VARIATIONS ON A QUESTION OF LARSEN AND LUNTS

JULIEN SEBAG

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ABSTRACT. Let k be a field of characteristic zero. Let X and X' be two k -schemes of finite type having the same class in the Grothendieck ring of varieties. Is it true that X and X' are piecewise isomorphic? This question, originally asked by Larsen and Lunts, and some of its consequences constitute the subject of this article.

1. INTRODUCTION

Assume that k is a field. Let us define two notions which are central in the present article:

a) Let X, X' be two k -schemes of finite type. We say that X and X' are *piecewise isomorphic* if there exist a finite partition of X into locally closed parts $(X_i)_{i \in I}$, a finite partition of X' into locally closed parts $(X'_i)_{i \in I}$, and a bijection $\sigma : I \rightarrow I$ such that, for each $i \in I$, $(X_i)_{\text{red}}$ is isomorphic to $(X'_{\sigma(i)})_{\text{red}}$ (as k -schemes).

b) Let $\mathbb{Z}[Var_k]$ be the free abelian group generated by the isomorphism classes of all the k -schemes of finite type. We denote by $\{X\}$ the class in $\mathbb{Z}[Var_k]$ of X . Let $K_0(Var_k)$ be the quotient of $\mathbb{Z}[Var_k]$ by the subgroup generated by the elements of the form $\{X\} - \{Y\} - \{X \setminus Y\}$, for all the k -schemes of finite type X and all the closed subschemes Y of X . In this group, we denote by $[X]$ the class of X . The fibered product over k endows $K_0(Var_k)$ with a ring structure, by setting $[X \times_k X'] = [X] \cdot [X']$, for all the k -schemes of finite type X and X' . For this structure, the neutral element is $1 = [\text{Spec}(k)]$. This ring is called the *Grothendieck ring of varieties*. We denote by $\mathbb{L} := [\mathbb{A}_k^1]$ the class in $K_0(Var_k)$ of the affine line $\mathbb{A}_k^1$ over k .

Such an object, and more generally such notions, is an important piece of the theory of motivic integration (see, for example, [8]). Thus a good understanding of the Grothendieck ring of varieties, which is naturally and closely related to cohomological theories and motives, seems to us to be an interesting and important question of algebraic geometry. Yet little information is known about it. In [5], Larsen and Lunts ask the following question. Up to now, this question remains open in the general case (see Remark 3.6).

Question LL. *Let k be a field of characteristic zero. Let X and X' be two k -schemes of finite type such that $[X] = [X']$ in $K_0(Var_k)$. Is it true that X and X' are piecewise isomorphic?*

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Question LL (and so the geometric understanding of the Grothendieck ring of varieties) is the subject of the present article. The heart of this article is contained in §4. In this section, we motivate such a question by setting, and studying, two new challenging problems (see Questions R and Cr), linked to question LL. More precisely, if question LL admits a positive answer, our questions R and Cr admit a positive answer too. Furthermore, we give partial (*i.e.*, in dimensions 1 or 2), but unconditional, solutions to each of these two problems (see Propositions 4.2 and 4.6). In §3, we complete, for k -varieties of dimension one or two and in the non-“smooth and projective” context (see Theorem 3.3), the previous answers to question LL, which we obtained in [7]. The results of the present article use the birational cancellation techniques, we introduced in [7], and we recall in §2 (see Theorem 2.3). An important class of varieties still gets away from our strategy of cancellation: *the uniruled k -varieties*. In our mind, giving an answer to question LL for uniruled varieties is a deep and difficult problem. To justify this last point of view, we exhibit an original link between question LL and that of determining if $\mathbb{L}$ is (or is not) a zero divisor in $K_0(\text{Var}_k)$. This last problem has remained, up to now, totally open, but is considered to be important in the “world” of motivic integration. We show that a positive answer to question LL has an impact on the description of the kernel of the multiplication by $\mathbb{L}$ (see Proposition 4.9). Besides, we prove that “curves” are not in the kernel of this multiplication if and only if the (uni)ruled surfaces verify the condition of question LL (see Theorem 3.3 and Remark 4.11).

Finally, we remark that all the results in §3 and §4 are original, and those of §3 cannot be deduced from our previous ones (see Remark 3.6). All these results underline the fact that the question of the existence of rational curves on varieties, a main question in birational geometry, also seems crucial in order to understand the Grothendieck ring of varieties.

2. NOTATION AND TECHNICAL LEMMAS

Notation. a) In this article, k is a field of characteristic zero. A k -variety X denotes a reduced k -scheme of finite type and separated. We say that two k -schemes X and X' of finite type are *stably birational* if there exist two integers $m, m' \in \mathbb{N}$ such that $X \times_k \mathbb{P}_k^m$ is birational to $X' \times_k \mathbb{P}_k^{m'}$. An integral k -variety X of dimension d is *uniruled* if there exists an integral k -variety Y of dimension $d - 1$ and a dominant rational map $Y \times_k \mathbb{P}_k^1 \dashrightarrow X$. A k -curve (surface) is a k -variety of dimension 1 (dimension 2).

b) Let k be an algebraically closed field of characteristic zero. Recall that if X is a smooth proper k -variety, its *Poincaré polynomial* is defined by $P_X(u) = \sum_{n=0}^{2 \dim X} \dim_k H_{\text{dR}}^n(X) u^n \in \mathbb{Z}[u]$. Let us set $b_n(X) := \dim_k H_{\text{dR}}^n(X)$. One knows (see, for example, [2]) that the assignment of Poincaré polynomials induces a ring morphism $K_0(\text{Var}_k) \rightarrow \mathbb{Z}[u]$, sending a smooth proper k -variety X to its Poincaré polynomial $P_X(u)$.

c) Let $\mathbf{a} \in K_0(\text{Var}_k)$. There exists a finite number of connected smooth projective k -varieties X_i , $i \in I$, and Y_j , $j \in J$, such that $\mathbf{a} = \sum_{i \in I} [X_i] - \sum_{j \in J} [Y_j]$. We call the *dimension of $\mathbf{a}$* the smallest integer n such that all the X_i and all the Y_j are of dimension less than or equal to n .

Technical lemmas. One can show:

Lemma 2.1. *Let k be a field of characteristic zero. Let X, X' be two k -varieties. Assume that one of the following properties is true:*

- a) X is piecewise isomorphic to X' ;
- b) k is algebraically closed and $[X] = [X']$ in $K_0(\text{Var}_k)$.

Then X and X' have the same dimension and the same number of irreducible components of maximal dimension.

Proof. Remark that the dimension of an irreducible k -variety S is also the transcendence degree (over k) of its functions field. The assertion for a) then follows directly from the definition of a piecewise isomorphism. For b), conclude by [7, Corollary 5]. $\square$

We denote by $\mathbf{Z}[\mathbf{sb}]$ the free abelian group generated by the stable birational equivalence classes of connected smooth projective k -varieties.

Theorem 2.2 (Larsen-Lunts, [5]; see also [1]). *Let k be a field of characteristic zero. There exists a unique group morphism $SB : K_0(\text{Var}_k) \rightarrow \mathbf{Z}[\mathbf{sb}]$, sending the class of a connected smooth proper k -variety to its stable birational equivalence class. The kernel of SB is the ideal generated by $\mathbb{L}$.*

Theorem 2.3 (Liu-Sebag, [7]). *Let k be a field of characteristic zero. Let X, X' be integral k -varieties of the same dimension. Assume that X and X' are stably birational and that X is non-uniruled. Then X' is non-uniruled. Furthermore, X and X' are birational.*

As an application of the two previous theorems, we have shown in [7, Proposition 5, Proposition 6]:

Theorem 2.4 (Liu-Sebag, [7]). a) *Let k be a field of characteristic zero. Let X, X' be zero-dimensional k -varieties such that $[X] = [X']$ in $K_0(\text{Var}_k)$. Then X is isomorphic to X' .* b) *Let k be an algebraically closed field of characteristic zero. Let X and X' be k -varieties such that $\dim X \leq 1$ and $[X] = [X']$ in $K_0(\text{Var}_k)$. Then X is piecewise isomorphic to X' .*

The following remark will be used throughout the article. We recall here some elementary results for the convenience of the reader.

Remark 2.5. a) In Lemma 2.1, under assumption a), we also see that the irreducible components of maximal dimension of X and those of X' are pairwise birational. Indeed, by the definition of a piecewise isomorphism, there is a bijection σ between the set of generic points of maximal dimension of X and that of X' , and if $(\eta_X, \sigma(\eta_X))$ is a pair of generic points, then $\kappa(\eta_X) \simeq \kappa(\sigma(\eta_X))$ (over k).

b) By definition, constructing a piecewise isomorphism can be realized by a finite number of steps. Let X and X' be two k -varieties of the same dimension d . *Step 1:* Construct a k -isomorphism $U \simeq U'$ on open sets of X and X' containing all the points of dimension d . *Step 2:* Apply Step 1 to the k -varieties $(X \setminus U)_{\text{red}}$ and $(X' \setminus U')_{\text{red}}$, and so on. Each time, the dimension is decreased by 1. Then, after exactly d steps, we have constructed the required partitions of X and X' .

c) We can complete Theorem 2.3 by the following statements (see [7, Lemma 9]), in small dimensions: if k is an algebraically closed field of characteristic zero and if X, X' are k -curves (surfaces) that are stably birational, then X is birational to

X' . Such a result uses crucially the so-called Lüroth problem (*i.e.*, Castelnuovo's criterion in dimension 2).

3. A NEW ANSWER TO LARSEN-LUNTS' QUESTION

In this section, we complete, for k -varieties of dimension one or two, the answer to question LL, which we have given in [7] (see Remark 3.6). In particular, if k is an algebraically closed field of characteristic zero, we link question LL for (uni)ruled surfaces to a condition on the nature of $\mathbb{L}$ in $K_0(\text{Var}_k)$.

Lemma 3.1. *Let k be a field of characteristic zero. Let X be a k -variety of dimension d and let $\overline{X}$ be a smooth projective model of X . Then there exists a finite number of connected smooth projective k -varieties C_i , $i \in I$, of dimension at most $d - 1$, such that*

$$[\overline{X}] - [X] = \sum_{i \in I} \varepsilon_i [C_i],$$

with $\varepsilon_i \in \{-1, 1\}$ for each $i \in I$.

Proof. The proof comes directly from the relations in $K_0(\text{Var}_k)$ and from Hironaka's theorem on the existence of resolutions of singularities. $\square$

Proposition 3.2. *Let k be a field of characteristic zero and let $n \geq 1$ be an integer. Let X_i , X'_i , $1 \leq i \leq n$, be irreducible k -varieties of dimension d . Assume that there exists a finite number of k -varieties V_s , $s \in S$, and W_t , $t \in T$, of dimension at most $d - 1$, such that*

$$(*) \quad \sum_{1 \leq i \leq n} [X_i] - \sum_{1 \leq i \leq n} [X'_i] = \sum_{s \in S} [V_s] - \sum_{t \in T} [W_t]$$

in $K_0(\text{Var}_k)$. Assume that one of the following properties holds:

- a) for each i , $1 \leq i \leq n$, X_i is non-uniruled;
- b) k is algebraically closed, $d = 2$, and, for each i , $1 \leq i \leq n$, X_i is either non-(uni)ruled or rational;
- c) k is algebraically closed, $d = 2$, and the kernel of the multiplication by $\mathbb{L}$ in $K_0(\text{Var}_k)$ does not contain any (non-trivial) element of dimension 1.

Then there exists a bijection $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ such that X_i and $X'_{\sigma(i)}$ are birational, for each i , $1 \leq i \leq n$. In particular, under the same assumption, if $\sum_{1 \leq i \leq n} [X_i] = \sum_{1 \leq i \leq n} [X'_i]$ in $K_0(\text{Var}_k)$, then X_i is birational to $X'_{\sigma(i)}$, for each i , $1 \leq i \leq n$.

Proof. Thanks to Lemma 3.1, one sees that, at the same time, we can replace each k -variety of $(*)$ by a smooth projective model and preserve a relation analogous to $(*)$. Thus, for proving the announced birationality property, we can assume that all the X_i , all the X'_i , all the V_s , and all the W_t are connected smooth and projective. Now apply SB to the (new) relation $(*)$. Since $\mathbb{Z}[\mathfrak{sb}]$ is a free abelian group, one can remark that $|S| = |T|$. Let i be such that $1 \leq i \leq n$. Two cases occur:

- i) There exists some $s \in S$, such that $SB(X_i) = SB(V_s)$. In this case, X_i is uniruled, by Theorem 2.3.
- ii) There exists some j , $1 \leq j \leq n$, such that $SB(X_i) = SB(X'_j)$.

In the rest of the proof, we are going to study more precisely these two cases with respect to the different assumptions, in order to conclude, up to renumbering, that X_i is birational to X'_i , for each i , $1 \leq i \leq n$.

a) Since X_i is non-uniruled, $SB(X_i)$ cannot be equal to $SB(V_s)$, for any $s \in S$. So X_i is stably birational to X'_j , for some j , $1 \leq j \leq n$, by the definition of SB . Since the number of the X_i is equal to that of the X'_j , this correspondence induces a bijection and, up to renumbering, each X_i is stably birational to X'_i . By Theorem 2.3 and by using again the non-uniruledness of X_i , we conclude that X_i is birational to X'_i , for each i , $1 \leq i \leq n$.

b) By assumption, we can suppose that there exists an integer $0 \leq n' \leq n$ such that, up to renumbering, X_i is non-uniruled for each i , $1 \leq i \leq n'$, and that X_i is rational for each i , $1 + n' \leq i \leq n$. Applying the arguments used in a), we conclude that up to renumbering, X_i is birational to X'_i , for each i , $1 \leq i \leq n'$. Thus there exists an open set U_i of X_i , and a dense open set U'_i of X'_i such that $[U_i] = [U'_i]$ in $K_0(Var_k)$. It follows that $[X_i] - [X'_i] = [(X_i \setminus U_i)_{\text{red}}] - [(X'_i \setminus U'_i)_{\text{red}}]$. By substituting these equalities in (*), the relation becomes:

$$(**) \quad \sum_{1 \leq i \leq n'} [X_i] - \sum_{1 \leq i \leq n'} [X'_i] = \sum_{s \in S'} [V'_s] - \sum_{t \in T'} [W'_t],$$

where all the X_i are rational, all the X'_i are uniruled, and all the V'_s and all the W'_t are connected smooth and projective. Assume that there exists i , $1 \leq i \leq n'$, such that $SB(X_i) = SB(X'_j)$, for some j , $1 \leq j \leq n'$. We then conclude by Remark 2.5c) (see also [7, Lemma 9]) that X_i is birational to X'_j . Up to renumbering, and by using the arguments above, we obtain a new relation:

$$(***) \quad \sum_{1 \leq i \leq n''} [X_i] - \sum_{1 \leq i \leq n''} [X'_i] = \sum_{s \in S''} [V''_s] - \sum_{t \in T''} [W''_t].$$

This time, all the X_i are rational, all the X'_i are uniruled, all the V''_s and all the W''_t are connected smooth and projective, of dimension less than or equal to 1, and, for each j , $1 \leq j \leq n''$, $SB(X'_j) \neq SB(X_i) = 1 = SB(\mathbb{P}_k^2)$, for any i , $1 \leq i \leq n''$. By Remark 2.5c), we can assume that $S'' = T'' = \{1, \dots, n\}$. Let us fix j , $1 \leq j \leq n''$. Let us set $X' := X_j$. There exists $t \in T''$ such that $SB(X') = SB(W''_t)$. Let us set $W := W''_t$. By definition, X' is stably birational to W , which is a *non-rational* connected smooth projective curve. By Remark 2.5c), we conclude that X' is birational to $W \times_k \mathbb{P}_k^1$. By eliminating indeterminacies from a birational map $X' \dashrightarrow W \times_k \mathbb{P}_k^1$, we remark that there exists an integer $\gamma \in \mathbb{Z}$ such that

$$[X'] = [W] + \mathbb{L}([W] + \gamma).$$

By applying these arguments to each j , $1 \leq j \leq n''$, by substituting the corresponding equalities in (***), we find integers $\gamma_j \in \mathbb{Z}$, $1 \leq j \leq n''$, such that the following relation holds in $K_0(Var_k)$:

$$\sum_{1 \leq i \leq n''} [X_i] = \mathbb{L} \sum_{1 \leq j \leq n''} [W''_j] + \sum_{1 \leq s \leq n''} [V''_s] + \mathbb{L} \sum_{1 \leq j \leq n''} \gamma_j.$$

Since $P_{\mathbb{A}_k^1}(u) = u^2$, by comparing the terms of degree 3 of the Poincaré polynomials of the right hand side and the left hand side, we find that $2 \sum_{1 \leq j \leq n''} g(W''_j) = 0$. Thus, for each j , $1 \leq j \leq n''$, W''_j is a rational k -curve. This is a contradiction. It follows that X'_i is rational, for each i , $1 \leq i \leq n'$. *Conclusion:* In any case, up to renumbering, X_i is birational to X'_i , for all the i , and X is birational to X' .

c) We are going to follow the strategy of the previous cases. By the arguments used in a), we can assume that all the X_i and all the X'_i are uniruled. By Remark 2.5c), we can assume that, for each i , $1 \leq i \leq n$, there exists an index $s \in S$

such that $SB(X_i) = SB(V_s)$; that for each j , $1 \leq j \leq n$, there exists an index $t \in T$ such that $SB(X'_j) = SB(W_t)$; and that $S = T = \{1, \dots, n\}$. At last, up to renumbering, we can assume also that $SB(X_i) = SB(V_i)$ and $SB(X'_i) = SB(W_i)$, for each i , $1 \leq i \leq n$. As in b), the relation $(*)$ becomes

$$\mathbb{L}(\sum_{1 \leq i \leq n} [V_i] - \sum_{1 \leq i \leq n} [W_i]) = \mathbb{L}\gamma.$$

Computing the Poincaré polynomials in this relation and by comparing, this time, the terms of degree 2, we conclude that $\gamma = 0$. So, by assumption,

$$\sum_{1 \leq i \leq n} [V_i] = \sum_{1 \leq i \leq n} [W_i].$$

By Theorem 2.4b), it follows that, up to renumbering, V_i is k -isomorphic to W_i . Thus, for each i , $1 \leq i \leq n$, X_i is birational to X'_i . $\square$

Theorem 3.3. *Let k be a field of characteristic zero. Let X, X' be two k -varieties such that $[X] = [X']$ in $K_0(\text{Var}_k)$.*

a) *Let X, X' be geometrically irreducible k -curves. Then X and X' are piecewise isomorphic.*

b) *Assume that k is algebraically closed and that the dimension of X is 2.*

i) *If all the irreducible components of X of dimension 2 are either non-uniruled or rational, then X and X' are piecewise isomorphic.*

ii) *If the kernel of the multiplication by $\mathbb{L}$ in $K_0(\text{Var}_k)$ does not contain any (non-trivial) element of dimension 1, then X and X' are piecewise isomorphic.*

Proof. By Remark 2.5b), in each case, we have only to prove the birationality of X and X' . We will then conclude by induction, using Theorem 2.3a) for proving a), and using Theorem 2.3b) for proving b).

a) Let us show that X is birational to X' . Let $\overline{X}$ and $\overline{X}'$ be smooth projective models of X and X' respectively. There exists a finite number of finite field extensions k_i , $i \in I$, and k'_j , $j \in J$, of k such that

$$[\overline{X}] - [\overline{X}'] = \sum_{i \in I} [x_i] - \sum_{j \in J} [x'_j],$$

where $x_i := \text{Spec}(k_i)$ and $x'_j := \text{Spec}(k'_j)$, for all i, j . Two cases occur. If $\overline{X}$ is non-uniruled, then, by Proposition 3.2a), we conclude that $\overline{X}$ is k -isomorphic to $\overline{X}'$. Thus X is birational to X' . If $SB(\overline{X}) \neq SB(\overline{X}')$, then there exist $i \in I$ and $j \in J$ such that $SB(\overline{X}) = SB(x_i)$ and $SB(\overline{X}') = SB(x'_j)$. As $\overline{X}$ and $\overline{X}'$ are geometrically irreducible, then $\overline{X} \simeq \mathbb{P}_k^1 \simeq \overline{X}'$. So X and X' are birational. Lastly, assume that $\overline{X}$ is uniruled and that $SB(\overline{X}) = SB(\overline{X}')$. Then, $|I| = |J|$ and, up to renumbering, $SB(x_i) = SB(x'_i)$, for each $i \in I$. It follows that $k_i \simeq k'_i$ and $[x_i] = [x'_i]$ for each $i \in I$. Thus $[\overline{X}] = [\overline{X}']$ in $K_0(\text{Var}_k)$. If $\overline{X} \simeq \mathbb{P}_k^1$, it follows, by Lüroth's problem, that $\overline{X}' \simeq \mathbb{P}_k^1$. If $\overline{X}$ has no rational point, we conclude by [3, Theorem 1.2].

b) Let X_i , $1 \leq i \leq n$, and X'_i , $1 \leq i \leq n'$, be the irreducible components of dimension 2 of X and X' respectively. By Lemma 2.1b), $n = n'$. Let us show that X and X' are birational. There exists a finite number of k -varieties V_s , $s \in S$, and W_t , $t \in T$, of dimension less than or equal to 1 such that $\sum_{1 \leq i \leq n} [X_i] - \sum_{1 \leq i \leq n} [X'_i] = \sum_{s \in S} [V_s] - \sum_{t \in T} [W_t]$. For i), we conclude by Proposition 3.2b)

that, up to renumbering, X_i is birational to X'_i , for each i . So, X is birational to X' . For *ii*), we conclude by Proposition 3.2c). $\square$

Remark 3.4. By the arguments developed in the proof of Theorem 3.3 above, we see that Larsen and Lunts' question can also be formulated in terms of the study of the graded ring associated to the filtration by the dimensions in $K_0(\text{Var}_k)$.

Corollary 3.5. *Let k be an algebraically closed field of characteristic zero. Let X, X' be two k -surfaces such that $[X] = [X']$. Assume that $\mathbb{L}$ is not a zero divisor in $K_0(\text{Var}_k)$. Then X is piecewise isomorphic to X' .*

Remark 3.6. Up to now, the only progress on question LL was contained in [7, §5], where we have proved that this question admits a positive answer in the following cases:

- a) if k is an arbitrary field of characteristic zero, for zero-dimensional k -varieties (see [7, §5.1]), or for k -varieties X such that $X_{\bar{k}}$ does not contain any irreducible rational curve (here $\bar{k}$ is an algebraic closure of k ; see [7, §5.4]);
- b) if k is an algebraically closed field of characteristic zero, for curves, or for smooth and projective surfaces (see [7, §§5.2, 5.3]), or for k -varieties containing only finitely many rational curves (see [7, §5.4]).

In particular, observe that:

- a') for a (Severi-Brauer) conic C , $C_{\bar{k}}$ contains (trivially) a rational $\bar{k}$ -curve;
- b') one knows that there exist non-(uni)ruled smooth projective surfaces with infinitely many rational curves. In particular, there exist open subvarieties of such surfaces having the same property.

For these reasons, Theorem 3.3 cannot be deduced from results of [7, §5].

4. GEOMETRIC PROBLEMS IN THE CIRCLE OF LARSEN AND LUNTS' QUESTION

In this section, we set two new challenging problems, linked to question LL. We also see the consequence of a positive answer to question LL on the (more classical) piecewise cancellation problem.

4.1. Larsen and Lunts' question and rational points on varieties. We ask the following question:

Question R. *Let k be a field of characteristic zero and let $m \in \mathbb{N} \setminus \{0\}$. Let X be a k -scheme of finite type. Are the following assertions equivalent?*

- i) X has m rational points;*
- ii) there exists a k -scheme of finite type Y such that $[X] = [Y] + m$ in $K_0(\text{Var}_k)$.*

In dimension zero, this is true by Theorem 2.4a). Observe also that *i*) $\Rightarrow$ *ii*) (in any characteristic).

Lemma 4.1. *Let k be a field of characteristic zero. Let X be a connected smooth proper k -variety. Then X has a rational point if and only if there exists a connected smooth proper k -variety $\overline{Y}$ and a smooth divisor D on $\overline{Y}$ such that $Y := \overline{Y} \setminus D$ satisfies the relation $[X] = 1 + [Y]$ in $K_0(\text{Var}_k)$.*

Proof. We have only to prove $\Leftarrow$. More generally, we are going to prove that if there exists an integer $m \geq 1$ such that $[X] = m + [Y]$ (with Y as in the statement), then X has a rational point. Assume that $D := \sum_{i \in I} D_i$. By assumption,

$$[X] + \sum_{i \in I} [D_i] = m + [\bar{Y}].$$

Applying SB to this relation, we see that $|I| = m$ (in particular, D is irreducible if and only if $m = 1$). If $SB(X) = 1 = SB(\mathbb{P}_k^1)$, then X has a rational point. If $SB(D_i) = 1$ for some i , then $SB(X) = SB(\bar{Y})$ and D_i has a rational point. It follows that $\bar{Y}$ also has a rational point. By definition of SB , there exists an integer s such that $X \times_k \mathbb{P}_k^s$ is birational to $\bar{Y} \times_k \mathbb{P}_k^s$. As $X \times_k \mathbb{P}_k^s$ and $\bar{Y} \times_k \mathbb{P}_k^s$ are smooth and proper, by the Lang–Nishimura Lemma, the variety $X \times_k \mathbb{P}_k^s$ has a rational point. Thus X has a rational point too. $\square$

Proposition 4.2. *Let k be a field of characteristic zero and let $m \geq 1$ be an integer. Let X be a geometrically irreducible smooth k -curve. Then X has m rational points if and only if there exists a geometrically irreducible smooth k -curve Y such that $[X] = m + [Y]$ in $K_0(Var_k)$.*

Remark 4.3. Proposition 4.2 implies easily the following consequence: if X' is another geometrically irreducible smooth k -curve such that $[X] = [X']$ in $K_0(Var_k)$, then X has m rational points (respectively m effective 0-cycles of degree d) if and only if X' has m rational points (respectively m effective 0-cycles of degree d).

Proof. We only have to show $\Leftarrow$. We are going to show it by induction on m .

a) Assume first that $m = 1$. Let $\bar{X}$ (respectively $\bar{Y}$) be a smooth projective model of X (respectively Y). There exists a finite number of finite field extensions k_i , $i \in I$, and k'_j , $j \in J$, of k such that

$$[\bar{X}] - \sum_{i \in I} [x_i] = 1 + [\bar{Y}] - \sum_{j \in J} [x'_j],$$

with $x_i := \text{Spec}(k_i)$, for $i \in I$, and $x'_j := \text{Spec}(k'_j)$, for $j \in J$. Applying SB to this relation, remark first that $|J| = |I| + 1$. Besides, two cases occur. In the first one, $SB(\bar{X}) = SB(\mathbb{P}_k^1) = 1$ or $SB(\bar{X}) = SB(x_i)$, for some $i \in I$. Since $\bar{X}$ is geometrically irreducible, it follows that $\bar{X} \simeq \mathbb{P}_k^1$, and X has a rational point. In the second one, $\bar{X}, \bar{Y}$ are stably birational, and $SB(x'_j) = SB(\text{Spec}(k)) = 1$, for some $j \in J$. So $x'_j \in \bar{Y}(k)$ and $\bar{X}$ has a rational point, say $x \in \bar{X}(k)$. If $x \notin X(k)$, necessarily $x = x_s$, for some $s \in I$, and we can write:

$$[X] = [\bar{X}] - 1 - \sum_{i \in I, x_i \neq x} [x_i] = 1 + [Y] = [\bar{Y}] - \sum_{\ell \in J \setminus \{j\}} [x'_\ell].$$

We conclude by induction on $|I|$, using Lemma 4.1 for the initialization, when $|I| = 0$.

b) Assume that $m \geq 1$. If there exists a smooth geometrically irreducible k -curve such that $[X] = (1 + m) + [Y]$, then, by the arguments that we have developed in a) (use the proof of Lemma 4.1 for the initialization), we conclude that X has a rational point, say $x \in X(k)$. Since $X \setminus \{x\}$ is also a geometrically irreducible smooth k -curve such that $[X \setminus \{x\}] = m + [Y]$ in $K_0(Var_k)$, we conclude, by the induction hypothesis, that $X \setminus \{x\}$ has m rational points. So X has $m + 1$ rational points. The result follows. $\square$

4.2. Larsen and Lunts' question and a piecewise analog of the "complement problem". Let $N \geq 1$ be an integer. Let $f, g \in \mathbb{C}[\underline{x}] := \mathbb{C}[x_1, \dots, x_N]$ be two *irreducible* polynomials. Assume that there exists a $\mathbb{C}$ -isomorphism $\mathbb{C}[\underline{x}]_f \rightarrow \mathbb{C}[\underline{x}]_g$. The *complement problem* can be expressed by the following question: *Is it true that the hypersurfaces $V(f)$ and $V(g)$ of $\mathbb{A}_{\mathbb{C}}^N$ are $\mathbb{C}$ -isomorphic?* One knows no complete answer to this classical question of affine geometry, even in dimension one (see [4]). We introduce the following piecewise analog.

Remark 4.4. In a sense, the piecewise analog below is more natural. When $N = 2$, if f, g are not assumed to be irreducible, it is easy to construct counterexamples to the complement problem. Yet, in this situation, our piecewise analog is true.

Question Cr. Assume that k is a field of characteristic zero. Let N be an integer. Assume that $\varphi : \mathbb{P}_k^N \dashrightarrow \mathbb{P}_k^N$ is a birational map. Can φ be extended in a piecewise isomorphism?

Let $N \geq 2$ be an integer and let $\varphi : \mathbb{P}_k^N \dashrightarrow \mathbb{P}_k^N$ be a birational map. There exist open sets U, V of $\mathbb{P}_k^N$ such that φ induces a k -isomorphism $U \simeq V$. Below, we are interested in extending the (fixed) k -isomorphism $\varphi : U \rightarrow V$ in a piecewise automorphism of $\mathbb{P}_k^N$.

Lemma 4.5. *Let k be a field of characteristic zero and let $n \geq 1$ be an integer. With the notation above, assume that $\mathbb{P}_k^N \setminus U$ is of dimension $r \in \mathbb{N}$ and that all the irreducible components X_i of dimension r , $1 \leq i \leq n$, of $(\mathbb{P}_k^N \setminus U)_{\text{red}}$ are non-uniruled. Let X'_i , $1 \leq i \leq n$, be the reduced irreducible components of dimension r of $(\mathbb{P}_k^N \setminus V)_{\text{red}}$. Then there exists a bijection $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ such that, for all i , $1 \leq i \leq n$, X_i is birational to $X'_{\sigma(i)}$.*

Proof. Remark that $[(\mathbb{P}_k^N \setminus U)_{\text{red}}] = [\mathbb{P}_k^N \setminus U] = [\mathbb{P}_k^N] - [U] = [\mathbb{P}_k^N \setminus V]$ in $K_0(\text{Var}_k)$. Thus $(\mathbb{P}_k^N \setminus U)_{\text{red}}$ and $(\mathbb{P}_k^N \setminus V)_{\text{red}}$ have the same dimension r and the same number n of irreducible components of dimension r . There exists a finite number of k -varieties V_s , $s \in S$, and W_t , $t \in T$, of dimension less than and or equal to $r - 1$ such that

$$\sum_{1 \leq i \leq n} [X_i] - \sum_{1 \leq i \leq n} [X'_i] = \sum_{s \in S} [V_s] - \sum_{t \in T} [W_t],$$

in $K_0(\text{Var}_k)$. We conclude by Proposition 3.2a). $\square$

Proposition 4.6. *Let k be an algebraically closed field of characteristic zero. Let $N \geq 1$ be an integer and let $\varphi : \mathbb{P}_k^N \dashrightarrow \mathbb{P}_k^N$ be a birational map.*

- a) *Assume that $N \leq 2$. Then φ can be extended to a piecewise isomorphism.*
- b) *Assume that $N \geq 3$. With the notation above, if $\mathbb{P}_k^N \setminus U$ is of dimension $r \leq 1$ or if $r = 2$ and if the reduced irreducible components of dimension 2 of $(\mathbb{P}_k^N \setminus U)_{\text{red}}$ are either non-uniruled or rational, then φ can be extended in a piecewise isomorphism.*

Proof. For a), this comes directly from Theorem 2.4b); for b), it comes directly from Theorem 3.3b)i). $\square$

4.3. Larsen and Lunts' question and a piecewise analog of Zariski's cancellation problem. If X, X' are k -varieties such that $X \times_k \mathbb{A}_k^1 \simeq X' \times_k \mathbb{A}_k^1$, is it true that $X \simeq X'$ (over k)? This is the so-called *Zariski cancellation problem* (which is not true in general). Motivic integration has suggested the following piecewise analog. Recall that an element a of a ring A is said to be *regular* if it is not a zero divisor in A .

Remark 4.7. The known counterexamples of Zariski's cancellation problem, such as Danielewski's varieties, which are piecewise isomorphic, give no information for solving the problem stated in question Z below. We ask a more refined question: *Is it possible to find two irreducible, not birational, k -varieties X and X' such that $X \times_k \mathbb{A}_k^1 \simeq X' \times_k \mathbb{A}_k^1$?* Up to now, this last question is open.

Question Z. Assume that k is a field of characteristic zero. Is $\mathbb{L}$ a regular element in $K_0(\text{Var}_k)$?

Lemma 4.8. Let k be a field of characteristic zero. Let X_i , $1 \leq i \leq n$, X'_j , $1 \leq j \leq m$, V_s , $s \in S$, and W_t , $t \in T$, be k -varieties of dimension d such that

$$\mathbb{L}\left(\sum_{1 \leq i \leq n} [X_i] - \sum_{1 \leq j \leq m} [X'_j]\right) = \sum_{s \in S} [V_s] - \sum_{t \in T} [W_t]$$

in $K_0(\text{Var}_k)$. Assume that question LL admits a positive answer for all the quasi-projective k -varieties of dimension less than or equal to $d+1$. Then the number of irreducible components of dimension d of all the X_i is equal to that of the irreducible components of dimension d of all the X'_j . Besides, for each i , $1 \leq i \leq n$, each irreducible component of dimension d of X_i is stably birational to an irreducible component of dimension d of some X_j , for j , $1 \leq j \leq m$.

Proof. Let us denote by $(X_{i,\alpha})_\alpha$ the irreducible components of dimension d of X_i , and by $(X'_{j,\beta})_\beta$ the irreducible components of dimension d of X'_j . There exist k -varieties V'_s , $s \in S'$, W'_t , $t \in T'$, of dimension $d-1$ such that

$$[Z := (\bigsqcup_{i,\alpha} X_{i,\alpha} \times_k \mathbb{A}_k^1) \sqcup (\bigsqcup_s V'_s)] = [(\bigsqcup_{j,\beta} X'_{j,\beta} \times_k \mathbb{A}_k^1) \sqcup (\bigsqcup_t W'_t)] =: [Z'].$$

By assumption, Z and Z' are of dimension $d+1$. The assertions come directly from Lemma 2.1 and Remark 2.5a). $\square$

Proposition 4.9. Let k be an algebraically closed field of characteristic zero and let $r \leq 3$ be an integer. Assume that question LL admits a positive answer for all the quasi-projective k -varieties of dimension less than or equal to r . Then the kernel of the multiplication by $\mathbb{L}$ in $K_0(\text{Var}_k)$ has no (non-trivial) element of dimension less than or equal to $r-1$.

Notice that this property is *unconditional* for $r=1$. Let $\mathbf{a} \in K_0(\text{Var}_k)$ be an element of dimension 0. Then there exists an integer m such that $\mathbf{a} = m$ in $K_0(\text{Var}_k)$. If $m\mathbb{L} = 0$, by computing the Poincaré polynomials in this relation, we conclude that $mu^2 = 0$ in $\mathbb{Z}[u]$. Thus $m = 0$.

Proof. Let $\mathbf{a} \in K_0(\text{Var}_k)$ of dimension $r-1$ such that $\mathbf{a}\mathbb{L} = 0$ in $K_0(\text{Var}_k)$. There exists a finite number of connected smooth projective k -varieties X_i , $i \in I$, and Y_j ,

$j \in J$, of dimension less than or equal to $r - 1$ such that $\mathbf{a} = \sum_{i \in I} [X_i] - \sum_{j \in J} [Y_j]$ and

$$(\diamond) \quad \mathbb{L} \sum_{i \in I} [X_i] = \mathbb{L} \sum_{j \in J} [Y_j].$$

Let us show that $[\bigsqcup_{i \in I} X_i] = \sum_{i \in I} [X_i] = \sum_{j \in J} [Y_j] = [\bigsqcup_{j \in J} Y_j]$. The strategy is to prove that $\bigsqcup_{i \in I} X_i$ is piecewise isomorphic to $\bigsqcup_{j \in J} X_j$. By Lemma 4.8, we see that there exists an integer $n \geq 1$ such that the family of the X_i and the family of the Y_j contain the same number n of k -varieties of maximal dimension (it can also be seen *unconditionally* by computing Poincaré polynomials). We can assume that $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$ are these k -varieties. Besides, by Lemma 4.8, up to renumbering, for each i , $1 \leq i \leq n$, $X_i \times_k \mathbb{A}_k^1$ is birational to some $Y_i \times_k \mathbb{A}_k^1$.

a) Assume that $r = 2$. By Remark 2.5c), we conclude that there exist integers $m, \ell \in \mathbb{N}$ such that

$$\bigsqcup_{i \in I} X_i \simeq \left(\bigsqcup_{i=1}^m \operatorname{Spec}(k) \right) \sqcup \left(\bigsqcup_{i=1}^{n-\ell} \mathbb{P}_k^1 \right) \sqcup \left(\bigsqcup_{i=1}^{\ell} X_i \right) \simeq \bigsqcup_{j \in J} Y_j.$$

b) Assume that $r = 3$. Again, by Remark 2.5c) (see also [7, Lemma 9]), we conclude that X_i is birational to Y_i , for each i . Thus, for each i , $1 \leq i \leq n$, there exist open sets U_i of X_i and V_i of Y_i such that $U_i \simeq V_i$ (over k). Let us set $C_i := X_i \setminus U_i$ and $D_i := Y_i \setminus V_i$. We can assume that all the C_i and all the D_i are reduced. So, the relation $(\diamond)$ takes the following form:

$$\mathbb{L}(m_1 + \sum_{i \in I'} [C'_i] + \sum_{1 \leq i \leq n} [C_i]) = \mathbb{L}(m_1 + \sum_{j \in J'} [D'_j] + \sum_{1 \leq i \leq n} [D_i]),$$

where $m_1, m_2 \in \mathbb{N}$; the C'_i , for $i \in I'$, and the D'_j , $j \in J'$ are connected smooth projective k -curves; and the C_i and the D_i , for $1 \leq i \leq n$, are projective k -varieties of dimension less than or equal to 1. Following Remark 2.5b), we have to prove that $Z := (\bigsqcup_{i \in I'} C'_i) \sqcup (\bigsqcup_{1 \leq i \leq n} C_i)$ is birational to $(\bigsqcup_{j \in J'} D'_j) \sqcup (\bigsqcup_{1 \leq i \leq n} D_i)$. The result then follows, by Lemma 4.8 and Remark 2.5c). The relation $(\diamond)$ can be reduced now to a relation with components of dimension zero. We conclude by computing Poincaré polynomials. By Remark 2.5b), we have constructed the required piecewise isomorphism. $\square$

Remark 4.10. The strategy, used in the proof of Proposition 4.9, consists in simplifying, “birationally”, by $\mathbb{A}_k^1$. So we can extend the result of Proposition 4.9 by induction to other classes of varieties (or elements of $K_0(\operatorname{Var}_k)$) as soon as we know how to simplify.

Remark 4.11. The statement for $r = 2$ and Theorem 3.3 imply that if k is an algebraically closed field of characteristic zero, the k -surfaces satisfy the condition of question LL if and only if the kernel of the multiplication by $\mathbb{L}$ in $K_0(\operatorname{Var}_k)$ does not contain any (non-trivial) element of dimension 1.

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INSTITUT DE MATHÉMATIQUES DE BORDEAUX, UNIVERSITÉ BORDEAUX 1, 351 COURS DE LA LIBÉRATION, 33405 TALENCE CEDEX, FRANCE

Current address: Institut de Formation et de Recherche Mathématiques and Institut de Recherche Mathématiques de Rennes, 263 Avenue du Général Leclerc, CS 74205, 35042 Rennes cedex, France

E-mail address: `julien.sebag@univ-rennes1.fr`