SOME REMARKS ON BOUNDED EARTHQUAKES

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ABSTRACT. We show that an earthquake of a geometrically infinite hyperbolic surface induces an asymptotically conformal change in the hyperbolic metric if and only if the measured lamination associated with the earthquake is asymptotically trivial on the surface. Then we show that the contraction along earthquake paths is continuous in the Teichmüller space of any hyperbolic surface. Finally, we show that if a measured lamination vanishes while approaching infinity at a rate higher than the distance to the boundary, then it must be trivial.

1. INTRODUCTION

An earthquake E of a hyperbolic surface M is obtained by partitioning M into the strata and by moving one stratum at a time. The support of an earthquake E is a geodesic lamination λ on M . Each stratum of E is either a geodesic from λ or a component of the complement of λ in M . The earthquake E does not change the shape of each stratum (i.e. E is a hyperbolic isometry on each stratum) and it moves each stratum to the left relative to any other stratum. For all the details, see Thurston [11] or Section 2.

The relative movement to the left between any two strata of E is best described by a transverse measure to λ which is invariant under homotopies relative to λ . A geodesic lamination together with a transverse measure is called a measured lamination. An earthquake E is uniquely determined (up to post-composition with a hyperbolic isometry) by its measured lamination. Any earthquake E of M onto another hyperbolic surface M_1 induces a homeomorphism between the boundaries of their corresponding universal coverings (we take the upper half-plane $\mathbf{H}$ as universal coverings); i.e. E induces a homeomorphism of the extended real line $\hat{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$ which conjugates the uniformizing Fuchsian group of M onto the uniformizing Fuchsian group of M_1 . Thurston [11] showed that any homeomorphism of $\hat{\mathbf{R}}$ is induced by an earthquake E of the upper half-plane $\mathbf{H}$, and if a homeomorphism conjugates a Fuchsian group G onto another Fuchsian group, then E is an earthquake of $\mathbf{H}/G$.

An earthquake E of the upper half-plane $\mathbf{H}$ induces a quasisymmetric homeomorphism of $\hat{\mathbf{R}}$ if and only if the transverse measure of E is uniformly bounded on all hyperbolic arcs of length 1 (see [9], [5], [7], [4], [10] and [11], or Section 2). Moreover, an earthquake E of $\mathbf{H}$ induces a symmetric homeomorphism of $\hat{\mathbf{R}}$ if and

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only if the transverse measure of E tends to zero uniformly on hyperbolic arcs of length 1 which converge to the boundary $\hat{\mathbf{R}}$ of $\mathbf{H}$ (see [5], [7] and [10], or Section 2).

A symmetric homeomorphism of $\hat{\mathbf{R}}$ extends to an asymptotically conformal quasiconformal map of $\mathbf{H}$, and an asymptotically conformal quasiconformal map of $\mathbf{H}$ extends to a symmetric homeomorphism of $\hat{\mathbf{R}}$ (see [6], [3] or Section 2). A proper generalization of the space of all symmetric homeomorphisms of $\hat{\mathbf{R}}$ for a geometrically infinite hyperbolic surface M is the space of all asymptotically conformal quasiconformal maps from M onto variable hyperbolic surfaces M_1 up to bounded homotopy (see [6], [2]). A lift to $\mathbf{H}$ of an asymptotically conformal quasiconformal map of M does not induce a symmetric homeomorphism of $\hat{\mathbf{R}}$ (unless $M = \mathbf{H}$). Our first result is a characterization of earthquakes of M whose lifts to $\mathbf{H}$ agree on the boundary $\hat{\mathbf{R}}$ with the continuous extensions of lifts of asymptotically conformal quasiconformal maps of M (see Theorem 3.1).

Let M be a geometrically infinite hyperbolic surface and let E be an earthquake of M . Let $\tilde{E} : \mathbf{H} \rightarrow \mathbf{H}$ be a lift of E to the universal covering and let $\tilde{E}|_{\hat{\mathbf{R}}}$ be its continuous extension to the boundary $\hat{\mathbf{R}}$. Let $f : M \rightarrow M_1$ be a quasiconformal map. Denote by $\tilde{f} : \mathbf{H} \rightarrow \mathbf{H}$ a lift of f to the universal covering $\mathbf{H}$ and by $\tilde{f}|_{\hat{\mathbf{R}}}$ the continuous extension to $\hat{\mathbf{R}}$. The measured lamination associated with E is said to be *asymptotically trivial* on M if the supremum of the measures of hyperbolic arcs of length 1 outside a compact subset K of M is approaching zero as K exhausts M .

Theorem 1. *An earthquake E of a geometrically infinite hyperbolic surface M induces a homeomorphism $\tilde{E}|_{\hat{\mathbf{R}}}$ of $\hat{\mathbf{R}}$ which agrees with an induced homeomorphism $\tilde{f}|_{\hat{\mathbf{R}}}$ of an asymptotically conformal quasiconformal map $f : M \rightarrow M_1$ if and only if the transverse measure of E is asymptotically trivial on M .*

Let $ML_b(\mathbf{H})$ denote the space of all bounded measured laminations on $\mathbf{H}$. Each quasimetric homeomorphism $h : \hat{\mathbf{R}} \rightarrow \hat{\mathbf{R}}$ is obtained by continuously extending to $\hat{\mathbf{R}}$ an earthquake E of $\mathbf{H}$ with associated bounded measured lamination. This gives an *earthquake measure map* $\mathcal{EM} : T(\mathbf{H}) \rightarrow ML_b(\mathbf{H})$, where $T(\mathbf{H})$ is the universal Teichmüller space; namely, $T(\mathbf{H})$ is the space of all quasimetric maps of $\hat{\mathbf{R}}$ modulo post-composition with $PSL_2(\mathbf{R})$. If $\mathbf{H}$ is replaced by a compact hyperbolic surface S , then the weak* topology on $ML(S)$ makes the earthquake measure map $\mathcal{EM} : T(S) \rightarrow ML(S)$ a homeomorphism (see [8]). The question concerning a natural topology on $ML_b(\mathbf{H})$ which makes $\mathcal{EM} : T(\mathbf{H}) \rightarrow ML_b(\mathbf{H})$ into a homeomorphism is still open (by a natural topology on $ML_b(\mathbf{H})$ we mean a topology given in terms of measured laminations on $\mathbf{H}$ without reference to the earthquake measure map). However, we show that the earthquake measure map is natural for the multiplication of the measured laminations by a positive parameter (see Theorem 4.1).

Theorem 2. *The scaling measure map $H : [0, 1] \times T(\mathbf{H}) \rightarrow T(\mathbf{H})$ given by*

$$H(t, [h]) = [E^{(1-t)\mathcal{EM}([h])}]|_{\hat{\mathbf{R}}}]$$

is continuous.

The above theorem is true for Teichmüller spaces of an arbitrary hyperbolic surface M as well. A direct corollary to the above theorem is that the Teichmüller

space $T(M)$ is contractible (also the closed subspace of $T(M)$ of classes with asymptotically conformal representatives is contractible). This was originally proved by Douady and Earle [1]. We obtained a new proof of this fact using earthquakes (see Corollary 4.1).

Finally, we consider measured laminations whose measures vanish as they approach the boundary. Gardiner-Hu-Lakic [5] and Hu [7] considered earthquakes whose measured laminations μ satisfy $\mu = o(\delta^\alpha)$, for $\alpha > 0$. (By the definition, $\mu = o(\delta^\alpha)$ if the supremum of $\mu(I)$ over all hyperbolic arcs of length 1 which are at a distance at most δ from the boundary is $o(\delta^\alpha)$.) We show that if $\alpha = 1$, i.e. $\mu = o(\delta)$, then μ is a trivial measured lamination, i.e. $\mu = 0$ (see Proposition 5.1).

2. PRELIMINARIES: BOUNDED EARTHQUAKES

Let $\mathbf{H}$ be the upper half-plane with the metric $\rho(z) = \frac{|dz|}{y}$. Denote by $\hat{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$ the boundary at infinity of $\mathbf{H}$. We identify the hyperbolic plane with the model $(\mathbf{H}, \rho)$.

Let M be a Riemann surface of infinite geometric type. The universal covering of M is the hyperbolic plane $\mathbf{H}$. The Riemann surface M supports a unique hyperbolic metric in its conformal class which is the projection of ρ under the universal covering by $\mathbf{H}$. Let G be a subgroup of $PSL_2(\mathbf{R})$ such that M is conformal and isometric to $\mathbf{H}/G$. From now on, we fix one such G and the identification $M = \mathbf{H}/G$.

Definition 2.1. A geodesic lamination λ on $\mathbf{H}$ is a closed subset $|\lambda|$ of $\mathbf{H}$ together with an assignment of a foliation of $|\lambda|$ by geodesics of $\mathbf{H}$. A geodesic lamination λ_G on $M = \mathbf{H}/G$ is a closed subset $|\lambda_G|$ of $\mathbf{H}$ which is foliated by geodesics of $\mathbf{H}$ such that the subset $|\lambda_G|$ and the foliation of $|\lambda_G|$ are invariant under G . A stratum of λ is either a geodesic of the foliation of $|\lambda|$ or a connected component of $\mathbf{H} \setminus |\lambda|$.

Remark. A component of $\mathbf{H} \setminus |\lambda|$ is an open hyperbolic polygon whose boundary in $\mathbf{H}$ consists of geodesics of the foliation. The polygon is possibly infinite-sided and can contain open intervals of $\hat{\mathbf{R}}$ on its boundary at infinity.

Remark. For simplicity of notation, a geodesic lamination on $\mathbf{H}$ which is invariant under the Fuchsian group G is said to be a geodesic lamination on $\mathbf{H}/G$.

Definition 2.2. Let λ be a geodesic lamination on $\mathbf{H}$. A measured lamination μ on $\mathbf{H}$ with support λ is an assignment of a positive, Radon measure to each closed, finite hyperbolic arc I in $\mathbf{H}$ whose support is $I \cap |\lambda|$ and which is invariant under homotopies which preserve the foliation of $|\lambda|$. (We do not require that the homotopies preserve the endpoints of I .) A measured lamination on $\mathbf{H}/G$ is an assignment of a measured lamination μ_G on $\mathbf{H}$ such that the support λ_G is a measured lamination on $\mathbf{H}/G$ (i.e. λ_G is invariant under G) and that the measure μ_G is invariant under the action of G .

Remark. For simplicity of notation, a measured lamination on $\mathbf{H}$ which is invariant under the Fuchsian group G is said to be a measured lamination on $\mathbf{H}/G$.

We define an earthquake of the hyperbolic plane $\mathbf{H}$ following Thurston [11].

Definition 2.3. An earthquake E of $\mathbf{H}$ whose support is a geodesic lamination λ is a bijective map $E : \mathbf{H} \rightarrow \mathbf{H}$ such that the restriction of E to any stratum of λ is in $PSL_2(\mathbf{R})$ and such that E moves strata of λ to the left relative to each other. Namely, for any stratum g of λ we have that $E|_g \in PSL_2(\mathbf{R})$, and for any two

strata g_1, g_2 of λ we have that $(E|_{g_1})^{-1} \circ E|_{g_2}$ is a hyperbolic isometry whose axis is weakly separating the two strata g_1 and g_2 , and which translates g_2 to the left as seen from g_1 . An earthquake E of $\mathbf{H}$ is said to be an earthquake of $\mathbf{H}/G$ if the support λ is invariant under G , and for every $\gamma \in G$ there exists $\gamma_E \in PSL_2(\mathbf{R})$ such that $E \circ \gamma = \gamma_E \circ E$.

An earthquake E of $\mathbf{H}$ naturally induces a measured lamination μ with support λ as follows. The measure μ on each compact hyperbolic arc I is approximated by the sum of the translation lengths between the comparison isometries of n consecutive strata of the support λ of E which intersect I such that the distance between any two consecutive strata is small for n large (see [11], [5]). The measure $\mu(I)$ is obtained by taking the limit of the sum of translation lengths as $n \rightarrow \infty$. It is clear that μ is homotopy invariant. An earthquake E is uniquely determined by its measured lamination up to post-composition with $PSL_2(\mathbf{R})$ [11]. A measured lamination corresponding to an earthquake is called an *earthquake measure*. Moreover, E is an earthquake on $\mathbf{H}/G$ if and only if the earthquake measure of E is a measured lamination on $\mathbf{H}/G$.

An earthquake of $\mathbf{H}$ is not necessarily a homeomorphism of $\mathbf{H}$. A discontinuity of an earthquake appears at the support geodesics of the earthquake measure μ which are atoms of μ . However, an earthquake always extends by continuity to a homeomorphism of the boundary $\hat{\mathbf{R}}$ of $\mathbf{H}$. Thurston [11] proved that any homeomorphism of $\hat{\mathbf{R}}$ can be obtained as the continuous extension of an earthquake of $\mathbf{H}$ to the boundary $\hat{\mathbf{R}}$. In addition, any homeomorphic deformation of the hyperbolic structure on $M = \mathbf{H}/G$ is obtained by an earthquake of M [11].

Definition 2.4. An earthquake measure μ is said to be bounded if

$$\|\mu\| = \sup_I \mu(I) < \infty,$$

where the supremum is over all length 1 closed hyperbolic arcs I in $\mathbf{H}$.

An earthquake E of $\mathbf{H}$ extends by continuity to a quasisymmetric map of $\hat{\mathbf{R}}$ if and only if the earthquake measure μ of E is bounded (see [9], [5], [11], [4], [10]).

Definition 2.5. An earthquake measure μ is said to be asymptotically trivial if for every $\epsilon > 0$ there exists a compact subset $K \subset \mathbf{H}$ such that

$$\sup_{I \cap K = \emptyset} \mu(I) < \epsilon,$$

where the supremum is over all length 1 hyperbolic arcs which do not intersect K .

A quasisymmetric homeomorphism of $\hat{\mathbf{R}}$ is said to be symmetric if it distorts symmetric triples into almost symmetric triples on small scales [6]. Equivalently, a quasisymmetric homeomorphism $h : \hat{\mathbf{R}} \rightarrow \hat{\mathbf{R}}$ is symmetric if and only if there exists a quasiconformal extension $f : \mathbf{H} \rightarrow \mathbf{H}$ of h such that for every $\epsilon > 0$ there exists a compact set $K \subset \mathbf{H}$ with $\|Belt(f)|_{\mathbf{H} \setminus K}\|_\infty < \epsilon$, where $Belt(f) = \bar{\partial}f/\partial f$ is the Beltrami coefficient of f .

It was proved by Gardiner, Hu and Lakic [5] that an earthquake E extends to a symmetric homeomorphism of $\hat{\mathbf{R}}$ if and only if the earthquake measure μ of E is asymptotically trivial (see also [7] and [10] for different proofs).

3. ASYMPTOTICALLY CONFORMAL STRUCTURES ON INFINITE SURFACES VIA EARTHQUAKES

Let $M = \mathbf{H}/G$ be a geometrically infinite hyperbolic surface. A quasiconformal map $f : M \rightarrow M_1$ is said to be *asymptotically conformal* on $M = \mathbf{H}/G$ if for any $\epsilon > 0$ there exists a compact set $K \subset M$ such that $\|Belt(f)|_{M \setminus K}\|_\infty < \epsilon$, where $Belt(f) = \bar{\partial}f/\partial f$ is the Beltrami coefficient of f .

Definition 3.1. An earthquake measure μ_G on $\mathbf{H}/G$ is said to be *asymptotically trivial* on $\mathbf{H}/G$ if for any $\epsilon > 0$ there exists a compact subset K of $\mathbf{H}/G$ such that $\sup_{I \cap \tilde{K} = \emptyset} \mu_G(I) < \epsilon$, where the supremum is over all closed hyperbolic arcs I of length 1 which do not intersect the lift $\tilde{K}$ of K to $\mathbf{H}$.

Theorem 3.1. Let $f : M \rightarrow M_1$ be a quasiconformal map between Riemann surfaces $M = \mathbf{H}/G$ and $M_1 = \mathbf{H}/G_1$, where $G, G_1 < PSL_2(\mathbf{R})$. Fix a lift $\tilde{f} : \mathbf{H} \rightarrow \mathbf{H}$ of f which conjugates G onto G_1 and denote by $h : \hat{\mathbf{R}} \rightarrow \hat{\mathbf{R}}$ the continuous extension of $\tilde{f}$ to $\hat{\mathbf{R}}$. Let $E : \mathbf{H} \rightarrow \mathbf{H}$ be the earthquake on $\mathbf{H}/G$ whose continuous extension to $\hat{\mathbf{R}}$ equals h . Then the earthquake measure μ_h of E is asymptotically trivial on $\mathbf{H}/G$ if and only if f is homotopic to an asymptotically conformal map on $\mathbf{H}/G$ through a bounded homotopy.

Proof. Let $f : M \rightarrow M_1$ be homotopic to an asymptotically conformal map through a bounded homotopy and let $h : \hat{\mathbf{R}} \rightarrow \hat{\mathbf{R}}$ be a continuous extension to $\hat{\mathbf{R}}$ of a lift $\tilde{f} : \mathbf{H} \rightarrow \mathbf{H}$ of f . Let F be the Douady-Earle extension of the quasisymmetric map $h : \hat{\mathbf{R}} \rightarrow \hat{\mathbf{R}}$ to $\mathbf{H}$ (see [1]). Then F induces an asymptotically conformal map on $\mathbf{H}/G$ (see [3]); namely, for every $\epsilon > 0$ there exists a compact set $K \subset \mathbf{H}/G$ such that $\|Belt(F)|_{\mathbf{H} \setminus \tilde{K}}\|_\infty < \epsilon$, where $\tilde{K}$ is the lift of K to $\mathbf{H}$ and $Belt(F) = \frac{\bar{\partial}F}{\partial F}$. Assume on the contrary that the earthquake measure μ_h of E is not asymptotically trivial on $M = \mathbf{H}/G$. Namely, there exists $\epsilon_0 > 0$ and a sequence $I_n \subset \mathbf{H}$ of hyperbolic arcs of length 1 such that $\mu(I_n) \geq \epsilon_0 > 0$ and the preimage in $\mathbf{H}$ of any compact subset of $\mathbf{H}/G$ intersects only finitely many I_n 's.

Let $i \in \mathbf{H}$ be an imaginary unit and let $\gamma_n \in PSL_2(\mathbf{R})$ be such that $\gamma_n(I_n) \ni i$. Let $\mu_n = (\gamma_n)_*(\mu_h)$ be the push-forward of μ_h by γ_n . Note that the support of μ_n is $\gamma_n(\lambda_h)$, where λ_h is the support of μ_h . Then μ_n is the earthquake measure of an earthquake E^{μ_n} which agrees with $h \circ \gamma_n^{-1}$ on $\hat{\mathbf{R}}$ up to post-composition with an element of $PSL_2(\mathbf{R})$. Since $\|\mu_n\| = \|\mu_h\|$ for each n , it follows that there is a subsequence μ_{n_k} of the sequence μ_n which converges in the weak* sense to a measured lamination μ_* with $\|\mu_*\| \leq \|\mu_h\|$. Note that μ_* is a non-zero earthquake measure because $\mu_h(I_n) = \mu_n(\gamma_n(I_n)) \geq \epsilon_0$. By abuse of notation we write μ_n in place of μ_{n_k} .

The restriction $E^{\mu_n}|_{\hat{\mathbf{R}}}$ to $\hat{\mathbf{R}}$ of a properly normalized sequence of earthquakes E^{μ_n} converges pointwise to the restriction $E^{\mu_*}|_{\hat{\mathbf{R}}}$ of a properly normalized earthquake E^{μ_*} (see [10, Proposition 3.3]). Let $\delta_n \in PSL_2(\mathbf{R})$ be such that $\delta_n \circ h \circ \gamma_n^{-1}|_{\hat{\mathbf{R}}} = E^{\mu_n}|_{\hat{\mathbf{R}}}$ for all n . Let F_n be the Douady-Earle extension of $\delta_n \circ h \circ \gamma_n^{-1}$. Since the I_n 's leave the preimage in $\mathbf{H}$ of every compact subset of $\mathbf{H}/G$ and by the conformal naturality of the Douady-Earle extension, we have that $Belt(F_n)$ converges to zero uniformly on compact subsets of $\mathbf{H}$ because F is asymptotically conformal by our assumption. In addition, since $\delta_n \circ h \circ \gamma_n^{-1}$ pointwise converges, it follows that F_n converges to a Möbius map by the continuity properties of the Douady-Earle extension (see [1] and [3]). On the other hand, $\delta_n \circ h \circ \gamma_n^{-1} = E^{\mu_n}|_{\hat{\mathbf{R}}}$

weakly converges to $E^{\mu_*}|_{\hat{\mathbf{R}}}$ with μ_* non-trivial. This implies that the Douady-Earle extension of $E^{\mu_*}|_{\hat{\mathbf{R}}}$ is not Möbius, which is a contradiction with continuity properties of the Douady-Earle extension.

Let μ_h be an asymptotically trivial earthquake measure on $\mathbf{H}/G$ corresponding to h . We need to show that the Douady-Earle extension F of h is asymptotically conformal. Assume on the contrary that there exists a sequence $z_n \in \mathbf{H}$ which leaves the preimage in $\mathbf{H}$ of every compact subset of $\mathbf{H}/G$ such that $|Belt(F)(z_n)| \geq \epsilon_0$, for fixed $\epsilon_0 > 0$. Let $\gamma_n \in PSL_2(\mathbf{R})$ be such that $\gamma_n(z_n) = i \in \mathbf{H}$. Let $\mu_n = (\gamma_n)_* \mu_h$. Then there exists a subsequence μ_{n_k} of μ_n which converges in the weak* sense to a measured lamination μ_* with $\|\mu_*\| \leq \|\mu_h\|$. Without loss of generality we assume that the sequence μ_n converges. By the choice of γ_n , it follows that μ_* is the trivial measured lamination, i.e. $\mu_* = 0$. This implies that there exists $\delta_n \in PSL_2(\mathbf{R})$ such that $\delta_n \circ h \circ \gamma_n^{-1} = E^{\mu_n}|_{\hat{\mathbf{R}}}$ converges weakly to the identity, where $E^{\mu_n}|_{\hat{\mathbf{R}}}$ is a properly normalized earthquake [10]. Let F_n be the Douady-Earle extension of $\delta_n \circ h \circ \gamma_n^{-1}$. Then $Belt(F_n)$ converges uniformly to 0 on compact subsets of $\mathbf{H}$. However, $|Belt(F_n)(i)| = |Belt(F)(z_n)| \geq \epsilon_0 > 0$, which is a contradiction. $\square$

4. THE EARTHQUAKE MEASURE MAP AND THE CONTRACTIBILITY

Given a quasisymmetric map $h : \hat{\mathbf{R}} \rightarrow \hat{\mathbf{R}}$, there is a unique earthquake E_h of $\mathbf{H}$ whose continuous extension to $\hat{\mathbf{R}}$ coincides with h . Let μ_h be the earthquake measure of E_h (μ_h is bounded [10]). Thus we obtain a one-to-one assignment $[h] = PSL_2(\mathbf{R}) \circ h \mapsto \mu_h$ from the universal Teichmüller space $T(\mathbf{H})$ onto the space of bounded measured laminations $ML_b(\mathbf{H})$ of $\mathbf{H}$. (Note that $[h]$ denotes the class $PSL_2(\mathbf{R}) \circ h$ of a quasisymmetric map h and that $T(\mathbf{H})$ is the space of all such classes of quasisymmetric maps.) We denote this map by $\mathcal{EM} : T(\mathbf{H}) \rightarrow ML_b(\mathbf{H})$ and call it the *earthquake measure map*. The earthquake measure map restricts to a map from the Teichmüller space of a hyperbolic Riemann surface M onto the space of bounded measured laminations on M (see [11]).

If M is a compact surface of genus at least two, then the weak* topology on $ML_b(M)$ makes the earthquake measure map $\mathcal{EM} : T(M) \rightarrow ML_b(M)$ a homeomorphism (see [8]). For a geometrically infinite Riemann surface M (including the possibility that $M = \mathbf{H}$), it is an open problem to find a natural topology on $ML_b(M)$ which would make $\mathcal{EM} : T(M) \rightarrow ML_b(M)$ a homeomorphism. (One can induce a topology on $ML_b(M)$ by the pull-back of the topology on $T(M)$, but there is no information about such an assigned topology on $ML_b(M)$. A “natural” topology on $ML_b(M)$ should be given in terms of the measured laminations on the surface M without the use of the quasisymmetric maps.) However, we show below that the earthquake measure map $\mathcal{EM} : T(M) \rightarrow ML_b(M)$ is natural for the scaling of the earthquake measure.

Theorem 4.1. *The scaling measure map $H : [0, 1] \times T(\mathbf{H}) \rightarrow T(\mathbf{H})$ given by*

$$H(t, [h]) = [E^{(1-t)\mathcal{EM}([h])}|_{\hat{\mathbf{R}}}]$$

is continuous.

Remark. The map $H : [0, 1] \times T(\mathbf{H}) \rightarrow T(\mathbf{H})$ restricts to a continuous map $H : [0, 1] \times T(\mathbf{H}/G) \rightarrow T(\mathbf{H}/G)$, for any Fuchsian group G .

Proof. Let $[h] \in T(\mathbf{H})$. We show that H is continuous at $(t, [h])$. Let $[h_n] \in T(\mathbf{H})$ be a sequence converging to $[h]$ in the Teichmüller metric of $T(\mathbf{H})$ and let $t_n \rightarrow t$ as $n \rightarrow \infty$. We need to show that $H(t_n, [h_n]) \rightarrow H(t, [h])$ as $n \rightarrow \infty$.

We give a lemma which is used in the proof. Let $\mathcal{G} = \hat{\mathbf{R}} \times \hat{\mathbf{R}} \setminus \text{diag}$ be the space of (oriented) geodesics in $\mathbf{H}$. Let $Q_* = [a, b] \times [c, d] \subset \mathcal{G}$, for a, b, c, d in counterclockwise order on $\hat{\mathbf{R}}$, be a fixed “box” of geodesics such that the Liouville measure $L(Q_*)$ of the box Q_* is 1, namely,

$$L(Q_*) = \log \frac{(c-a)(d-b)}{(d-a)(c-b)} = 1.$$

Lemma 4.1. *Let $[h_n] \rightarrow [h]$ in $T(\mathbf{H})$, $\mu_n = \mathcal{EM}([h_n])$ and $\mu = \mathcal{EM}([h])$. Then $\mu_n \rightarrow \mu$ in the following sense:*

Let $Q_ = [a, b] \times [c, d]$ be a fixed box with $L(Q_*) = 1$. If Q is any box with $L(Q) = 1$, then there exists $\gamma_Q \in PSL_2(\mathbf{R})$ such that $\gamma_Q(Q) = Q_*$. For any continuous function $\varphi : \mathcal{G} \rightarrow \mathbf{R}$ whose support is in the fixed box Q_* , we have that*

$$\mathcal{S}_\varphi(\mu_n, \mu) = \sup_{L(Q)=1} \left| \iint_{Q_*} \varphi d(\gamma_Q)_*(\mu_n) - \iint_{Q_*} \varphi d(\gamma_Q)_*(\mu) \right| \rightarrow 0$$

as $n \rightarrow \infty$, where the supremum is over all boxes Q with $L(Q) = 1$.

Proof. Assume on the contrary that $\mu_n \not\rightarrow \mu$ as $n \rightarrow \infty$. Then there exist a continuous function $\varphi : \mathcal{G} \rightarrow \mathbf{R}$ whose support is in Q_* and a sequence Q_{n_k} with $L(Q_{n_k}) = 1$ such that

$$\left| \iint_{Q_*} \varphi d(\gamma_{Q_{n_k}})_*(\mu_n) - \iint_{Q_*} \varphi d(\gamma_{Q_{n_k}})_*(\mu) \right| \geq \epsilon_0 > 0.$$

We write n instead of n_k for simplicity. Let $\nu_n = (\gamma_{Q_n})_*(\mu_n)$ and $\zeta_n = (\gamma_{Q_n})_*(\mu)$ be the push-forwards by γ_{Q_n} of μ_n and μ , respectively. By the above, at least one of the measures ν_n and ζ_n are not zero on Q_* . Let t_{ν_n} be a geodesic in Q_* in the support of ν_n or in the interior of a stratum of ν_n . Define t_{ζ_n} similarly.

Let $\delta_n, \delta'_n \in PSL_2(\mathbf{R})$ be such that $E^{\nu_n}|_{t_{\nu_n}} = \delta_n \circ E^{\mu_n} \circ \gamma_{Q_n}^{-1}|_{t_{\nu_n}} = id$ and that $E^{\zeta_n}|_{t_{\zeta_n}} = \delta'_n \circ E^\mu \circ \gamma_{Q_n}^{-1}|_{t_{\zeta_n}} = id$. Note that $\|\nu_n\|$ and $\|\zeta_n\|$ are bounded uniformly in n (because $[h_n] \rightarrow [h]$ in the Teichmüller metric). It follows that there exist subsequences of $E^{\nu_n}|_{\hat{\mathbf{R}}}$ and $E^{\zeta_n}|_{\hat{\mathbf{R}}}$ which converge weakly on $\hat{R}$ to homeomorphisms f and g , respectively (see [10]). Moreover, there are subsequences of ν_n and ζ_n which converge in the weak* sense to bounded measured laminations ν_* and ζ_* , respectively. We can assume without loss of generality that both sequences converge. It is clear that $\nu_* \neq \zeta_*$ because ν_n and ζ_n differ on Q_* by a definite amount (according to the assumption). This implies that $g \circ f^{-1}$ is not in $PSL_2(\mathbf{R})$.

Let $\tilde{F}_n$ and $\tilde{G}_n$ be the Douady-Earle extensions of $E^{\nu_n}|_{\hat{\mathbf{R}}}$ and $E^{\zeta_n}|_{\hat{\mathbf{R}}}$, respectively. Let $\tilde{F}$ and $\tilde{G}$ be the Douady-Earle extensions of f and g , respectively. Then

$$|Belt(\tilde{F}_n) - Belt(\tilde{G}_n)| \rightarrow 0$$

uniformly on compact subsets of $\mathbf{H}$ because $E^{\nu_n}|_{\hat{\mathbf{R}}} = \delta_n \circ h_n \circ \gamma_{Q_n}^{-1}$ and $E^{\zeta_n}|_{\hat{\mathbf{R}}} = \delta'_n \circ h \circ \gamma_{Q_n}^{-1}$, $[h_n] \rightarrow [h]$ and by the conformal naturality of the Douady-Earle extension. This implies that $Belt(\tilde{G}) = Belt(\tilde{F})$, which is a contradiction with $g \circ f^{-1} \notin PSL_2(\mathbf{R})$. $\square$

We continue with the proof of the theorem. Let $d([h_1], [h_2])$ denote the Teichmüller distance between $[h_1], [h_2] \in T(\mathbf{H})$. Let F_n and G be the Douady-Earle extensions of $E^{(1-t_n)\mu_n}|_{\hat{\mathbf{R}}}$ and $E^{(1-t)\mu}|_{\hat{\mathbf{R}}}$, respectively. Assume on the contrary that $d(E^{(1-t_n)\mu_n}|_{\hat{\mathbf{R}}}, E^{(1-t)\mu}|_{\hat{\mathbf{R}}}) \not\rightarrow 0$ as $n \rightarrow \infty$. This is equivalent to the statement that there exists $z_n \in \mathbf{H}$ such that

$$(1) \quad |\text{Belt}(F_n)(z_n) - \text{Belt}(G)(z_n)| \geq \epsilon_0 > 0.$$

Let $\gamma_n \in PSL_2(\mathbf{R})$ be such that $\gamma_n(z_n) = i \in \mathbf{H}$. There exist subsequences of $(\gamma_n)_*((1-t_n)\mu_n)$ and of $(\gamma_n)_*((1-t)\mu)$ which converge in the weak* sense. We can assume without loss of generality that both sequences converge in the weak* sense and the above lemma implies that their limits are equal. Thus there exist $\delta_n, \delta'_n \in PSL_2(\mathbf{R})$ such that $f_n = \delta_n \circ E^{(1-t_n)\mu_n}|_{\hat{\mathbf{R}}} \circ \gamma_n^{-1}$ and $g_n = \delta'_n \circ E^{(1-t)\mu}|_{\hat{\mathbf{R}}} \circ \gamma_n^{-1}$ pointwise converge to the same homeomorphism of $\hat{\mathbf{R}}$. Let $\tilde{F}_n$ and $\tilde{G}_n$ be the Douady-Earle extensions of f_n and g_n , respectively. Consequently $\text{Belt}(\tilde{F}_n)$ and $\text{Belt}(\tilde{G}_n)$ are close on compact subsets of $\mathbf{H}$ for n large. This contradicts (1) by the conformal naturality of the Douady-Earle extension and the fact that $\gamma_n(z_n) = i \in \mathbf{H}$ (see the proof of the above lemma for similar reasoning). $\square$

The above theorem gives a new proof that the Teichmüller space of an arbitrary Riemann surface is contractible. This was originally proved by Douady and Earle [1] (see [3] for contractibility of the subspace of asymptotically conformal classes).

Corollary 4.1. *The Teichmüller space $T(\mathbf{H}/G)$ of a hyperbolic surface $\mathbf{H}/G$ is contractible. If $\mathbf{H}/G$ is geometrically infinite, then the subspace of $T(\mathbf{H}/G)$ consisting of all asymptotically conformal classes is also contractible.*

Remark. The above theorem implies contractibility of any subspace of $T(\mathbf{H})$ as long as it is closed under earthquake paths.

5. MINIMAL RATE OF DECREASE TOWARD THE BOUNDARY

We establish a sufficient condition for a measured lamination μ on the hyperbolic plane to be trivial in terms of its rate of decrease when approaching the boundary. It is convenient to work with the unit disk $\mathbf{D}$ model of the hyperbolic plane, where the hyperbolic metric is $\rho(z) = \frac{2|dz|}{1-|z|^2}$. Let I be a closed hyperbolic arc of length 1 in $\mathbf{D}$. Denote by $\delta(I)$ the minimal Euclidean distance from I to the unit circle $S^1 = \partial\mathbf{D}$. We say that μ decreases of the order $o(\delta)$ and write $\mu = o(\delta)$ if

$$\mu(I)/\delta(I) \rightarrow 0$$

as $\delta(I) \rightarrow 0$ uniformly in I .

Proposition 5.1. *Let μ be a measured lamination on $\mathbf{D}$. If $\mu = o(\delta)$, then μ is trivial, i.e. $\mu = 0$.*

Proof. Consider the Euclidean circle of radius $1-1/n$ with center 0. The hyperbolic length of the circle is $\frac{4\pi(n-1)}{2-1/n}$. The circle is at a Euclidean distance $1/n$ from S^1 . The assumption on μ implies that the total μ -measure of the circle is $o(1/n) \cdot \frac{4\pi(n-1)}{2-1/n} \rightarrow 0$ as $n \rightarrow \infty$. Since each compact subset of the support of μ intersects all such circles for n large enough, it follows that the μ -measure of each compact set in its support is zero. $\square$

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