AN OPTIMAL LIMITING 2D SOBOLEV INEQUALITY

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ABSTRACT. The main goal of this paper is to prove an optimal limiting Sobolev inequality in two dimensions for Hölder continuous functions. Additionally, from this inequality we derive the double logarithmic inequality

$$\|u\|_{L^\infty} \leq \frac{\|\nabla u\|_{L^2}}{\sqrt{2\pi\alpha}} \sqrt{\ln\left(1 + 6\sqrt{2\pi\alpha} \frac{\|u\|_{\dot{C}^\alpha}}{\|\nabla u\|_{L^2}} \sqrt{\ln(1 + \sqrt{2\pi\alpha} \frac{\|u\|_{\dot{C}^\alpha}}{\|\nabla u\|_{L^2}})}\right)}$$

for functions $u \in W_0^{1,2}(B_1)$ on the unit disk B_1 in $\mathbb{R}^2$, $\alpha \in (0, 1]$.

1. INTRODUCTION

The Sobolev embeddings in two dimensions,

$$W^{1,p} \subset L^{\frac{2p}{2-p}}, \quad \text{for } p \in [1, 2) \quad \text{and} \quad W^{1,p} \subset C^{1-2/p}, \quad \text{for } p \in (2, +\infty),$$

fail in the limiting case $p = 2$ (see e.g. [1, chapter 5]). In the setting of a bounded domain, we have the inclusion $W^{1,2} \subset L^q$ for any $q < \infty$, but not for $q = \infty$. The function $f(x) = \ln(1 - \min(0, \ln|x|))$ gives a counterexample to the limiting inclusion.

However, with a small additional regularity condition, functions in $W^{1,2}$ are known to be bounded, with a bound that can be obtained from the so-called logarithmic Sobolev inequalities. Brezis-Gallouet [4] firstly presented this type of inequality.¹ In particular they prove the following inequality:

$$(1) \quad \|u\|_{L^\infty} \leq C \|u\|_{W^{1,2}} \left(1 + \sqrt{\ln(1 + \|u\|_{W^{2,2}}/\|u\|_{W^{1,2}})}\right).$$

There are a number of articles in which similar inequalities were proven in various other settings, including cases of different norms and in different space dimensions; see, for example, [5], [7], [13], [3].

The inequality (1) can be rewritten as $\|u\|_{L^\infty} \leq \|u\|_{W^{1,2}} g(\|u\|_{W^{2,2}}/\|u\|_{W^{1,2}})$, where $g(t) = C(1 + \sqrt{\ln(1+t)})$. It is natural to ask what is the optimal (i.e. minimal) function g for which this inequality holds. In the present article we study a similar inequality when the $W^{2,2}$ -norm is replaced with the α -Hölder seminorm.

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¹Another kind of logarithmic Sobolev inequality was introduced in [9].

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We give a complete answer for the optimality question when the domain is a disk. A similar problem, without however the optimality question, is studied in [10].

Let Ω be an open domain in $\mathbb{R}^2$. Define $W_0^{1,2}(\Omega)$ to be the closure of smooth and compactly supported functions in the Sobolev space $W^{1,2}(\Omega)$. For all real numbers $\alpha \in (0, 1]$ let $\dot{C}^\alpha$ be the (homogeneous) space of α -Hölder continuous functions, equipped with the seminorm

$$\|u\|_{\dot{C}^\alpha} = \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

For $\alpha = 1$ this is the Lipschitz seminorm and $\dot{C}^1$ is the space of Lipschitz continuous functions.

The main result of this paper is the following theorem.

Theorem 1. *Let B_1 be the unit disk in $\mathbb{R}^2$, and consider $\alpha \in (0, 1]$. For $u \in W_0^{1,2}(B_1)$,*

$$(2) \quad \|u\|_{L^\infty}^2 \leq \frac{1}{2\pi\alpha} \|\nabla u\|_{L^2}^2 F(\sqrt{2\pi\alpha} \frac{\|u\|_{\dot{C}^\alpha}}{\|\nabla u\|_{L^2}}),$$

where the function $F: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is defined implicitly by

$$(3) \quad F\left(\frac{\exp(y)}{\sqrt{y+1/2}}\right) = \frac{(1+y)^2}{y+1/2}, \quad y \geq 0 \quad \text{and} \quad F(t) = t^2 \quad \text{for } t < \sqrt{2},$$

using the convention that $F(+\infty) = +\infty$.

Here and in what follows, $\mathbb{R}_+$ stands for the set of nonnegative real numbers.

Remarks. The map F is indeed well defined as a function since the map $y \mapsto e^y(\frac{1}{2} + y)^{-1/2}$ is continuously increasing for $y \geq 0$, ranging in $[\sqrt{2}, +\infty)$. Since $\exp(y) \geq 1 + y$ we have $F(t) \leq t^2$ for any $t \geq 0$. Therefore (2) is consistent with the simple bound $\|u\|_{L^\infty(B_1)} \leq \|u\|_{\dot{C}^\alpha(B_1)}$.

Our second main result establishes the optimality of the function F .

Theorem 2 (Optimality). *Let $\alpha \in (0, 1]$. Let $G_\alpha: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a function such that the inequality*

$$(4) \quad \|u\|_{L^\infty}^2 \leq \frac{1}{2\pi\alpha} \|\nabla u\|_{L^2}^2 G_\alpha\left(\frac{\|u\|_{\dot{C}^\alpha}}{\|\nabla u\|_{L^2}}\right)$$

holds for any function $u \in W_0^{1,2}(B_1) \cap \dot{C}^\alpha(B_1)$. Then for $\alpha \in (0, 1)$ and for any $s > 0$ we have

$$(5) \quad G_\alpha(s) \geq F(s\sqrt{2\pi\alpha}).$$

For $\alpha = 1$, inequality (5) holds for $s \geq \frac{1}{\sqrt{\pi}}$, and we have no restrictions on $G_1(s)$ for $s < \frac{1}{\sqrt{\pi}}$.

The α -homogeneity in (2), i.e. the fact that the same function F provides the optimal inequality for all α , can be explained by the fact that the set of extremes of (2), as well as the quantities $\|\nabla u\|_{L^2}/\sqrt{\alpha}$, $\|u\|_{\dot{C}^\alpha}$ and $\|u\|_{L^\infty}$ on the set of extremes, are preserved under the transformation

$$u(x) \mapsto \tilde{u}(x) = u(x|x|^{\lambda-1}), \quad \alpha \mapsto \tilde{\alpha} = \lambda\alpha,$$

provided $\alpha, \lambda\alpha \in (0, 1]$. See section 2 for more details.

The function F admits simple bounds. Indeed, if the argument t of the function F is greater than or equal to $\sqrt{2}$, then according to definition (3) we can write $t = \frac{\exp(y)}{\sqrt{y+1/2}}$, $y \geq 0$. Equivalently, the nonnegative parameter y satisfies the following relation:

$$(6) \quad y = \ln t + \frac{1}{2} \ln(y + \frac{1}{2}).$$

Using the inequality $\ln a \leq a - 1$ with $a = y + \frac{1}{2}$ we conclude that $y + 1/2 \leq 2 \ln t$. Therefore for $t \geq \sqrt{2}$ we have the following upper bounds:

$$F(t) \leq 2 + y = 2 + \ln(t) + \frac{1}{2} \ln(y + \frac{1}{2}) \leq 2 + \ln(t) + \frac{1}{2} \ln(2 \ln t) \leq \ln(11 t \sqrt{\ln(t)}).$$

Similarly, the nonnegativity of the parameter y and the relation (6) imply that $y + 1/2 \geq \ln t$. Using the inequality $F(t) \geq \frac{3}{2} + y$ and (6) again, we obtain the following lower bounds for $t \geq \sqrt{2}$:

$$(7) \quad F(t) \geq \frac{3}{2} + y \geq \frac{3}{2} + \ln(t) + \frac{1}{2} \ln(\ln t) = \ln(e^{3/2} t \sqrt{\ln(t)}) \geq \ln(4.48 t \sqrt{\ln(t)}).$$

In section 3 we will prove more involved estimates, as well as the following theorem:

Theorem 3. *For any $t \geq 0$ the function F from Theorem 1 satisfies the following inequality:*

$$(8) \quad F(t) \leq \ln(1 + 6t\sqrt{\ln(1+t)}).$$

The constant 6 is close to the optimal constant; for example, it cannot be replaced with 5.95. Numerical experimentation shows that the optimal constant is about 5.97 ... ; however, the precise value of this constant falls outside the scope of the present article.

Not only is the function F optimal, but the constant $\frac{1}{2\pi\alpha}$ and the square root of the inner logarithm are also optimal in the following sense.

Theorem 4 (Optimality of the constant and the square root). *Consider $\alpha \in (0, 1]$. Let λ and β be nonnegative real numbers, and let $G : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a function of the form*

$$G(t) = \ln(c_1 + c_2 t (\ln(c_3 + c_4 t))^\beta)$$

such that the inequality

$$(9) \quad \|u\|_{L^\infty}^2 \leq \lambda \|\nabla u\|_{L^2}^2 G\left(\frac{\|u\|_{\dot{C}^\alpha}}{\|\nabla u\|_{L^2}}\right)$$

holds for all functions $u \in W_0^{1,2}(B_1) \cap \dot{C}^\alpha(B_1)$. Then

$$\lambda \geq \frac{1}{2\pi\alpha}$$

and

$$\text{if } \lambda = \frac{1}{2\pi\alpha}, \quad \text{then } \beta \geq \frac{1}{2}.$$

Proof. The result follows from Theorem 2 and inequality (7). $\square$

Remarks. In the case $\lambda = 1/(2\pi\alpha)$ and $\beta = 1/2$, we can further obtain that $c_2 > \sqrt{2\pi\alpha} e^{3/2}$. The inequality $c_2 \geq \sqrt{2\pi\alpha} e^{3/2}$ follows from Theorem 2 and the lower bound (7). To show the strict inequality we should use lower bounds on F from section 3.

This article is organized as follows. In section 2 we prove Theorems 1 and 2. In section 3 we give bounds for the optimal function F and prove Theorem 3. In section 4 we give two results concerning the case of an arbitrary domain and the case of the whole space $\mathbb{R}^2$.

2. PROOF OF THEOREMS 1 AND 2

Proof of Theorem 1. Let $g(y) = e^y(\frac{1}{2} + y)^{-1/2}$ and $h(y) = \frac{(1+y)^2}{1/2+y}$. Both functions g and h are positive and increasing for $y > 0$ and

$$\frac{g'(y)}{g(y)} = \frac{y}{y + 1/2}, \quad h'(y) = \frac{(y+1)y}{(y+1/2)^2}.$$

Therefore the function $F(t)$, which for $t \geq \sqrt{2}$ can be parametrically given as $F(g(y)) = h(y)$, is monotonically increasing. Using the chain rule we can compute F' at any point $t = g(y)$, $y \geq 0$, as follows:

$$F'(g(y)) = \frac{h'(y)}{g'(y)} = \frac{y+1}{e^y \sqrt{y+1/2}},$$

where for $y = 0$ ($t = \sqrt{2}$) the one-sided derivative is considered. First we show that the map

$$(z, D) \mapsto z^2 F\left(\frac{D}{z}\right), \quad z > 0, \quad D \geq 0$$

is monotonically nondecreasing in both variables z and D . The monotonicity in the D variable is clear, following immediately from the fact that F is increasing. To show the monotonicity in the z argument, compute the partial derivative

$$\frac{\partial}{\partial z} \left(z^2 F\left(\frac{D}{z}\right) \right) = z \left(2F\left(\frac{D}{z}\right) - \frac{D}{z} F'\left(\frac{D}{z}\right) \right).$$

Thus we need to show that $2F(t) - tF'(t) \geq 0$ for any $t \geq 0$. For the interval $t \in [0, \sqrt{2}]$ this is clear. For the case $t \geq \sqrt{2}$, even more is true: $F(t) - tF'(t) \geq 0$. Indeed, substituting $t = g(y)$, $y \geq 0$ we have

$$F(t) - tF'(t) = h(y) - g(y) \frac{h'(y)}{g'(y)} = \frac{(1+y)y}{y+1/2} \geq 0.$$

We conclude that for any positive number α the map

$$(10) \quad (z, D) \mapsto z^2 F\left(\sqrt{2\pi\alpha} \frac{D}{z}\right)$$

is monotonically nondecreasing in both positive arguments z and D .

Without loss of generality we assume that the function u is nonnegative. Indeed, we may assume that $\|u\|_{L^\infty} = \|u_+\|_{L^\infty}$, where $u_+ = \max\{u, 0\}$. The case of $u_- = \max\{-u, 0\}$ is similar. Since

$$\|\nabla u_+\|_{L^\infty} \leq \|\nabla u\|_{L^\infty} \quad \text{and} \quad \|u_+\|_{\dot{C}^\alpha} \leq \|u\|_{\dot{C}^\alpha},$$

then the monotonicity property for the function (10) would complete the argument.

For any nonnegative function $u \in W_0^{1,2}$ both (semi)norms $\|\nabla u\|_{L^2}$ and $\|u\|_{\dot{C}^\alpha}$ are nonincreasing under symmetric decreasing rearrangement (also known as Schwarz symmetrization [6, p. 70]). The nonincreasing property of the L^2 -norm of the gradient can be found, e.g., in [14, appendix A], [12], [15]. The nonincreasing property for $\|\cdot\|_{\dot{C}^\alpha}$ follows from the fact that the symmetrization increases (more precisely, does not decrease) distances between the level sets of a function (see the Brunn-Mikowski inequality, Theorem III.2.2 in [6]). Therefore the continuity modulus and all Hölder (semi)norms do not increase under such rearrangements.

Since the L^∞ -norm remains unchanged under symmetric decreasing rearrangement, we conclude that it is sufficient to prove (2) in the class of nonnegative, radially symmetric and nonincreasing functions.

Without loss of generality we can normalize $\|u\|_{L^\infty} = 1$, and this normalization implies that $\|u\|_{\dot{C}^\alpha} \geq 1$, since u vanishes on the boundary.

Let $W_{0,\text{rad}}^{1,2}$ be the space of all radially symmetric nonincreasing functions in $W_0^{1,2}$. To prove the theorem, it is sufficient to show that

$$(11) \quad 2\pi\alpha \leq \inf_{D \geq 1} \inf_{u \in W_{0,\text{rad}}^{1,2}, \|u\|_{L^\infty}=1, \|u\|_{\dot{C}^\alpha}=D} \|\nabla u\|_{L^2}^2 F(\sqrt{2\pi\alpha} \frac{D}{\|\nabla u\|_{L^2}}).$$

For any real parameter $D \geq 1$, let S_D be the closed convex subset of $W_{0,\text{rad}}^{1,2}$ that is given by

$$S_D = \{u \in W_{0,\text{rad}}^{1,2} : u(r) \geq 1 - Dr^\alpha, r \in [0, 1]\}.$$

Note that the set $\{u \in W_{0,\text{rad}}^{1,2}, \|u\|_{L^\infty} = 1, \|u\|_{\dot{C}^\alpha} = D\}$ is a subset of S_D . Therefore (11) will follow, provided we prove that for any $D \geq 1$ we have

$$(12) \quad 2\pi\alpha = \inf_{u \in S_D} \|\nabla u\|_{L^2}^2 F(\sqrt{2\pi\alpha} \frac{D}{\|\nabla u\|_{L^2}}).$$

We will prove that the infimum on the right hand side of (12) is actually a minimum. Moreover the minimizer belongs to $\{u \in W_{0,\text{rad}}^{1,2}, \|u\|_{L^\infty} = 1, \|u\|_{\dot{C}^\alpha} = D\}$. This means that we actually have equality in (11).

Keeping in mind the monotonicity of (10), we consider the problem of minimization

$$I[u] := \|\nabla u\|_{L^2(B_1)}^2$$

over all functions u in the set S_D . This is a variational problem with an obstacle (one-sided constraint). It is well-known (see, e.g., [8, Sect. 8.4.2] and [11]) that it has a unique minimizer u^* , which is variationally characterized by

$$\int_{B_1} \nabla u^* \cdot \nabla (v - u^*) dx \geq 0 \quad \text{for all } v \in S_D.$$

Also the minimizer u^* is continuous and moreover is at least continuously differentiable at points of regularity of the constraint $x \mapsto 1 - D|x|^\alpha$, i.e. for $x \neq 0$. We also have $u^* \in W^{2,\infty}(B_1 \setminus \{0\})$; see [11]. Hence the radially symmetric set

$$\mathcal{O} := \{x \in B_1 : u^*(x) > 1 - D|x|^\alpha\}$$

is open and u^* is harmonic in $\mathcal{O}$. We now claim that $\mathcal{O}$ has at most one connected component and moreover that

$$\mathcal{O} := \{x \in B_1 : |x| > a\} \quad \text{for certain } a \in (0, 1].$$

Indeed, any harmonic, radially symmetric function in $\mathbb{R}^2$ is of the form $r \mapsto c_1 + c_2 \ln r$ for some parameters c_1 and c_2 , and therefore it can only have a unique tangent point with the function $r \mapsto 1 - Dr^\alpha$ at some point $r = a$. Note also that u^* cannot be harmonic near $r = 0$; otherwise it is a positive constant there, which implies that u^* is constant in the whole unit disk, which leads to a contradiction with the boundary conditions. Keeping in mind the boundary condition at $r = 1$ we conclude that there exists a unique $a \in (0, 1]$ such that

$$(13) \quad u^*(r) = \begin{cases} 1 - Dr^\alpha & \text{for } r \in [0, a], \\ (1 - Da^\alpha) \frac{\ln r}{\ln a} & \text{for } r \in [a, 1]. \end{cases}$$

If $D > 1$, then because of the boundary conditions, the set $\mathcal{O}$ is nonempty, and we can find a from the tangent condition

$$-D\alpha r^{\alpha-1} = (1 - Da^\alpha) \frac{1}{r} \frac{1}{\ln a} \quad \text{at } r = a.$$

This gives a relation between a and D :

$$Da^\alpha(1 - \ln a^\alpha) = 1.$$

If $D = 1$, then we necessarily have $a = 1$, which is consistent with the tangent condition. Now we need to compute the norms of u^* . Clearly $\|u^*\|_{L^\infty} = 1$ and $\|u^*\|_{\dot{C}^\alpha} = D$. To find $\|\nabla u^*\|_{L^2}^2$ we need to compute two integrals:

$$2\pi \int_0^a r D^2 \alpha^2 r^{2\alpha-2} dr = \pi \alpha D^2 a^{2\alpha}$$

and

$$2\pi \int_a^1 \left(\frac{1 - Da^\alpha}{\ln a} \right)^2 \frac{dr}{r} = -2\pi \left(\frac{1 - Da^\alpha}{\ln a} \right)^2 \ln a.$$

Using the tangent condition in the form $\frac{1 - Da^\alpha}{-\ln a} = \alpha Da^\alpha$ we obtain

$$\|\nabla u^*\|_{L^2}^2 = 2\pi \alpha D^2 a^{2\alpha} \left(\frac{1}{2} - \ln a^\alpha \right) = 2\pi \alpha \frac{1/2 - \ln a^\alpha}{(1 - \ln a^\alpha)^2}.$$

We introduce a new parameter $y = -\ln a^\alpha \in [0, +\infty)$. From the tangent condition we have $D = \frac{e^y}{1+y}$, and therefore we can write

$$\|\nabla u^*\|_{L^2}^2 = 2\pi \alpha \frac{1/2 + y}{(1+y)^2}, \quad \sqrt{2\pi \alpha} \frac{D}{\|\nabla u^*\|_{L^2}} = \frac{e^y}{1/2 + y}.$$

Using the definition of F we see that the function u^* (which minimizes the r.h.s. of (12)) gives equality in (12). Thus we have proved (12) and therefore Theorem 1. $\square$

Proof of Theorem 2. First we consider the case $s \geq \frac{1}{\sqrt{\pi\alpha}}$, i.e., $\sqrt{2\pi\alpha}s \geq \sqrt{2}$. Find $y \geq 0$ from $\frac{e^y}{1/2+y} = \sqrt{2\pi\alpha}s$ and put $D = \frac{e^y}{1+y}$. Then the minimizer (13) satisfies

$$\|u^*\|_{\dot{C}^\alpha} / \|\nabla u^*\|_{L^2} = s$$

and gives the equality in (2). Therefore $G_\alpha(s) \geq F(s\sqrt{2\pi\alpha})$ in this case.

For $\alpha = 1$ we always have $\|u\|_{\dot{C}^1} / \|\nabla u\|_{L^2} \geq 1/\sqrt{\pi}$. Therefore $G_1(s)$ can be arbitrary for $s < 1/\sqrt{\pi}$.

Let $\alpha \in (0, 1)$ and $\frac{1}{s} > \sqrt{\pi\alpha}$. Let $u^*(r) = 1 - r^\alpha$. We have $\|\nabla u^*\|_{L^2} = \sqrt{\pi\alpha}$ and $\|u^*\|_{\dot{C}^\alpha} = 1$. For any positive integer n and nonnegative real number μ let $u_{n,\mu} = u^* + \varphi_{n,\mu}$, where $\varphi_{n,\mu}$ denotes the following radially symmetric function:

$$\varphi_{n,\mu}(r) = \frac{\mu}{4n} \text{dist}(4nr, \mathbb{Z}) \quad \text{if } r \in [1/2, 3/4], \quad \varphi_{n,\mu}(r) = 0 \quad \text{otherwise.}$$

Here $\text{dist}(x, \mathbb{Z}) = \inf_{m \in \mathbb{Z}} |x - m|$ denotes the distance to the nearest integer. It is clear that $\varphi_{n,\mu}$ is a Lipschitz continuous function with Lipschitz constant μ . Moreover,

$$\|\nabla \varphi_{n,\mu}\|_{L^2} = \mu \frac{\sqrt{5\pi}}{4} \quad \text{and} \quad \lim_{n \rightarrow \infty} \|\varphi_{n,\mu}\|_{\dot{C}^\alpha} = 0 \quad \text{for } \alpha \in (0, 1).$$

Choose n sufficiently large so that $\|u_{n,\mu}\|_{\dot{C}^\alpha} = 1$ for all $\mu \in [0, \frac{8}{s\sqrt{5\pi}}]$. Secondly, by the intermediate value principle one can find $\mu \in [0, \frac{8}{s\sqrt{5\pi}}]$ such that

$\|\nabla u_{n,\mu}\|_{L^2} = 1/s$. Finally we note that the function $u_{n,\mu}$ gives equality in (2) and satisfies $\|u_{n,\mu}\|_{\dot{C}^\alpha}/\|\nabla u_{n,\mu}\|_{L^2} = s$. Therefore $G_\alpha(s) \geq F(s\sqrt{2\pi\alpha})$ in this case. $\square$

3. BOUNDS FOR THE FUNCTION F

The objective of this section is to find upper and lower bounds for F that asymptotically recover F with a good accuracy as $t \rightarrow \infty$, yet are also good for finite t , and are sufficiently simple.

We will obtain such bounds for the interval $t \in [\sqrt{2}, +\infty)$. As a corollary we have the result of Theorem 3.

In this section the variable t always relates to $y \geq 0$ as long as it is greater than or equal to $\sqrt{2}$, and vice versa, as $t = g(y)$, where the function $g(\cdot)$ is defined in section 2. Equivalently, we have

$$(14) \quad y = \ln t + \frac{1}{2} \ln(y + \frac{1}{2}).$$

One can check that the inequality

$$\frac{1/2}{y + 1/2} \leq \ln\left(y + 1 + \frac{1}{2} \ln(y + 1) - \frac{1}{2} \ln(y + \frac{1}{2})\right) - \ln(y + \frac{1}{2})$$

is valid for all $y > \frac{1}{35}$. Dividing it by 2 and adding y to both the left hand side and the right hand side, we obtain the following inequality for all $y > \frac{1}{35}$:

$$(15) \quad y + \frac{1/4}{y + 1/2} \leq \ln t + \frac{1}{2} \ln(1 + \ln t + \frac{1}{2} \ln(y + 1)).$$

Since $\ln$ is a concave function, we have $\ln(y + 1/2) + \frac{1/2}{y + 1/2} \geq \ln(y + 1)$, and therefore

$$(16) \quad y + \frac{1/4}{y + 1/2} = \ln t + \frac{1}{2} \ln(y + \frac{1}{2}) + \frac{1/4}{y + 1/2} \geq \ln t + \frac{1}{2} \ln(y + 1).$$

Adding 3/2 to (15) and (16) and using (14), we obtain the following bounds for $F(t) = y + \frac{3}{2} + \frac{1/4}{y + 1/2}$:

$$(17) \quad F(t) \leq \frac{3}{2} + \ln(t) + \frac{1}{2} \ln(1 + \ln t + \frac{1}{2} \ln(y + 1)), \quad \text{for } y > \frac{1}{35} \quad (t > g(\frac{1}{35})),$$

$$(18) \quad F(t) \geq \frac{3}{2} + \ln(t) + \frac{1}{2} \ln(1 + \ln t + \frac{1}{2} \ln(y + \frac{1}{2})), \quad \text{for } y \geq 0 \quad (t \geq \sqrt{2}).$$

Consider the following family of functions, which depends on a parameter c :

$$P_c(t) = \frac{3}{2} + \ln(t) + \frac{1}{2} \ln\left(1 + \ln t + \frac{1}{2} \ln(\ln t + \frac{1}{2} \ln \ln t + c)\right).$$

Lemma 5. For all real numbers $t \geq \sqrt{2}$ we have

$$P_{1/2}(t) < F(t) < P_{3/2}(t).$$

Proof. First we check the right inequality manually for $t \in [\sqrt{2}, g(1/35)]$. This is not hard since $F(g(1/35)) < P_{3/2}(\sqrt{2})$ and both functions are increasing. Using the inequalities

$$-1 \leq \ln(y + \frac{1}{2}) \leq y - \frac{1}{2}$$

we conclude from (14) that

$$\ln t \leq y + \frac{1}{2} \leq 2 \ln t.$$

Using (14) again, we obtain

$$\ln t + \frac{1}{2} \ln \ln t \leq y \leq \ln t + \frac{1}{2} \ln \ln t + \frac{\ln 2}{2}.$$

Now (17) and (18) complete the proof. $\square$

One can further compute that

$$\lim_{t \rightarrow \infty} (P_c(t) - F(t)) \ln^2 t = \frac{1}{4}(c - \frac{3}{4}),$$

$$P_{3/4}(t) < F(t) \text{ for all } t \geq \sqrt{2} \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{(P_{3/4}(t) - F(t)) \ln^3 t}{\ln \ln(t)} = -\frac{1}{16}.$$

Proof of Theorem 3. The inequality

$$P_{3/2}(t) < \ln(1 + 6t\sqrt{\ln(1+t)})$$

gives a proof for $t \geq \sqrt{2}$. The concavity of the function

$$\tau \mapsto \ln(1 + 6\sqrt{\tau}\sqrt{\ln(1+\sqrt{\tau})})$$

completes the proof for $t < \sqrt{2}$. $\square$

We remark that $P_{3/4}$ is just a truncation of the general expansion

$$P(t) = \frac{3}{2} + \ln t + \frac{1}{2} \ln \left(c_1 + \ln t + \frac{1}{2} \ln \left(c_2 + \ln t + \frac{1}{2} \ln \left(c_3 + \ln t + \frac{1}{2} \ln \{ \dots \} \right) \right) \right).$$

Indeed, making a change of variables $x = \frac{1/2}{y+1/2}$, $x \in [0, 1]$ and using the identity

$$2x(c_i + \ln t + \frac{1}{2} \ln \{***\}) = 1 + (2c_i - 1)x + x \ln(2x\{***\}),$$

we obtain

$$2(P(t) - F(t)) = \ln \left(1 + a_1 x + x \ln \left(1 + a_2 x + x \ln \left(1 + a_3 x + x \ln \{ \dots \} \right) \right) \right) - x,$$

where $a_i = 2c_i - 1$. We then can find a_i 's uniquely from the condition that this difference is zero.

4. ARBITRARY DOMAIN AND WHOLE SPACE CASES

In this section we extend Theorem 1 to the case of an arbitrary bounded domain and to the case of the whole space $\mathbb{R}^2$.

Theorem 6. *Let R be a positive real number. Let $\Omega \subset \mathbb{R}^2$ be any domain with area $A = \pi R^2$. Consider $\alpha \in (0, 1]$ and $u \in W_0^{1,2}(\Omega)$. Then*

$$(19) \quad \|u\|_{L^\infty}^2 \leq \frac{1}{2\pi\alpha} \|\nabla u\|_{L^2}^2 F(\sqrt{2\pi\alpha} R^\alpha \frac{\|u\|_{\dot{C}^\alpha}}{\|\nabla u\|_{L^2}}),$$

where the function $F: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is defined by (3).

Proof. Applying the Pólya-Szegő symmetrization principle as in the proof of Theorem 1, we see that it is sufficient to prove the theorem for the case $\Omega = B_R$, the disk of the radius R in $\mathbb{R}^2$. Now (19) follows from Theorem 1 by a standard scaling argument. $\square$

Theorem 7. *Let $\alpha \in (0, 1]$ and let $u \in H^1(\mathbb{R}^2) \cap C^\alpha(\mathbb{R}^2)$. Then for any $\varepsilon > 0$ we have*

$$(20) \quad \|u\|_{L^\infty}^2 \leq \frac{1}{2\pi\alpha} (\|\nabla u\|_{L^2}^2 + \varepsilon^2 \|u\|_{L^2}^2) F\left(\sqrt{2\pi\alpha} \frac{(2/\varepsilon)^\alpha \|u\|_{\dot{C}^\alpha} + 2^\alpha \|u\|_{L^\infty}}{\sqrt{\|\nabla u\|_{L^2}^2 + \varepsilon^2 \|u\|_{L^2}^2}}\right),$$

where the function $F: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is defined by (3).

Proof. Without loss of generality, we can assume that $\|u\|_{L^\infty} = u(0)$. Let Ω be a domain. For any smooth function $\varphi \in C_0^\infty(\Omega)$ we have

$$\|\nabla(u\varphi)\|_{L^2(\Omega)}^2 = \int_\Omega \varphi^2 (\nabla u)^2 dx - \int_\Omega u^2 \varphi \Delta \varphi dx.$$

Choose $\varphi \in C_0^\infty(B_2)$ such that $\varphi(0) = 1$, $0 \leq \varphi \leq 1$, and $-\varphi \Delta \varphi \leq 1$. Let $\varphi_\varepsilon(x) = \varphi(\varepsilon x)$. We obtain

$$\|\nabla(u\varphi_\varepsilon)\|_{L^2}^2 \leq \|\nabla u\|_{L^2}^2 + \varepsilon^2 \|u\|_{L^2}^2, \quad \text{supp } u\varphi_\varepsilon \subset B_{2/\varepsilon}.$$

We can assume in addition that $\sup |\nabla \varphi| \leq 1$. For example we choose $\varphi(x) = 1 - |x|^2/4$. The bound on the gradient of φ and the condition $0 \leq \varphi \leq 1$ imply the following:

$$\|\varphi\|_{\dot{C}^\alpha} \leq 1 \quad \text{for each } \alpha \in (0, 1].$$

Therefore $\|\varphi_\varepsilon\|_{\dot{C}^\alpha} \leq \varepsilon^\alpha$. Using inequality $\|u\varphi_\varepsilon\|_{\dot{C}^\alpha} \leq \|u\|_{\dot{C}^\alpha} \|\varphi_\varepsilon\|_{L^\infty} + \|\varphi_\varepsilon\|_{\dot{C}^\alpha} \|u\|_{L^\infty}$, we obtain

$$\|u\varphi_\varepsilon\|_{\dot{C}^\alpha} \leq \|u\|_{\dot{C}^\alpha} + \varepsilon^\alpha \|u\|_{L^\infty}.$$

Using Theorem 6 with $R = 2/\varepsilon$ and the monotonicity of the map $(z, D) \mapsto z^2 F(\sqrt{2\pi\alpha} \frac{D}{z})$, we arrive at the statement of the theorem. $\square$

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