HEEGAARD SPLITTINGS WITH (DISK, ESSENTIAL SURFACE) PAIRS THAT INTERSECT IN ONE POINT

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ABSTRACT. We consider a Heegaard splitting $M = H_1 \cup_S H_2$ of a 3-manifold M having an essential disk D in H_1 and an essential surface F in H_2 with $|D \cap F| = 1$.

From $H_1 \cup_S H_2$, we obtain another Heegaard splitting $H'_1 \cup_{S'} H'_2$ by removing a neighborhood of F from H_2 and attaching it to H_1 . As an application, by using a theorem due to Casson and Gordon, we give examples of 3-manifolds admitting two Heegaard splittings of distinct genera, where one of them is a strongly irreducible non-minimal genus splitting and it is obtained from the other by the above construction.

We also show that all Heegaard splittings of a Seifert fibered space are related via the above construction.

1. INTRODUCTION

Every compact 3-manifold M admits a Heegaard splitting and there are various Heegaard splittings as the genus varies. If g is the minimal genus of Heegaard splittings of M , then for each $g' > g$ there exists at least one Heegaard splitting of genus g' , i.e., a splitting obtained by stabilizations.

From a Heegaard splitting, we can obtain another Heegaard splitting of different genus which is not just a stabilization if the original one has certain embedded surfaces that intersect in one point. A stabilized Heegaard splitting $H_1 \cup_S H_2$, which has essential disks $D_1 \subset H_1$ and $D_2 \subset H_2$ with $|D_1 \cap D_2| = 1$, can be destabilized and the genus goes down.

Concerning (disk, annulus) pairs, many people, ([14], [13], [10], [5]) considered several notions on Heegaard splittings. In [5], the author considered a Heegaard splitting $H_1 \cup_S H_2$ having an essential disk $D \subset H_1$ and an essential annulus $A \subset H_2$ with $|D \cap A| = 1$, and it was shown that such a Heegaard splitting has the disjoint curve property, a notion which was introduced by Thompson [19], and another Heegaard splitting $H'_1 \cup_{S'} H'_2$ can be obtained from $H_1 \cup_S H_2$ by removing a neighborhood of A from H_2 and attaching it to H_1 . In this case the genus of the Heegaard splitting remains unchanged.

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In this paper, we consider a Heegaard splitting $H_1 \cup_S H_2$ of a 3-manifold M having an essential disk $D \subset H_1$ and an essential surface $F \subset H_2$ with $|D \cap F| = 1$. We denote it as a *strong* (D, F) pair for consistency of terminology with [10]. Here a surface properly embedded in a compression body is said to be *essential* if it is incompressible and not boundary parallel. (We mainly deal with the case that $\partial F \subset S$ when H_2 is a compression body with $\partial_- H_2 \neq \emptyset$, but also consider the case that F is a spanning annulus.) First we show that if F has genus g and n boundary components, the distance $d(S)$ of $H_1 \cup_S H_2$ is bounded above by $2g + n$ (Theorem 2.3).

From $H_1 \cup_S H_2$ we can obtain another Heegaard splitting $H'_1 \cup_{S'} H'_2$ by removing a neighborhood of F from H_2 and attaching it to H_1 .

Theorem 1.1. *Let $H_1 \cup_S H_2$ be a Heegaard splitting of a 3-manifold with a strong (D, F) pair. Let H'_1 be obtained from H_1 by attaching $F \times I$ along $\partial F \times I$ and let H'_2 be obtained from H_2 by cutting along F . Then $H'_1 \cup_{S'} H'_2$ is a Heegaard splitting of genus $g(S) + 2g + n - 2$.*

The construction of a new Heegaard surface in Theorem 1.1 resembles quite a bit the *Haken sum* in Moriah, Schleimer, and Sedgwick's paper [8]. In that paper, they considered the Haken sum of a Heegaard surface with copies of an incompressible surface in the manifold and obtained infinitely many distinct Heegaard splittings. There are also related works by Kobayashi [3] and by Lustig and Moriah [6]. However, in our case the essential surface lives in one of the compression bodies (Figure 1).

FIGURE 1. Haken sum and compressing along an essential surface in a compression body

In Theorem 3.5, it is shown that $H'_1 \cup_{S'} H'_2$ has the disjoint curve property if F is not a disk.

In most of the paper, we are considering the case $\partial F \subset \partial_+ H_2$ when H_2 is a compression body with $\partial_- H_2 \neq \emptyset$. In the last part of Section 3, we briefly consider the case that F is a spanning annulus in a compression body (Corollary 3.6), which will be used in Section 5.

For $g \geq 1$ or $n \geq 3$, the genus of $H'_1 \cup_{S'} H'_2$ is greater than that of $H_1 \cup_S H_2$. We give examples of 3-manifolds admitting two Heegaard splittings of distinct genera where one of the two Heegaard splittings is a strongly irreducible non-minimal genus splitting and is obtained from the other by the method in Theorem 1.1. The

examples are constructed by doing $1/q$ -Dehn surgery ($|q| \geq 6$) on certain knots, and a theorem due to Casson and Gordon is used to show strong irreducibility.

Theorem 1.2. *Let K be a knot in a 3-manifold M with the following properties:*

- *There exists a free incompressible Seifert surface F of genus g for K .*
- *Every tunnel of a given unknotting tunnel system $\{t_1, t_2, \dots, t_k\}$ for K can be isotoped to lie on F and be mutually disjoint.*
- *$k + 1 \leq 2g$.*

Let $K(1/q)$ be the manifold obtained by doing $1/q$ -Dehn surgery ($|q| \geq 6$) on K in M . Then $K(1/q)$ has a genus $2g$ strongly irreducible Heegaard splitting and a genus $k + 1$ Heegaard splitting, and the two are related by a sequence of constructions in Theorem 1.1.

Remark 1.3. Note that there exist knots in S^3 which bound only non-free incompressible Seifert surfaces [4], [12]. Also note that the free genus of a knot can be strictly larger than the usual genus of the knot [4], [7]. So the surface F in Theorem 1.2 can possibly be a non-minimal genus Seifert surface for K .

In particular, if K is a torus knot and F is a minimal Seifert surface, then $K(1/q)$ is a Seifert fibered space over S^2 with three exceptional fibers ([10], [18]). Hence Theorem 1.2 gives some insight into the relation of a vertical splitting and a horizontal splitting of such manifolds.

In Section 5, we consider vertical splittings and horizontal splittings of Seifert fibered spaces in more detail. Any irreducible Heegaard splitting of a Seifert fibered space is either vertical or horizontal [9]. We show that any two Heegaard splittings of a Seifert fibered spaces are related via the constructions in Theorem 1.1.

Theorem 1.4. *Any two vertical splittings of a Seifert fibered space are equivalent via repeated operations: drilling out an exceptional fiber, the construction in Theorem 1.1 or its inverse using a strong (D, A) pair, and refilling the fiber.*

Any horizontal splitting can be obtained from a certain vertical splitting or its stabilization by the construction in Theorem 1.1.

2. HEEGAARD SPLITTING WITH A STRONG (D, F) PAIR

We say that a surface properly embedded in a 3-manifold is *essential* if it is incompressible and not boundary parallel. First we show that for a (D, F) pair with $|D \cap F| = 1$, in fact the essentiality of F follows automatically from incompressibility. In other words, we have the following.

Proposition 2.1. *For a Heegaard splitting $H_1 \cup_S H_2$, let D be an essential disk in H_1 and F be an incompressible surface in H_2 with $\partial F \neq \emptyset$ and $\partial F \subset S$. If $|D \cap F| = 1$, then F is essential in H_2 .*

Proof. Suppose F is not essential in H_2 . Then F is parallel to a subsurface $F' \subset S$ (rel. ∂F). When we go around ∂D , we pass through $\partial F'$ from $S - F'$ to the interior of F' at some time. We should pass through $\partial F'$ at least once more to go around all of ∂D . This is a contradiction since $|D \cap F| = |D \cap \partial F| = 1$ and $\partial F = \partial F'$. $\square$

Now we consider the distance, due to Hempel [2], of a Heegaard splitting with a strong (D, F) pair. The *distance* $d(S)$ of a Heegaard splitting $H_1 \cup_S H_2$ is the smallest number $n \geq 0$ so that there is a sequence of essential simple closed curves

$\alpha_0, \dots, \alpha_n$ in S with α_0 bounding a disk in H_1 , α_n bounding a disk in H_2 and for each $1 \leq i \leq n$, α_{i-1} and α_i can be isotoped in S to be disjoint.

We need the following technical lemma on boundary compression by Morimoto to get an upper bound for distance.

Lemma 2.2 (Lemma 5.1 of [11]). *Let W be a compact orientable 3-manifold, and let F be an essential surface properly embedded in W such that $\partial F \neq \emptyset$ and ∂F is contained in a single component of ∂W . Let F' be the 2-manifold obtained from F by a boundary compression. Then F' is incompressible and has a component which is not boundary parallel. Hence F' is essential.*

Theorem 2.3. *Let $H_1 \cup_S H_2$ be a genus ≥ 2 Heegaard splitting with a strong (D, F) pair. If F has genus g and n boundary components, then the distance $d(S) \leq 2g + n$.*

Proof. Let $\partial F = \beta_1 \cup \dots \cup \beta_n$ and $|D \cap \beta_1| = 1$. Since F is incompressible and not boundary parallel and $\partial F \subset S$, F intersects a meridian disk system of H_2 . By standard innermost disk and outermost arc arguments, we may assume that there is a boundary compressing disk Δ for F , where the boundary compression occurs toward S . Let $\partial \Delta = \alpha \cup \beta$, where α is an essential arc in F and β is an arc in S . We construct a sequence of essential simple closed curves $\alpha_0, \dots, \alpha_k$ with α_0 bounding a disk in H_1 , α_k bounding a disk in H_2 and for each $1 \leq i \leq k$, α_{i-1} and α_i can be isotoped in S to be disjoint, dividing into two cases according to n .

Case 1. $n = 1$.

Take two parallel copies of D in H_1 and connect them with a band along β_1 and push the band slightly into the interior of H_1 to make a disk $D' \subset H_1$. Since $\partial D'$ bounds a once-punctured torus in S and $H_1 \cup_S H_2$ is a genus ≥ 2 Heegaard splitting, D' is an essential disk in H_1 . Note that $\partial D'$ is disjoint from β_1 . Take $\partial D'$ as α_0 and β_1 as α_1 .

Case 2. $n > 1$.

In this case, take ∂D as α_0 and any β_i ($i \neq 1$) as α_1 .

Both in Case 1 and Case 2, boundary compress F along Δ to get an essential surface $F_{(1)}$ by Lemma 2.2. All the boundary components of $F_{(1)}$ can be made disjoint from ∂F . Take any component of $\partial F_{(1)}$ as α_2 . Boundary compress $F_{(1)}$ to get an essential surface $F_{(2)}$ by Lemma 2.2. All the boundary components of $F_{(2)}$ can be made disjoint from $\partial F_{(1)}$. Take any component of $\partial F_{(2)}$ as α_3 . In this way, we successively boundary compress until we get an essential disk in H_2 by Lemma 2.2. We can check that the possible maximum number of boundary compressions is $2g + n - 1$. So the possible maximum length sequence of α_i 's would be $\alpha_0, \alpha_1, \dots, \alpha_{2g+n}$. So we conclude that $d(S) \leq 2g + n$. $\square$

3. OBTAINING NEW HEEGAARD SPLITTINGS

We consider attaching $F \times I$ to a handlebody along $\partial F \times I$. Let $g(X)$ denote the genus of X .

Lemma 3.1. *Let $\gamma_1, \dots, \gamma_n$ be mutually disjoint loops on the boundary of a handlebody H and let D be an essential disk of H such that $|\partial D \cap \gamma_1| = 1$ and $\partial D \cap \gamma_i = \emptyset$ ($i = 2, \dots, n$). Let F be a genus g surface with n ($n \geq 1$) boundary components $\beta_1, \dots, \beta_n$.*

If we attach $F \times I$ to H along $\partial F \times I$ so that $\beta_i \times I$ is attached to $N(\gamma_i; \partial H) \cong \gamma_i \times I$ ($i = 1, 2, \dots, n$), then the resulting manifold is a handlebody of genus $g(H) + 2g + n - 2$.

Proof. Let p be the intersection point $D \cap \gamma_1$. Consider the neighborhood $D \times I$ in H and $\gamma_1 \times I$ in ∂H . We can assume that $\partial(D \times I) \cap (\gamma_1 \times I)$ is a small rectangle R containing p . Let R' be the rectangle in $\beta_1 \times I$ that is attached to R .

Since F is a genus g surface with n boundary components, there are mutually disjoint essential arcs $a_1, \dots, a_{2g+n-1}$ in F such that F cut along $a_1 \cup \dots \cup a_{2g+n-1}$ is a disk. In particular, take such an essential arc system so as to satisfy that one of the two points of ∂a_1 is attached to p . More precisely, we take the rectangular parallelepiped neighborhood $a_1 \times I \times I$ of a_1 in $F \times I$ to be equal to $R' \times I$.

Let H' be $cl(H - (D \times I))$. Here the size of I is the same as the interval used for the rectangle R so that $R \subset (\partial D \times I)$ and $R \cap (\partial D \times I) = R$. In other words, the width of R fits into $\partial D \times I$. Since $|D \cap \gamma_1| = 1$, D is a non-separating essential disk in H . So H' is a handlebody of genus $g(H) - 1$. Attach a rectangular parallelepiped neighborhood $a_i \times I \times I$ of a_i taken in $F \times I$ to H' along $\partial a_i \times I \times I$ for each $i = 2, \dots, 2g+n-1$. Since each $a_i \times I \times I$ ($i = 2, \dots, 2g+n-1$) can be considered as a 1-handle, the resulting manifold H'' is a genus $g(H) + 2g + n - 3$ handlebody.

Observe that $cl(F \times I - (\bigcup_{i=1}^{2g+n-1} a_i \times I \times I))$ is homeomorphic to a 3-ball B , which is attached to H'' along two subdisks of its boundary. Then $H''' = H'' \cup B$ is a handlebody of genus $g(H) + 2g + n - 2$. Observe also that $(D \times I) \cup (R' \times I)$ is a 3-ball attached to H''' along $(D \times \partial I) \cup (\text{three faces of } \partial(R' \times I))$, which is a disk on the boundary of a 3-ball. So the genus remains unchanged after attaching $(D \times I) \cup (R' \times I)$ to H''' . Hence we conclude that the resulting manifold after attaching $F \times I$ to H along $\partial F \times I$ is a genus $g(H) + 2g + n - 2$ handlebody. $\square$

Now we consider removing a neighborhood of an incompressible surface from a compression body. The following lemma is well-known. It can be found, for example, as Lemma 2 in Schultens' paper [17].

Lemma 3.2. *Let F ($\partial F \neq \emptyset$) be an incompressible surface properly embedded in a compression body H with $\partial F \subset \partial_+ H$. Then F cuts H into compression bodies (or a compression body).*

Using Lemma 3.1 and Lemma 3.2, we give a proof of Theorem 1.1.

Proof of Theorem 1.1. By Lemma 3.1, H'_1 is a handlebody of genus $g(S) + 2g + n - 2$. By Lemma 3.2, H'_2 is a union of two compression bodies or a compression body. Since $\partial H'_1$ is the same as $\partial_+ H'_2$, H'_2 is connected. Therefore $H'_1 \cup_{S'} H'_2$ is a Heegaard splitting of genus $g(S) + 2g + n - 2$. $\square$

Corollary 3.3. *Let $H_1 \cup_S H_2$ be a Heegaard splitting of a 3-manifold M with a strong (D, A) pair, where A is an annulus. Let H'_1 be obtained from H_1 by attaching $A \times I \subset H_2$ along $\partial A \times I$ and let H'_2 be obtained from H_2 by cutting along A . Then $H'_1 \cup_{S'} H'_2$ is a Heegaard splitting of the same genus as $H_1 \cup_S H_2$.*

Remark 3.4. Corollary 3.3 is a generalization of Definition 14 of [17].

A Heegaard splitting $H_1 \cup_S H_2$ is said to have the *disjoint curve property* ([19]) if there are essential disks $D_1 \subset H_1$, $D_2 \subset H_2$ and an essential loop $\gamma \subset S$ such that $(\partial D_1 \cup \partial D_2) \cap \gamma = \emptyset$. It is equivalent to the distance $d(S)$ being less than or equal to two. The newly obtained Heegaard splitting $H'_1 \cup_{S'} H'_2$ of Theorem 1.1 has the disjoint curve property if the switch from $H_1 \cup_S H_2$ to $H'_1 \cup_{S'} H'_2$ is not a destabilization.

Theorem 3.5. *If F is not a disk, the Heegaard splitting $H'_1 \cup_{S'} H'_2$ obtained in Theorem 1.1 has the disjoint curve property.*

FIGURE 2. An essential disk E' in H'_1 satisfying the disjoint curve property

Proof. Recall the proof of Lemma 3.1. In the proof of Lemma 3.1, H''' was obtained from H'' by attaching a 1-handle. Consider a meridian disk (co-core) E of the 1-handle. Also remember that $H'_1 = H_1 \cup (F \times I)$ was obtained from H''' by attaching a 3-ball along a 2-disk on its boundary. Then E is enlarged to an essential disk E' in H'_1 , which can be taken as two parallel copies of D attached by a band in $F \times I$. See Figure 2. Since the band is equivalent to an $(\text{arc}) \times I$ in $F \times I$ with both endpoints of the arc in the same component of ∂F , we can take an essential loop $\gamma \subset F$ which is disjoint from E' . Since F is incompressible in H_2 , we can see that γ is an essential loop in the new Heegaard surface S' . Take a boundary compressing disk $\Delta \subset H_2$ for F . Let $\partial\Delta = \alpha \cup \beta$, where α is an essential arc in F . Then after cutting H_2 along F , Δ is an essential disk in H'_2 . We may assume that α belongs to $F \times \{0\}$ and γ belongs to $F \times \{1\}$. So Δ is disjoint from γ . So we conclude that the triple (E', Δ, γ) satisfies the disjoint curve property. $\square$

Now we consider the case when F is a spanning annulus in a compression body. Given a compression body H with $\partial_- H \neq \emptyset$, there exists a meridian disk system $\{D_1, \dots, D_k\}$ of H such that H cut along $\bigcup_{i=1}^k D_i$ is $\partial_- H \times I \cup$ (possibly empty) 3-balls. A *spanning annulus* of a compression body H is an annulus which can be expressed as $\gamma \times I$ in the compression body structure, where γ is an essential loop in $\partial_- H$. By showing the analogue of Lemma 3.1 and Lemma 3.2, we can state the following. It will be used in Section 5.

Corollary 3.6. *Let $H_1 \cup_S H_2$ be a Heegaard splitting of a 3-manifold M with a strong (D, A) pair, where A is a spanning annulus. Let $A = \gamma \times I$ in the compression body structure and let $\gamma \subset \Sigma \subset \partial_- H_2$, where Σ has genus g . Let H'_1 and H'_2 be obtained as follows (Figure 3).*

- H'_1 is obtained from H_1 by attaching $A \times I$ along $(\partial A - \gamma) \times I$ and attaching $\Sigma \times I$ along $\gamma \times I$.
- H'_2 is obtained from H_2 by cutting along A and shrinking the part adjacent to Σ .

Then $H'_1 \cup_{S'} H'_2$ is a Heegaard splitting of genus $g(S) + g - 1$. In particular, if Σ is a torus, it has the same genus as $H_1 \cup_S H_2$.

FIGURE 3. Compressing along a spanning annulus

Proof. By following the procedure as in the proof of Lemma 3.1, we can see that H'_1 is a compression body of genus $g(S) + g - 1$. One remarkable point is that the partition of components of ∂M is changed for the Heegaard splitting: Σ belongs to $\partial_- H_2$ before the change and to $\partial_- H'_1$ after the change.

By cutting H_2 along A , H'_2 becomes a genus $g(S) + g - 1$ compression body, and parts of Σ are connected to $\partial_+ H_2$. We conclude that $H'_1 \cup_{S'} H'_2$ is a Heegaard splitting of genus $g(S) + g - 1$. $\square$

4. EXAMPLES

Let K be a knot in M admitting an incompressible free Seifert surface F of genus g . Then F is incompressible in $cl(M - N(F))$. Also F is incompressible in the product neighborhood $N(F) = F \times I$. Since $N(F)$ and $cl(M - N(F))$ are handlebodies, this gives a Heegaard splitting $N(F) \cup_{\Sigma} cl(M - N(F))$ of M .

Now we are going to construct a strongly irreducible Heegaard splitting from Σ by Dehn surgery on K . Remove a neighborhood $N(K)$ from M . Let $K(1/q)$ denote the manifold obtained by $1/q$ -filling on $cl(M - N(K))$. Here a longitudinal slope is determined by ∂F . We can assume that the filling solid torus T is attached to $N(F) = F \times I$ along an annulus $\partial F \times I$. Note that if we perform $1/q$ -filling, a meridian curve $(1, 0)$ of the filling solid torus is mapped to a $(1, q)$ curve, and the longitude $(0, 1)$ of the filling solid torus is mapped to longitude $(0, 1)$. So $N(F) \cup T$ is a handlebody. Then we get the Heegaard splitting $(N(F) \cup T) \cup_{\Sigma'} cl(M - N(F))$ for $K(1/q)$. (Alternatively, we can regard Σ' as obtained from Σ by Dehn twists on K $|q|$ times.) By a theorem due to Casson and Gordon [1], Σ' is a strongly irreducible Heegaard splitting if $|q| \geq 6$. Here we refer to the statements in ([9], Appendix).

Theorem 4.1 (Casson-Gordon). *Suppose $M = H_1 \cup_{\Sigma} H_2$ is a weakly reducible Heegaard splitting for the closed manifold M . Let K be a simple closed curve in Σ such that $\Sigma - N(K)$ is incompressible in both H_1 and H_2 . Then Σ' , for all $|q| \geq 6$, is a strongly irreducible Heegaard splitting for the Dehn filled manifold $M(1/q)$.*

Proof of Theorem 1.2. $K(1/q)$ has a genus $2g$ Heegaard splitting $(N(F) \cup T) \cup_{\Sigma'} cl(M - N(F))$. The strong irreducibility of it is already shown above.

Note that $cl(M - N(K \cup (\bigcup_{i=1}^k t_i)))$ is a genus $k+1$ handlebody and this handlebody remains untouched during the $1/q$ -surgery on K . So $(T \cup (\bigcup_{i=1}^k N(t_i))) \cup_{\Sigma'} cl(M - N(K \cup (\bigcup_{i=1}^k t_i)))$ is a genus $k+1$ Heegaard splitting for the Dehn filled manifold $K(1/q)$. To simplify notation, let $V_0 = T \cup (\bigcup_{i=1}^k N(t_i))$ and $W_0 = cl(M - N(K \cup (\bigcup_{i=1}^k t_i)))$.

By the assumptions of Theorem 1.2, every tunnel of an unknotting tunnel system $\{t_1, t_2, \dots, t_k\}$ for K can be isotoped to lie on F and be mutually disjoint. If we

cut F along $\bigcup_{i=1}^k t_i$, then F can possibly be disconnected. Let $F' = F_1 \cup \cdots \cup F_n$ be the resulting 2-manifold. Since F_i is a subsurface of F and by the innermost disk arguments, F_i is incompressible in W_0 for all $i = 1, 2, \dots, n$. Without loss of generality, we may assume that the meridian of the filling solid torus T intersects F_1 in one point since the determinant of the matrix $\begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix}$ is 1. So by Proposition 2.1, the genus $k + 1$ Heegaard splitting $V_0 \cup_{S'} W_0$ has a strong (D, F_1) pair. Let $V_1 = V_0 \cup_{\partial F_1 \times I} N(F_1)$ and $W_1 = cl(K(1/q) - V_1)$. Then $V_1 \cup_{S'_1} W_1$ is another Heegaard splitting of $K(1/q)$.

Suppose $n > 1$. Let F_2 be one of the subsurfaces that is adjacent to F_1 . Relying upon the fact that F_i is a subsurface of F again, F_i is incompressible in W_1 for all $i = 2, \dots, n$. Note that the meridian of the filling solid torus T does not intersect $F_2 \cup \cdots \cup F_n$. However, we can find some alternative essential disk that intersects F_2 in one point. Since $\bigcup_{i=1}^k t_i$ cuts F into $F_1 \cup \cdots \cup F_n$, there exists a tunnel t_i that is adjacent to both F_1 and F_2 for some i ($1 \leq i \leq k$). Then F_2 intersects a meridian disk of t_i in one point. Again recall the proof and the notation of Lemma 3.1. In the proof of Lemma 3.1, the handlebodies H', H'', H''', H'_1 were obtained as follows:

- (1) $H \rightarrow H'$: compressing,
- (2) $H' \rightarrow H''$: adding 1-handles,
- (3) $H'' \rightarrow H'''$: adding a final 1-handle,
- (4) $H''' \rightarrow H'_1$: adding a 3-ball along a 2-disk.

The corresponding operations for steps (3) and (4) in attaching $V_0 \cup_{\partial F_1 \times I} N(F_1)$ change the shape of the meridian disk of t_i , but it still intersects F_2 in one point. So by Proposition 2.1, $V_1 \cup_{S'_1} W_1$ has a strong (D, F_2) pair. Let $V_2 = V_1 \cup_{\partial F_2 \times I} N(F_2)$ and $W_2 = cl(K(1/q) - V_2)$. Then $V_2 \cup_{S'_2} W_2$ is another Heegaard splitting of $K(1/q)$.

In this way, we successively apply Theorem 1.1 and finally we can get the genus $2g$ strongly irreducible Heegaard splitting $(N(F) \cup T) \cup_{\Sigma'} cl(M - N(F))$. This completes the proof of Theorem 1.2. $\square$

As an illustrative example, if K is a torus knot and F is a minimal Seifert surface, then $K(1/q)$ is a Seifert fibered space over S^2 with three exceptional fibers. The splitting induced by an unknotting tunnel is the “vertical” splitting and the strongly irreducible non-minimal genus splitting is the “horizontal” splitting ([10], [18]). Hence Theorem 1.2 gives some insight into the relation of a vertical splitting

FIGURE 4. Genus two Heegaard splitting with a strong (D, F_1) pair

FIGURE 5. Strongly irreducible Heegaard splitting of genus $2g$

and a horizontal splitting of such manifolds (Figure 4), (Figure 5). We consider Seifert fibered spaces in more detail in the next section.

5. SEIFERT FIBERED SPACES

For convenience, we refer to the definitions as described in [16]. Let M be a closed orientable Seifert fibered space over an orientable base surface P of genus g with k exceptional fibers $e_1, \dots, e_k$. Let p be the natural projection map $p : M \rightarrow P$. Let D_i be a closed neighborhood of the exceptional point $p(e_i)$ in P . Let B be $cl(P - \bigcup D_i)$ and x_0 be a point in $int(B)$. For $i = 1, \dots, k$, let d_i be the union of ∂D_i and an arc α_i in B connecting x_0 to ∂D_i . Let δ_i be the union of α_i and a radius of D_i . Let $\{a_1, b_1, \dots, a_g, b_g\}$ be a collection of loops based at x_0 which cut P into a disk. We assume that (except for $\delta_i \cap d_i = \alpha_i$), all arcs and loops are disjoint except at x_0 .

5.1. Vertical splittings. When $k \geq 2$, let $(i_1, \dots, i_j) \subset \{1, \dots, k-1\}$ be a non-empty collection of distinct indices. Let $(l_1, \dots, l_{k-j-1})$ be the complementary set in $\{1, \dots, k-1\}$ to $(i_1, \dots, i_j)$. Let $\Omega(i_1, \dots, i_j)$ be a 1-complex in M obtained by lifting

$$a_1 \cup b_1 \cup \dots \cup a_g \cup b_g \cup \delta_{i_1} \cup \dots \cup \delta_{i_j} \cup d_{l_1} \cup \dots \cup d_{l_{k-j-1}}$$

to M . Let V_i be $p^{-1}(D_i)$. Set

$$H_1(i_1, \dots, i_j) = N(\Omega(i_1, \dots, i_j)) \cup V_{i_1} \cup \dots \cup V_{i_j},$$

$$H_2 = cl(M - H_1).$$

Then (H_1, H_2) is a *vertical Heegaard splitting* with splitting surface $F(i_1, \dots, i_j)$.

Now suppose M has no exceptional fiber (respectively, one exceptional fiber). Let $a_1, b_1, \dots, a_g, b_g$ and x_0 be as above and Ω be a 1-complex in M obtained by lifting $a_1 \cup b_1 \cup \dots \cup a_g \cup b_g$ to M . Let V be a closed neighborhood of the regular fiber which projects to x_0 (respectively, a closed neighborhood of the exceptional fiber, containing the regular fiber projecting to x_0). Set

$$H_1 = N(\Omega) \cup V, \quad H_2 = cl(M - H_1).$$

Then (H_1, H_2) is a *vertical splitting* of M .

We can also define a vertical splitting for a Seifert fibered space with a non-empty boundary. Note that a vertical splitting of a Seifert fibered space is uniquely determined by a partition of the collection of exceptional fibers.

5.2. Horizontal splittings. Not all Seifert fibered spaces admit a horizontal splitting. If the following procedure yields a Heegaard splitting of M , it is called a *horizontal Heegaard splitting* of M . It is similar in spirit to the construction used in Section 4.

Let e be an exceptional fiber of the closed orientable Seifert fibered space M . Then $\widehat{M} = cl(M - N(e))$ is a surface bundle over S^1 . Also let Q be the fiber surface that has one boundary component. If $I_1 \cup I_2$ is a partition of S^1 into two intervals, then $\widehat{M} = (Q \times I_1) \cup (Q \times I_2)$. Set

$$H_1 = (Q \times I_1) \cup N(e), \quad H_2 = (Q \times I_2).$$

H_2 is a handlebody. In H_1 , $N(e)$ is glued to $(Q \times I_1)$ along an annulus. If the annulus is primitive in $N(e)$, then H_1 is a handlebody. Then (H_1, H_2) is a horizontal splitting of M .

It is known that any irreducible Heegaard splitting of a Seifert fibered space is either vertical or horizontal [9].

Proof of Theorem 1.4. Any Seifert fibered space with a non-empty boundary can be obtained from a closed Seifert fibered space by drilling out some fibers. So it is enough to show the first statement of Theorem 1.4 for closed Seifert fibered spaces.

Any two vertical splittings can be distinguished by the index set $(i_1, \dots, i_j)$ of the exceptional fibers used in the definition. So without loss of generality, we only need to show that two Heegaard surfaces $F(i_1, \dots, i_j)$ and $F(i_1, \dots, i_{j-1})$ are related by the operations in Theorem 1.4. The local picture of $\Omega(i_1, \dots, i_j)$ and $\Omega(i_1, \dots, i_{j-1})$ in P near $p(e_{i_j})$ is Figure 6. Drill out e_{i_j} . Then D_{i_j} is changed to an annulus A_{i_j} (Figure 7). The lift of A_{i_j} to M is a spanning annulus A in the compression body $cl(H_2(i_1, \dots, i_{j-1}) - N(e_{i_j}))$. In the neighborhood of the lift of d_{i_j} to M , there exist a meridian disk D that intersects A in one point. Hence the Heegaard splitting has a strong (D, A) pair. Now do a construction as in the Corollary 3.6 and refill $N(e_{i_j})$. Then the manifold is again M and the Heegaard surface is changed to $F(i_1, \dots, i_j)$. This completes the proof for the first statement.

FIGURE 6. Near $p(e_{i_j})$

Let g' be the genus of Q and let $\mathcal{A} = \{c_1, d_1, \dots, c_{g'}, d_{g'}\}$ be a collection of arcs properly embedded in Q that cuts Q into a disk. Recall that e is the exceptional fiber used in the definition of horizontal splitting. Let $\{c_{i_1}, \dots, c_{i_m}, d_{j_1}, \dots, d_{j_n}\}$ be a subcollection of $\mathcal{A}$ and let $H_1 = N(e \cup c_{i_1} \cup \dots \cup c_{i_m} \cup d_{j_1} \cup \dots \cup d_{j_n})$. Take a minimal subcollection of $\mathcal{A}$ such that $H_2 = cl(M - H_1)$ is a handlebody. At least

FIGURE 7. A_{i_j} lifts to a spanning annulus A

such a collection exists since $\mathcal{A}$ satisfies it. In fact, the splitting induced by $\mathcal{A}$ is a stabilization of the horizontal splitting induced by Q .

Since e is a fiber, $cl(M - N(e))$ is a Seifert fibered space with boundary and $(cl(H_1 - N(e)), H_2)$ is a Heegaard splitting. Since an irreducible Heegaard splitting of a Seifert fibered space with boundary is vertical [15], $(cl(H_1 - N(e)), H_2)$ is a vertical splitting or its stabilization. Hence (H_1, H_2) is also a vertical splitting of M or its stabilization.

Let F be the surface $cl(Q - N(c_{i_1} \cup \dots \cup c_{i_m} \cup d_{j_1} \cup \dots \cup d_{j_n}))$. Since a longitude of the solid torus $N(e)$ is parallel to ∂Q , a meridian disk D of $N(e)$ intersects F in one point. So (H_1, H_2) has a strong (D, F) pair. Hence the horizontal splitting induced by Q is obtained from the vertical splitting (H_1, H_2) by the construction in Theorem 1.1. $\square$

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