NOTE ON BETA ELEMENTS IN HOMOTOPY, AND AN APPLICATION TO THE PRIME THREE CASE

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ABSTRACT. Let $S_{(p)}^0$ denote the sphere spectrum localized at an odd prime p . Then we have the first beta element $\beta_1 \in \pi_{2p^2-2p-2}(S_{(p)}^0)$, whose cofiber we denote by W . We also consider a generalized Smith-Toda spectrum V_r characterized by $BP_*(V_r) = BP_*/(p, v_1^r)$. In this note, we show that an element of $\pi_*(V_r \wedge W)$ gives rise to a beta element of homotopy groups of spheres. As an application, we show the existence of β_{9t+3} at the prime three to complete a conjecture of Ravenel's: $\beta_s \in \pi_{16s-6}(S_{(3)}^0)$ exists if and only if s is not congruent to 4, 7 or 8 mod 9.

1. INTRODUCTION AND STATEMENTS OF RESULTS

Let BP denote the Brown-Peterson spectrum at an odd prime number p . Then, we have the Hopf algebroid BP_*BP over BP_* , where

$$BP_* = \pi_*(BP) = \mathbb{Z}_{(p)}[v_1, v_2, \dots] \quad \text{and} \quad BP_*BP = BP_*(BP) = BP_*[t_1, t_2, \dots]$$

for $v_i \in BP_{2p^i-2}$ and $t_i \in BP_{2p^i-2}BP$. This gives rise to the Adams-Novikov spectral sequence converging to homotopy groups $\pi_*(X)$ of a connective spectrum X with E_2 -term

$$E_2^{s,t}(X) = \text{Ext}_{BP_*BP}^{s,t}(BP_*, BP_*(X)).$$

We consider the sphere spectrum S^0 , the modulo p Moore spectrum M and a cofiber V_r of the map $\alpha^r : \Sigma^{rq}M \rightarrow M$ for a positive integer r and the Adams map $\alpha : \Sigma^qM \rightarrow M$, so that

$$(1.1) \quad S^0 \xrightarrow{p} S^0 \xrightarrow{i} M \xrightarrow{j} S^1 \quad \text{and} \quad \Sigma^{rq}M \xrightarrow{\alpha^r} M \xrightarrow{i_r} V_r \xrightarrow{j_r} \Sigma^{rq+1}M$$

are cofiber sequences. Here $q = 2p - 2$ as usual. Suppose that $v_1^t v_2^s \in E_2^{0, (sp+s+t)q}(V_r)$ for integers $r > 0$, $s > 0$ and $t \geq 0$. Then, we define the beta element $\beta_{s/r-t}$ in the E_2 -term by

$$(1.2) \quad \beta_{s/r-t} = \delta \delta_r(v_1^t v_2^s) \in E_2^{2, (sp+s+t-r)q}(S^0)$$

for the connecting homomorphisms $\delta_r : E_2^{s,t}(V_r) \rightarrow E_2^{s+1, t-rq}(M)$ and $\delta : E_2^{s,t}(M) \rightarrow E_2^{s+1, t}(S^0)$ associated to the cofiber sequences (1.1). We note that if $r - t = r' - t'$, then $\beta_{s/r-t} = \beta_{s/r'-t'}$, and abbreviate $\beta_{s/1}$ to β_s . In this paper, we study which

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of these elements survives to the homotopy groups $\pi_*(S^0)$ of spheres. For a prime greater than three, the following beta elements survive:

$$\begin{aligned} \beta_s & \text{ for } s > 0 \text{ by L. Smith [9],} \\ \beta_{tp/r} & \text{ for } t > 0 \text{ and } r \leq p \text{ with } (t, r) \neq (1, p) \text{ by S. Oka [4], [5], and} \\ \beta_{tp^2/r} & \text{ for } t > 0 \text{ and } r \leq 2p \text{ by S. Oka [6].} \end{aligned}$$

At the prime three, β_s survives if $s < 4$ by Toda [10], and does not if $s = 4$, and does if $s = 5$ by Oka [2]. Ravenel then conjectured that β_s survives in the spectral sequence if and only if $s \equiv 0, 1, 2, 3, 5, 6 \pmod{9}$. In [8], we proved the ‘only if’ part. For the survivors, we have

$$\beta_s \text{ for } s > 0 \text{ with } s \equiv 0, 1, 2, 5, 6 \pmod{9} \text{ by M. Behrens and S. Pemmaraju [1].}$$

We note that the element $\beta_1 \in \pi_{pq-2}(S^0)$ is given by

$$\beta_1 = jj_1[\beta i_1]i$$

for the maps i, j, i_1 and j_1 are the maps given in (1.1). Here, $[\beta i_1]$ denotes βi_1 for the self-map $\beta \in [V(1), V(1)]_{(p+1)q}$ ($V(1) = V_1$) constructed by Smith [9] at a prime greater than three, and the generator of the homotopy group $[M, V(1)]_{16}$ given in [11] at the prime three. We define W by the cofiber sequence

$$(1.3) \quad S^{pq-2} \xrightarrow{\beta_1} S^0 \xrightarrow{\iota} W \xrightarrow{\kappa} S^{pq-1}.$$

Then we have our main theorem:

Theorem 1.4. *Suppose that there is an element $B_s \in \pi_{s(p+1)q}(V_r \wedge W)$ detected by $v_2^s \in E_2^{0, s(p+1)q}(V_r \wedge W)$. Then, the beta element $\beta_{s/l}$ for $0 < l \leq r - 2$ survives to $\pi_{(sp+s-l)q-2}(S^0)$.*

As an example, we have

Lemma 1.5. *At an odd prime, there exists $B_{tp} \in \pi_{tp(p+1)q}(V_p \wedge W)$ for $t > 0$ detected by $v_2^{tp} \in E_2^{0, tp(p+1)q}(V_p \wedge W)$.*

Corollary 1.6. *The beta elements $\beta_{tp/l}$ for $t > 0$ and $0 < l \leq p - 2$ survive.*

This corollary shows that β_{3t} survives at the prime three and completes a proof of the conjecture.

2. PROOFS OF RESULTS

Applying the BP_* -homology to the first cofiber sequence of (1.1), we obtain

$$BP_*(M) = BP_*/(p).$$

We observe the E_2 -term $E_2^*(X)$ of the Adams-Novikov spectral sequence as the cohomology of the reduced cobar complex $\widetilde{\Omega}_{BP_*BP}^* BP_*(X)$. Then, we have a vanishing line for a (-1) -connected spectrum X :

$$(2.1) \quad E_2^{s,t}(X) = 0 \quad \text{if } t < sq.$$

The structure maps of the Hopf algebroid BP_*BP act on generators by

$$(2.2) \quad \begin{aligned} \eta_R(v_1) &= v_1 + pt_1, \\ \eta_R(v_2) &\equiv v_2 + v_1 t_1^p - v_1^p t_1 \pmod{p}, \quad \text{and} \\ \Delta(t_1) &= t_1 \otimes 1 + 1 \otimes t_1 \end{aligned}$$

(cf. [7]). By this, we see that $v_1 \in E_2^{0,q}(M)$. Since $E_2^{sq+1,(s+1)q}(M) = 0$ for $s > 0$ by (2.1), v_1 is a permanent cycle. The Adams map α in (1.1) is given by $\alpha = m(M \wedge v_1)$ for the multiplication m of M , and so it induces v_1 -multiplication on the BP_* -homology. It follows that

$$BP_*(V_r) = BP_*/(p, v_1^r)$$

for $r > 0$, and we see that

$$(2.3) \quad v_1 \in E_2^{0,q}(V_r) \quad \text{and} \quad v_2^p \in E_2^{0,(p^2+p)q}(V_p)$$

by (2.2). For later use, we notice that

$$(2.4) \quad \alpha i = m(i \wedge v_1) = v_1.$$

From [12], we get

Lemma 2.5. *Suppose that $t - s < (p^2 + p + 1)q$. In this range, the Adams-Novikov E_2 -term $E_2^{*,*}(M)$ is a subquotient of $\mathbb{Z}/p[v_1, v_2] \otimes \{h_0, h_1, h_2, g_0, k_0, k_0 h_0, h_0 h_2\} \otimes P(b_0, b_1, b_{20})$. Here the bi-degrees of the generators are:*

$$\begin{aligned} |h_i| &= (1, p^i q) \quad (i = 0, 1, 2), \quad |g_0| = (2, (p+2)q), \quad |k_0| = (2, (2p+1)q), \\ |b_0| &= (2, pq), \quad |b_1| = (2, p^2 q), \quad \text{and} \quad |b_{20}| = (2, (p^2 + p)q). \end{aligned}$$

Proof. We have short exact sequences $0 \rightarrow BP_*/(p) \xrightarrow{v_1} BP_*/(p) \rightarrow BP_*/(p, v_1) \rightarrow 0$ and $0 \rightarrow BP_*/(p, v_1) \xrightarrow{v_2} BP_*/(p, v_1) \rightarrow BP_*/(p, v_1, v_2) \rightarrow 0$, which give rise to Bockstein spectral sequences converging to the Adams-Novikov E_2 -terms $E_2^*(M)$ and $E_2^*(V_1)$ with E_1 -terms $E_2^*(V_1)$ and $\text{Ext}_{BP_*BP}^*(BP_*, BP_*/(p, v_1, v_2))$, respectively. In our range, we have $\text{Ext}_{BP_*BP}^*(BP_*, BP_*/(p, v_1, v_2)) = \text{Ext}_{\mathcal{P}}^*(\mathbb{Z}/p, \mathbb{Z}/p)$ for the subalgebra $\mathcal{P}$ of the Steenrod algebra generated by the reduced power operations. Thus, $E_2^*(M)$ is a subquotient of $\mathbb{Z}/p[v_1, v_2] \otimes \text{Ext}_{\mathcal{P}}^*(\mathbb{Z}/p, \mathbb{Z}/p)$. We now get the structure of $\text{Ext}_{\mathcal{P}}^*(\mathbb{Z}/p, \mathbb{Z}/p)$ from [12]. $\square$

Corollary 2.6. *In our range, we have a vanishing line: $E_2^{2s+\varepsilon, tq}(V) = 0$ for $V = M, V_r$, if $t < ps + \varepsilon$. Here, $\varepsilon = 0, 1$.*

Lemma 2.7. *Let $\delta: E_2^s(M) \rightarrow E_2^{s+1}(S^0)$ be the connecting homomorphism associated with the first cofiber sequence in (1.1). Then, it is a derivation and*

$$\delta(v_1) = h_0, \quad \delta(h_2) = -b_1 \quad \text{and} \quad \delta(b_0) = 0.$$

Proof. Note that h_i and b_i are represented by cocycles $t_1^{p^i}$ and $\sum_{k=1}^{p-1} \frac{1}{p} \binom{p}{k} t_1^{p-k} \otimes t_1^k$ of the cobar complex. By (2.2), we see that the differential d of the cobar complex acts on v_1 and $t_1^{p^i}$ as $d(v_1) = pt_1$ and $d(t_1^{p^i}) = -pb_{i-1}$ for $i > 0$. The lemma now follows from the definition of the connecting homomorphism. $\square$

The cofiber sequence (1.3) induces a split short exact sequence

$$0 \rightarrow E_2^{s,t}(V) \xrightarrow{\iota_*} E_2^{s,t}(V \wedge W) \xrightarrow{\kappa_*} E_2^{s,t-pq+1}(V) \rightarrow 0$$

of E_2 -terms for $V = M$ and V_r , and so

$$E_2^*(V \wedge W) = E_2(V) \oplus gE_2(V),$$

where g denotes a generator of degree $pq - 1$ such that $\kappa_*(xg) = x$. Since the E_3 -term is a homology of E_2 -terms and $d_2(g) = \beta_1$ for the element β_1 in (1.2), we have the long exact sequence

$$(2.8) \quad E_3^{s,t}(M) \xrightarrow{\partial} E_3^{s+2,t+pq}(M) \xrightarrow{\iota_*} E_3^{s+2,t+pq}(M \wedge W) \xrightarrow{\kappa_*} E_3^{s+2,t+1}(M)$$

with the connecting homomorphism ∂ given by $\partial(x) = x\beta_1$.

Lemma 2.9. *The element $v_2^p \in E_2^0(V_p \wedge W)$ in (2.3) survives to an element $B_p \in \pi_{p(p+1)q}(V_p \wedge W)$.*

Proof. Consider the cofiber sequence (1.1) with $r = p$. In the Adams-Novikov spectral sequence for computing $\pi_*(S^0)$, we have the Toda differential $d_{q+1}(b_1) = h_0 b_0^p \in E_2^{q+3, (p^2+1)q}(S^0)$ up to a nonzero scalar. By Lemma 2.5, $E_2^{q+2, (p^2+1)q}(M)$ is a subquotient of $\{v_1 b_0^p\}$. Since $\delta(v_1 b_0^p) = h_0 b_0^p$, we see that $d_{q+1}(h_2) = v_1 b_0^p \in E_2^{q+2, (p^2+1)q}(M)$ up to a nonzero scalar by Lemma 2.7. Note that $\beta_1 = b_0$. In the exact sequence (2.8), $v_1 b_0^p = \partial(v_1 b_0^{p-1})$, and so $d_{q+1}(\iota_*(h_2)) = 0$ in $E_3^{q+2, (p^2+1)q}(M \wedge W)$. Besides, Corollary 2.6 shows that $E_2^{sq+2, (p^2+s)q}(M) = 0$ for $s > 1$, and we see that $\iota_*(h_2) \in E_2^{1, p^2q}(M \wedge W)$ is a permanent cycle, which detects an element $\beta'_{p/p} \in \pi_{p^2q-1}(M \wedge W)$. Send it by α^p in (1.1). The element $\alpha^p \beta'_{p/p} \in \pi_{(p^2+p)q-1}(M \wedge W)$ is detected by an element of $E_2^{q+1, (p^2+p+1)q}(M \wedge W)$, since the E_2 -term $E_2^{sq+1, p(p+1)q+sq}(M \wedge W) = E_2^{sq+1, p(p+1)q+sq}(M)$ for $s > 1$ is zero by Corollary 2.6. The E_2 -term $E_2^{q+1, (p^2+p+1)q}(M)$ for $s = 1$ is a subquotient of

$$h_0 b_0 b_1 \ (p = 3), \ v_1^{p-1} v_2 h_0 b_0^{p-1}, \ v_1^{2p} h_0 b_0^{p-1}, \ v_2 h_1 b_0^{p-1}, \ v_1^{p+1} h_1 b_0^{p-1}, \ v_1^{p-1} k_0 h_0 b_0^{p-2}$$

by Lemma 2.5, and so the E_3 -term $E_3^{q+1, (p^2+p+1)q}(M \wedge W) = 0$ by (2.8). Therefore, $\alpha^p \beta'_{p/p} = 0$ and $\beta'_{p/p}$ is pulled back to an element B_p under the map j_p . $\square$

We call a spectrum R a ring spectrum if there exist a multiplication $\mu: R \wedge R \rightarrow R$ and a unit $\iota: S^0 \rightarrow R$ such that $\mu(\iota \wedge R) = 1_R = \mu(R \wedge \iota): R \rightarrow R$. By [3, Ex. 2.9] and [3, Ex. 5.7], we have

$$(2.10) \quad W \text{ and } V_r \text{ for } r > 1 \text{ are ring spectra.}$$

In particular, the spectrum $R_r = V_r \wedge W$ for $r > 1$ is a ring spectrum with multiplication $m_r = (\mu_r \wedge \mu_W)(V_r \wedge T \wedge W): R_r \wedge R_r = V_r \wedge W \wedge V_r \wedge W \rightarrow V_r \wedge V_r \wedge W \wedge W \rightarrow V_r \wedge W = R_r$, where T denotes the switching map and μ_r and μ_W are the multiplications of V_r and W , respectively.

Proof of Lemma 1.5. Since $R_p = V_p \wedge W$ is a ring spectrum, we obtain a self-map $[\beta^p]: R_p \xrightarrow{R_p \wedge B_p} R_p \wedge R_p \xrightarrow{m_p} R_p$ inducing v_2^p on BP_* -homology. Now put $B_{tp} = [\beta^p]^{t-1} B_p$ to see the lemma. $\square$

We consider the element $i_r \alpha^2 i \in \pi_{2q}(V_r) \cong \pi_{2q}(M) = \mathbb{Z}/p\{\alpha^2 i\}$ for $r > 2$ and for the maps in (1.1), which is detected by the element $v_1^2 \in E_2^0(V_r)$ by (2.4).

Lemma 2.11. *Let $r > 2$. There exists an element $\eta_r \in \pi_{(p+2)q-1}(V_r \wedge W)$ such that $\kappa_*(\eta_r) = i_r \alpha^2 i \in \pi_{2q}(V_r)$. Besides, it is detected by $v_1^2 g \in E_2^0(V_r \wedge W) = E_2^0(V_r) \oplus gE_2^0(V_r)$.*

Proof. Putting $\delta = ij$ for the maps i, j in (1.1), we have Yamamoto's relation $\alpha^2 \delta = 2\alpha\delta\alpha - \delta\alpha^2 \in [M, M]_{2q-1}$ (cf. [11]). We compute

$$\alpha^2 i \beta_1 = \alpha^2 \delta j_1 [\beta i_1] i = (\delta \alpha^2 + \alpha \delta \alpha) j_1 [\beta i_1] i = 0,$$

since $\alpha j_1 = 0$ by (1.1). It follows that $i_r \alpha^2 i \in \pi_{2q}(V_r)$ is pulled back to an element $\eta_r \in \pi_{(p+2)q-1}(V_r \wedge W)$ as desired. Since $i_r \alpha^2 i$ is detected by $v_1^2 \in E_2^0(V_r)$, η_r is detected by the element $v_1^2 g \in E_2^0(V_r \wedge W) = E_2^0(V_r) \oplus gE_2^0(V_r)$. $\square$

Proof of Theorem 1.4. Consider the product $B_s\eta_r \in \pi_*(V_r \wedge W)$ for the element η_r in Lemma 2.11. Then, it is detected by $v_1^2v_2^sg$, since η_r induces a BP_*BP -comodule map $(\eta_r)_*: BP_*(V_r) \rightarrow BP_*(V_r \wedge W)$ such that $(\eta_r)_*(x) = v_1^2xg$ and B_s is detected by v_2^s . The map $\kappa_*: E_2^0(V_r \wedge W) \rightarrow E_2^0(V_r)$ assigns $v_1^2v_2^sg$ to $v_1^2v_2^s$, which is a permanent cycle detected by $\kappa(B_s\eta_r)$. Now putting $\beta_{s/l} = jj_{l+2}a^{r,l+2}\kappa(B_s\eta_r) = j\alpha^{r-2-l}j_r\kappa(B_s\eta_r) \in \pi_*(S^0)$ for $l < r-2$, we see the theorem by the Geometric Boundary Theorem (*cf.* [7]). Here, $a^{r,k}$ denotes a map in the cofiber sequence $V_{r-k} \longrightarrow V_r \xrightarrow{a^{r,k}} V_k$ obtained from applying the 3×3 Lemma to the cofiber sequences of (1.1) for $r-k$, r and k . We note that it satisfies $j_ka^{r,k} = \alpha^{r-k}j_r$. $\square$

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