5,$$

where $V_5 \simeq H^0(F)$. In fact, we can take any $V_5 \subset V_7$ with $V_5 \cap H^0(L) = 0$. Then, the restriction of λ to $\wedge^2 V_7$ is also surjective. Let $K = \ker(\lambda)$ be the 11-dimensional subspace of $\wedge^2 V_7$. Consider an incidence variety $\mathcal{R} \subset Gr(5, V_7) \times \mathbb{P}(K)$,

$$\mathcal{R} = \{(V_5, [x]) \mid x \in \wedge^2 V_5 \cap K\}. \tag{6}$$

We have the following diagram:

$$\begin{tikzcd} & \mathcal{R} \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ Gr(5, V_7) & & \mathbb{P}(K). \end{tikzcd} \tag{7}$$

It is enough to show that the map pr_1 is not dominant, which means that for the general $V_5 \subset V_7$ not in the image of pr_1, we have the surjection in the assertion. Assume that pr_1 is dominant, then

$$\dim(\mathcal{R}) \geq 10.$$

If we consider again the map

$$\begin{tikzcd} \mathbb{P}(\wedge^2 \mathcal{E}) \arrow[r, "f"] \arrow[d] & \mathbb{P}(\wedge^2 V_7) \\ Gr(5, V_7), & \end{tikzcd}$$

then the image of pr_2 in $\mathbb{P}(K)$ is the intersection of $\mathrm{Im}(f) = \mathrm{Sec}(Gr(2,V_7))$ with $\mathbb{P}(K)$ in $\mathbb{P}(\wedge^2 V_7) \simeq \mathbb{P}_{20}$. Since $\mathrm{Im}(f)$ is 17-dimensional, we have

$$7 \le \dim \mathrm{Im}(pr_2) \le 10.$$

It is clear that $\mathbb{P}(K)$ contains a point in $\mathrm{Sec}(Gr(2,V_7))$, but not in $Gr(2,V_7)$. The fibre over this point in $\mathcal{R}$ is isomorphic to $Gr(1,3) \simeq \mathbb{P}_2$. Thus the dimension of $\mathrm{Im}(pr_2)$ is greater than 7.

Now assume that $\dim \mathbb{P}(K) \cap \mathrm{Sec}(Gr(2,7)) \ge 8$. In the proof of (2.2), we have

$$K \cap (W \wedge V_5) = (0),$$

if $V_5 \cap W = (0)$. If $V_5 \cap W \neq (0)$, the intersection is always $[\wedge^2 W]$. Let us consider the canonical map

$$s : W \otimes V_7/W \to \wedge^2 V_7/\wedge^2 W.$$

For all V_5 with $V_5 \cap W = (0)$, the images in $\wedge^2 V_7/\wedge^2 W$ are the same as a 10-dimensional vector space. If we take the preimage of this space in $\wedge^2 V_7$, then it is the union of $W \wedge V_5$ for all V_5, which is now an 11-dimensional space. Note that $K \cap (W \wedge V_5) = [\wedge^2 W]$ if $W \cap V_5 \neq (0)$. Let us denote by D the projectivization of the preimage of $s(W \otimes V_7/W)$ in $\wedge^2 V_7$. Then D is a 10-dimensional subvariety of $\mathbb{P}(\wedge^2 V_7)$ and it intersects with $\mathbb{P}(K)$ at the unique point $[\wedge^2 W]$. In fact, D is the projective tangent space $\mathbb{P}T_{[W]}Gr(2,V_7)$ of $Gr(2,V_7)$ at $[W]$ in $\mathbb{P}(\wedge^2 V_7)$. Recall that

$$\begin{aligned} &T_{[W]}Gr(2,V_7) = \mathrm{Hom}(W, V_7/W) \simeq W^* \otimes V_7/W \\ &T_{[\wedge^2 W]}\mathbb{P}(\wedge^2 V_7) = \mathrm{Hom}(\wedge^2 W, \wedge^2 V_7/\wedge^2 W). \end{aligned} \tag{8}$$

The differential map of the Plücker embedding at $[W]$ is defined as follows: $x = w^* \otimes e \in T_{[W]}Gr(2,V_7)$ is sent to the map

$$w_1 \wedge w_2 \mapsto s((w^*(w_1)w_2 - w_1 w^*(w_2)) \otimes e),$$

where $W = \langle w_1, w_2 \rangle$. This explains the assertion.

Now since the union of the secant lines of $Gr(2,V_7)$ passing through $[\wedge^2 W]$ is 11-dimensional and $\mathbb{P}(K) \cap \mathrm{Sec}(Gr(2,V_7))$ is of dimension ≥ 8, we can pick an element $[U] \in \mathbb{P}(K) \cap Gr(2,V_7)$, and then the secant line $\overline{[U][W]}$ lies in $\mathbb{P}(K)$. From the condition on W, U and W span a 4-dimensional subspace of V_7. In particular, general points on the secant line $\overline{[U][W]}$ are indecomposable. Let p be such a point. Since $\mathrm{Sing}(\mathrm{Sec}(Gr(2,V_7))) = Gr(2,V_7)$ [16], the dimension of $T_p(\mathrm{Sec}(Gr(2,V_7)))$ is 17. Note that

$$T_p(\mathrm{Sec}(Gr(2,V_7))) = \langle T_{[W]}G, T_{[U]}G \rangle. \tag{9}$$

Since

$$T_p(\mathbb{P}(K) \cap \mathrm{Sec}(Gr(2,V_7))) = \mathbb{P}(K) \cap T_p(\mathrm{Sec}(Gr(2,V_7)))$$

is at least 8-dimensional, $\mathbb{P}(K)$ intersects $T_{[W]}G$ along at least 1-dimensional subspace, which is a contradiction because $\mathbb{P}(K) \cap D$ is a single point. $\square$

From the previous lemma, we have the commutative diagram

$$\begin{tikzcd}
\mathbb{P}_2 \simeq \mathbb{P}H^0(K_C)^* \arrow[rr, hook, "v_3"] & & \mathbb{P}H^0(3K_C)^* \arrow[d, hook] \\
C \arrow[u, hook] \arrow[r] & Gr(H^0(E),2) \arrow[r, hook] \arrow[d, dashed] & \mathbb{P}(\wedge^2 H^0(E)^*) \arrow[d, dashed] \\
& Gr(V,2) \arrow[r, hook] & \mathbb{P}(\wedge^2 V^*),
\end{tikzcd} \tag{10}$$

where the composite of the two vertical maps on the right,

$$\mathbb{P}H^0(3K_C)^* \hookrightarrow \mathbb{P}(\wedge^2 H^0(E)^*) \dashrightarrow \mathbb{P}(\wedge^2 V^*), \tag{11}$$

is an isomorphism and v_3 is the 3-tuple Veronese embedding; i.e. v_3 is given by the complete linear system $|\mathcal{O}_{\mathbb{P}_2}(3)|$. In particular, C is embedded into $Gr(V,2)$. Note that C is non-degenerate in $\mathbb{P}_9 \simeq \mathbb{P}(\wedge^2 V^*)$ due to the Riemann-Roch theorem and the Noether theorem.

Corollary 2.6. *General element E in $SU_C(2,3K_C)$ is generated by a 5-dimensional subspace of $H^0(E)$.*

3. Embedding the projective plane into Grassmannian

In the diagram (10), the projective plane $\mathbb{P}H^0(K_C)^* \simeq \mathbb{P}_2$ is embedded into the projective space $\mathbb{P}(\wedge^2 V^*) \simeq \mathbb{P}_9$ by the Plücker embedding.

Lemma 3.1. *For general $E \in SU_C(2,3K_C)$, there exists a 5-dimensional vector subspace $V \subset H^0(E)$ such that $\mathbb{P}H^0(K_C)^*$ is embedded into $Gr(V,2)$ in the diagram* (10).

Proof. Let $V \subset H^0(E)$ be a 5-dimensional subspace selected in 2.5 and assume that $\mathbb{P}H^0(K_C)^*$ is not embedded into $Gr(V,2)$. Recall that $Gr(V,2)$ is cut out by the 4-dimensional projectively linear family of quadrics of rank 6 in $\mathbb{P}_9$ whose singular locus is $\mathbb{P}_3$ contained in $Gr(V,2)$ as the Schubert variety of lines through a point corresponding to the quadric in $\mathbb{P}_4$ [15]. Let $Q(p)$ be one of the quadrics of rank 6 containing $Gr(V,2)$ which does not contain S, where p is a point in $\mathbb{P}_4$ and S is the image of $\mathbb{P}_2$ by v_3. Since $v_3^{-1}(Q(p))$ is a plane sextic curve, we have

$$v_3^{-1}(Q(p)) = C + C',$$

where C' is a conic. First, assume that $Gr(V,2) \cap S = C + C'$. If we consider the incidence variety $Z_C = \{(l,x) | x \in l\} \subset C \times \mathbb{P}_4$, we have a diagram

$$\begin{tikzcd}
& Z_C \arrow[dl, "p"'] \arrow[dr, "q"] & \\
C & & \mathbb{P}_4.
\end{tikzcd}$$

Let S_C be the image of q in $\mathbb{P}_4$. If S_C is degenerate, i.e. there exists a hyperplane $\mathbb{P}_3 \subset \mathbb{P}_4$ containing S_C, then C is contained in some Grassmannian $Gr(4,2) \subset Gr(V,2)$ and, in particular, C is contained in $\mathbb{P}_5$, the Plücker space of $Gr(4,2)$, which is a contradiction to the non-degeneracy of C in $\mathbb{P}_9$. Similarly we can define $Z_{C'}$ and $S_{C'}$. Recall the well known fact that

$$\deg(C) = \deg(S_C) \cdot \deg(q).$$

If $\deg(S_C) = 1$, i.e. S_C is a plane in $\mathbb{P}_4$, then C must be contained in $\mathbb{P}_3(p)$, the singular locus of a quadric $Q(p)$ for $p \in S_C$, which is a contradiction to the fact that $C \subset \mathbb{P}_9$ is nondegenerate. Hence $\deg(S_C) \geq 2$ and so $\deg(q) \leq 6$. This implies that the number of points in $\mathbb{P}_3(p) \cap C$ is less than 7 for $p \in S_C$. Since the intersection of S_C and $S_{C'}$ is at most 1-dimensional in S_C, we still have 2-dimensional choices for p for which $\mathbb{P}_3(p) \cap (C + C') = \mathbb{P}_3(p) \cap C$ is less than 7 points. We can also have the same conclusion on the intersection number of $\mathbb{P}_3(p) \cap (C + C')$ in the case when $Gr(V, 2) \cap S$ is the proper subset of $C + C'$ since it still contains C. Now choose $p \in \mathbb{P}_4$ such that the singular locus $\mathbb{P}_3(p)$ of $Q(p)$ meets $C + C'$ with k points where $0 < k < 7$. We have the commutative diagram

$$\begin{tikzcd}
\mathbb{P}_3(p) \arrow[d, hook] & & & \\
Gr(V,2) \arrow[r, hook] \arrow[d, dashed] & \mathbb{P}(\wedge^2 V^*) \arrow[d, dashed] & S \arrow[l, hook'] \arrow[d, dashed] & C + C' \arrow[l, hook'] \arrow[d] \\
Gr(4,2) \arrow[r, hook] & \mathbb{P}_5 & \overline{S} \arrow[l, hook'] & \overline{C + C'}, \arrow[l, hook']
\end{tikzcd}$$

where $\overline{S}$, $\overline{C + C'}$ are the images of S, $C + C'$, respectively, via the projection, and the image of $Gr(V, 2)$ lies in the image of the quadric Q, i.e. the Grassmannian $Gr(4, 2) \subset \mathbb{P}_5$. Let Q' be another quadric cutting $Gr(V, 2)$ with singular locus $\mathbb{P}_3'$. Since $\mathbb{P}_3 \cap \mathbb{P}_3'$ is a single point, the image of Q' by the projection is $\mathbb{P}_5$. Thus the image of $Gr(V, 2)$ is $Gr(4, 2)$. Note that the degree of $\overline{C + C'}$ is $18 - k$ and the degree of $\overline{S}$ is $9 - k$ since $\mathbb{P}_3(p) \cap S = \mathbb{P}_3(p) \cap (C + C')$. If $Q(p)$ contains S for all such $p \in S_C$, then all quadrics containing $Gr(V, 2)$ of rank 6 should contain S since S_C is nondegenerate in $\mathbb{P}_4$. In particular, $Gr(V, 2)$ should contain S, which is against the assumption. So there exists a $p \in S_C$ for which S is not contained in $Q(p)$. Thus the image of S by the projection is also not contained in the image of $Q(p)$, i.e. $Gr(4, 2)$. But the degree of intersection $Gr(4, 2) \cap \overline{S}$ is $2 \times (9 - k) < 18 - k$, which is a contradiction to the fact that this intersection contains $\overline{C + C'}$. $\square$

Let U_V and $\overline{U_V}$ be the universal subbundle and quotient bundle of $Gr(V, 2)$, respectively. With the condition on V in the previous lemma, let

$$E_V := v_3^* \overline{U_V}, \tag{12}$$

which implies that the restriction of E_V to C is E, i.e. $E_V|_C = E$.

Lemma 3.2. *E_V is stable with the Chern classes $(3, 6)$, i.e. $E_V \in \overline{M}(3, 6)$.*

Proof. Since the first Chern class of $\overline{U_V}$ is the hyperplane section of $Gr(V, 2)$ in $\mathbb{P}(\wedge^2 V^*)$ and v_3 is the 3-tuple Veronese embedding, we get $c_1(E_V) = 3$.

By the choice of V, we have an exact sequence

$$0 \to G \to V \otimes \mathcal{O}_{\mathbb{P}_2} \to E_V \to 0, \tag{13}$$

where G is the kernel of the surjection $V \otimes \mathcal{O}_{\mathbb{P}_2} \twoheadrightarrow E_V$ and V is a 5-dimensional vector subspace of $H^0(E_V)$. In particular, $h^0(E_V) \geq 5$. By the choice of E, we have $h^0(E_V(-1)|_C) = 0$. From the long exact sequence of cohomology of the exact sequence

$$0 \to E_V(-5) \to E_V(-1) \to E_V(-1)|_C \to 0,$$

we have

$$H^0(E_V(-5)) \simeq H^0(E_V(-1)).$$

For a line $H \subset \mathbb{P}_2$, $E_V|_H \simeq \mathcal{O}_H(a) \oplus \mathcal{O}_H(3-a)$ for $a = 2$ or 3 since E_V is globally generated. In particular, $h^0(E_V(-k)|_H) = 0$ for $k \geq 4$. From the long exact sequence of cohomology of the exact sequence

$$0 \to E_V(-k-1) \to E_V(-k) \to E_V(-k)|_H \to 0,$$

we have $h^0(E_V(-k-1)) = h^0(E_V(-k))$ for all $k \geq 4$. Since $h^0(E_V(-k)) = 0$ for sufficiently large k, we have $h^0(E_V(-k)) = 0$ for $k \geq 4$ and in particular, $h^0(E_V(-1)) = h^0(E_V(-5)) = 0$; i.e. $h^0(E_V(-k)) = 0$ for all $k \geq 1$. Hence the vector bundle E_V is stable.

Again, let H be a line in $\mathbb{P}_2$. From the exact sequence

$$0 \to E_V(-1) \to E_V \to E_V|_H \to 0,$$

we get $h^0(E_V) \leq h^0(E_V|_H)$. Since $E_V|_H \simeq \mathcal{O}_H(a) \oplus \mathcal{O}_H(3-a)$ for $a = 2$ or 3, $h^0(E_V|_H) = 5$ and so $h^0(E_V) \leq 5$. Thus we obtain $h^0(E_V) = \dim V = 5$.

Now from the long exact sequence of cohomology of (13), we have $h^0(\mathbb{P}_2, G) = 0$. If we twist (13) by -1, we have $h^1(\mathbb{P}_2, G(-1)) = 0$. For any line $l \subset \mathbb{P}_2$, consider the exact sequence

$$0 \to G(-1) \to G \to G|_l \to 0.$$

From the above statement, we get $H^0(G|_l) = 0$. Since $c_1(G) = -c_1(E_V) = -3$, we have $G|_l \simeq \mathcal{O}_l(a) \oplus \mathcal{O}_l(b) \oplus \mathcal{O}_l(c)$ with $a + b + c = -3$. The only choice from the vanishing of $H^0(G|_l)$ is $(a, b, c) = (-1, -1, -1)$. Hence G is a uniform vector bundle of rank 3 on $\mathbb{P}_2$ with the splitting type $(-1, -1, -1)$. From the classification of such bundles [5], we have

$$G \simeq \mathcal{O}_{\mathbb{P}_2}(-1)^{\oplus 3}.$$

In particular, $c_2(G) = 3$ and so $c_2(E_V) = 6$. □

Since we can pick an element $E_V \in \overline{M}(3,6)$ mapping to a general element $E \in SU_C(2, 3K_C)$, the rational map

$$\overline{M}(3,6) \dashrightarrow SU_C(2, 3K_C) \tag{14}$$

is dominant. By twisting the map (14) with $\mathcal{O}_{\mathbb{P}_2}(-1)$ and $\mathcal{O}_C(-K_C)$, we have the following main theorem.

Theorem 3.3. *The restriction map*

$$\Phi_4 : \overline{M}(1,4) \dashrightarrow SU_C(2, K_C)$$

is dominant.

Remark 3.4. Dolgachev and Kapranov [4] showed that the logarithmic bundles $E(\mathcal{H})$ attached to the general hyperplane arrangement $\mathcal{H} = (H_1, \cdots, H_6)$ in $\mathbb{P}_2$ form an open Zariski subset $U \subset \overline{M}(3,6)$. For these bundles $E(\mathcal{H})$, we have a Steiner resolution

$$0 \to \mathcal{O}_{\mathbb{P}_2}(-1)^{\oplus 3} \to \mathcal{O}_{\mathbb{P}_2}^{\oplus 5} \to E(\mathcal{H}) \to 0.$$

From this, we have a 5-dimensional space $V = H^0(\mathbb{P}_2, E(\mathcal{H}))$. Tensoring the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}_2}(-4) \to \mathcal{O}_{\mathbb{P}_2} \to \mathcal{O}_C \to 0$$

by $E(\mathcal{H})$, we can consider V as a subspace of $H^0(C, E(\mathcal{H})|_C)$, which is 8-dimensional. As we have seen already in the proof of 3.2, the bundle E_V has a Steiner resolution, pulled back from the universal exact sequence on the Grassmannian $Gr(V, 2)$. This motivates the whole argument in this paper.

Since $\overline{M}(1, 4)$ is rational and the map Φ_4 is dominant, $SU_C(2, K_C)$ is unirational. It implies that $SU_C(2, K_C)$ is rationally connected and so rationally chain-connected. Let $\mathcal{H} = (H_0, \cdots, H_6)$ be a general arrangement of 6 lines on $\mathbb{P}_2$ and then we can associate a logarithmic bundle $E(\mathcal{H}) \in \overline{M}(3, 6)$ to $\mathcal{H}$. It is known [4] that the logarithmic bundles $E(\mathcal{H})$ form an open Zariski subset of $\overline{M}(3, 6)$ and, after twisting by $\mathcal{O}_{\mathbb{P}_2}(-1)$, $\overline{M}(1, 4)$. Let $\mathcal{F}$ be a family of arrangements of 6 lines on $\mathbb{P}_2$ and let $E(\mathcal{F})$ be the closure of the subvariety of $\overline{M}(1, 4)$ whose closed points correspond to $E(\mathcal{H}) \otimes \mathcal{O}_{\mathbb{P}_2}(-1)$ with $\mathcal{H} \in \mathcal{F}$.

Proposition 3.5. *$SU_C(2, K_C)$ is rationally chain-connected. In fact, any two general points in $SU_C(2, K_C)$ can be connected by at most 6 rational curves which can be described explicitly.*

Proof. Let us consider a special type of arrangement of 6 lines. Let $H_0, H_1, \cdots, H_5$ be 6 lines in general position on $\mathbb{P}_2$ and let p be a fixed point on H_0 in general position. If we fix $H_1, \cdots, H_5$, then we have a 1-dimensional family $\mathcal{F}$ of 6 lines with H_0 moving. Consider a map

$$\Psi : \mathbb{P}_1(\mathcal{F}) \to SU_C(2, K_C),$$

sending $\mathcal{H}$ to $E(\mathcal{H})(-1)|_C$. Since $SU_C(2, K_C)$ is projective, this map is a morphism [6]. Clearly Ψ is not a constant map; otherwise Φ_4 is also a constant map, which is not true. From the fact that logarithmic bundles associated to 6 lines in general position form an open Zariski subset of $\overline{M}(3, 6)$ and Φ_4 is dominant, we can find a 1-dimensional family of 6 lines $\mathcal{F}$ which maps to a rational curve on $SU_C(2, K_C)$ via Ψ for a general element of $SU_C(2, K_C)$. Furthermore, for two general elements $E_1, E_2 \in \overline{M}(3, 6)$, we can find 6 families of 6 lines $\mathcal{F}_i$, $1 \leq i \leq 6$, as above such that the arrangements corresponding to E_1, E_2 lie in $\mathcal{F}_1$, $\mathcal{F}_6$ respectively and $\mathcal{F}_i \cap \mathcal{F}_{i+1} \neq \emptyset$. From this fact with the dominance of Φ_4, we can find 6 rational curves passing through two general points on $SU_C(2, K_C)$. □

Remark 3.6. Note that we can choose these rational curves not contained in the singular locus of $SU_C(2, K_C)$ which is the Kummer variety of $\mathrm{Pic}^2(C)$. Let $\widetilde{S}$ be a desingularization by the blow-up [10] and consider the proper transform of the previous 6 rational curves on $SU_C(2, K_C)$. It shows the rationally chain-connectedness of $\widetilde{S}$, and since $\widetilde{S}$ is smooth, it implies the rational connectedness; i.e. the chain of these 6 curves can be deformed to a rational curve and its image on $SU_C(2, K_C)$ will give us a rational curve through two general points.

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