PRODUCTS OF TOEPLITZ OPERATORS ON THE HARMONIC BERGMAN SPACE

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(Communicated by Nigel J. Kalton)

ABSTRACT. In this paper, we first discuss some basic results concerning Toeplitz operators with quasihomogeneous symbols (i.e., symbols being of the form $e^{ip\theta}\varphi$, where φ is a radial function) on the harmonic Bergman space. Then we determine when the product of two Toeplitz operators with quasihomogeneous symbols is a Toeplitz operator.

1. INTRODUCTION

Let dA denote the Lebesgue area measure on the unit disk D , normalized so that the measure of D equals 1. $L^2(D, dA)$ is the Hilbert space of Lebesgue square integrable functions on D with the inner product

$$\langle f, g \rangle = \int_D f(z) \overline{g(z)} dA(z).$$

The harmonic Bergman space L_h^2 is the closed subspace of $L^2(D, dA)$ consisting of the harmonic functions on D . For $u \in L^\infty(D, dA)$, the Toeplitz operator T_u with symbol u is the operator on L_h^2 defined by

$$T_u f = Q(uf)$$

for $f \in L_h^2$, where Q is the orthogonal projection of $L^2(D, dA)$ onto L_h^2 . Each point evaluation is easily verified to be a bounded linear functional on L_h^2 . Hence, for each $z \in D$, there exists a unique function R_z (called the harmonic Bergman kernel) in L_h^2 that has the reproducing property

$$f(z) = \langle f, R_z \rangle$$

for every $f \in L_h^2$.

Given $z \in D$, let $K_z(w) = \frac{1}{(1-w\bar{z})^2}$ be the well-known reproducing kernel for the analytic Bergman space L_a^2 consisting of all L^2 -analytic functions on D . The well-known Bergman projection P is then the integral operator

$$Pf(z) = \int_D f(w) \overline{K_z(w)} dA(w)$$

Received by the editors June 30, 2009, and, in revised form, September 7, 2009.

2010 *Mathematics Subject Classification*. Primary 47B35; Secondary 47B38.

Key words and phrases. Toeplitz operators, harmonic Bergman space, quasihomogeneous symbols.

The second author was supported in part by the National Natural Science Foundation of China (Grant Nos. 10971153, 10671141).

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for $f \in L^2(D, dA)$. Since $L_h^2 = L_a^2 + \overline{L_a^2}$, it is easily checked that $R_z = K_z + \overline{K_z} - 1$. Thus, Q can be represented by

$$Qf = Pf + \overline{P\bar{f}} - Pf(0).$$

Recall that a function φ on D is radial if $\varphi(z)$ depends only on $|z|$. Then for each radial function φ , we define the function $\tilde{\varphi}$ on $[0, 1]$ by $\tilde{\varphi}(r) = \varphi(re^{i\theta})$, where (r, θ) are the polar coordinates of the complex variable z . It is obvious that $\tilde{\varphi}$ is well defined. In the following, we shall often identify a bounded radial function φ on D with the corresponding function $\tilde{\varphi}$ defined on $[0, 1]$.

A bounded function f is said to be quasihomogeneous of degree $p \in \mathbb{Z}$ if and only if

$$f(re^{i\theta}) = e^{ip\theta} \varphi(r),$$

where φ is a radial function (see [9]). In this case the associated Toeplitz operator T_f is also called a quasihomogeneous Toeplitz operator of degree p .

In 1964, Brown and Halmos [2] proved that $T_f T_g = T_h$ on the Hardy space H^2 of the unit circle if and only if: either (I) g is analytic, or (II) f is conjugate analytic. They also showed that, in both cases $h = fg$. In the setting of an analytic Bergman space, conditions (I) and (II) are still sufficient, but they are no longer necessary. Ahern and Čučković [1] showed that a Brown-Halmos type result holds for Toeplitz operators with harmonic symbols on L_a^2 . Later in [6], Louhichi, Strouse and Zakariasy gave necessary and sufficient conditions for the product of two quasihomogeneous Toeplitz operators to be a Toeplitz operator. Then Louhichi and Zakariasy [7] characterized commuting Toeplitz operators on L_a^2 with quasihomogeneous symbols. The main reason for them to study such a family of symbols is that any function f in $L^2(D, dA)$ has the polar decomposition

$$f(re^{i\theta}) = \sum_{k \in \mathbb{Z}} e^{ik\theta} f_k(r),$$

where f_k are radial functions in $L^2([0, 1], r dr)$. Recently, we studied some algebraic properties of quasihomogeneous Toeplitz operators on the analytic Bergman space of the unit ball in [5] and [10].

The theory of Toeplitz operators on L_h^2 is quite different from that on L_a^2 . For example, Choe and Lee [3] showed that two analytic Toeplitz operators on L_h^2 commute only when their symbols and the constant function 1 are linearly dependent, but analytic Toeplitz operators always commute on L_a^2 . In this paper, we shall consider when the product of two quasihomogeneous Toeplitz operators on L_h^2 is a Toeplitz operator.

We first give the following basic result concerning quasihomogeneous Toeplitz operators on L_h^2 .

Theorem 1.1. *Let $p \in \mathbb{Z}$ and let f be a bounded function on D . Then the following assertions are equivalent:*

- (a) *For each $k \in \mathbb{N}$, there exists $\lambda_k \in \mathbb{C}$ such that*

$$T_f(r^k e^{ik\theta}) = \lambda_k r^{|k+p|} e^{i(k+p)\theta}.$$

- (b) *f is a quasihomogeneous function of degree p .*

Next, we show a necessary condition for the product of two quasihomogeneous Toeplitz operators on L_h^2 to be a Toeplitz operator.

Theorem 1.2. *Let f_1 and f_2 be two bounded quasihomogeneous functions on D of degrees k_1 and k_2 respectively. If there exists a bounded function h such that $T_{f_1}T_{f_2} = T_h$, then h is a quasihomogeneous function of degree $k_1 + k_2$.*

Louhichi, Strouse and Zakariasy [6] gave lots of examples of radial functions ψ_1 and ψ_2 such that $T_{e^{ip\theta}\psi_1}T_{e^{-is\theta}\psi_2}$ on L_a^2 is not a Toeplitz operator. Moreover, Theorem 6.7 of [6] illustrates the difficulty in characterizing more precisely those pairs of Toeplitz operators whose product is a Toeplitz operator. The problem of determining when the product of two quasihomogeneous Toeplitz operators on L_a^2 is a Toeplitz operators still remains open. The following theorem characterizes when the product of two Toeplitz operators on L_h^2 with certain quasihomogeneous symbols is a Toeplitz operator.

Theorem 1.3. *Let $l_1, l_2 > 0$ and let $k_1, k_2 \in \mathbb{Z}$. Then $T_{e^{ik_1\theta}r^{l_1}}T_{e^{ik_2\theta}r^{l_2}}$ is equal to a Toeplitz operator if and only if $k_1 = k_2 = 0$. In this case:*

$$T_{r^{l_1}}T_{r^{l_2}} = \begin{cases} \frac{l_1}{l_1-l_2}T_{r^{l_1}} - \frac{l_2}{l_1-l_2}T_{r^{l_2}} & \text{if } l_1 \neq l_2 \\ T_{r^{l_1}(1+l_1 \log r)} & \text{if } l_1 = l_2. \end{cases}$$

2. SOME PRELIMINARY RESULTS

One of the most useful tools in the following calculations will be the Mellin transform.

The Mellin transform $\hat{\varphi}$ of a function $\varphi \in L^1([0, 1], r dr)$ is defined by the equation

$$\hat{\varphi}(z) = \int_0^1 \varphi(s) s^{z-1} ds.$$

It is clear that $\hat{\varphi}$ is well defined on the right half-plane $\{z : \operatorname{Re} z \geq 2\}$ and analytic on $\{z : \operatorname{Re} z > 2\}$. The use of the Mellin transform in the study of Toeplitz operators was introduced for the first time in [4]. A direct calculation gives the following lemma, which we shall use often.

Lemma 2.1. *Let $p \in \mathbb{Z}$ and let φ be a bounded radial function. Then for each $k \in \mathbb{N}$,*

$$\begin{aligned} T_{e^{ip\theta}\varphi}(z^k) &= \begin{cases} 2(k+p+1)\hat{\varphi}(2k+p+2)z^{k+p} & \text{if } k \geq -p \\ 2(-k-p+1)\hat{\varphi}(-p+2)\bar{z}^{-k-p} & \text{if } k < -p, \end{cases} \\ T_{e^{ip\theta}\varphi}(\bar{z}^k) &= \begin{cases} 2(k-p+1)\hat{\varphi}(2k-p+2)\bar{z}^{k-p} & \text{if } k \geq p \\ 2(p-k+1)\hat{\varphi}(p+2)z^{p-k} & \text{if } k < p. \end{cases} \end{aligned}$$

Proof. Since φ is a bounded radial function, we see that for each $k \in \mathbb{N}$,

$$\begin{aligned} P(e^{ip\theta}\varphi w^k)(z) &= \int_D \varphi(r) e^{ip\theta} w^k \overline{K_z(w)} dA(w) \\ &= \sum_{j=0}^{\infty} (j+1) z^j \int_D \varphi(r) e^{ip\theta} w^k \bar{w}^j dA(w) \\ &= \begin{cases} 2(k+p+1)\hat{\varphi}(2k+p+2)z^{k+p} & \text{if } k \geq -p \\ 0 & \text{if } k < -p \end{cases} \end{aligned}$$

and

$$\begin{aligned} P(e^{ip\theta}\varphi\overline{w}^k)(z) &= \int_D \varphi(r)e^{ip\theta}\overline{w}^k\overline{K_z(w)}dA(w) \\ &= \sum_{j=0}^{\infty} (j+1)z^j \int_D \varphi(r)e^{ip\theta}\overline{w}^{k+j}dA(w) \\ &= \begin{cases} 2(p-k+1)\widehat{\varphi}(p+2)z^{p-k} & \text{if } k \leq p \\ 0 & \text{if } k > p. \end{cases} \end{aligned}$$

Note that $\overline{\varphi}$ is still radial and

$$\overline{\widehat{\varphi}(n)} = \widehat{\overline{\varphi}}(n), \quad \forall n \in \mathbb{N}.$$

Thus,

$$\begin{aligned} T_{e^{ip\theta}\varphi}(w^k)(z) &= P(e^{ip\theta}\varphi w^k)(z) + \overline{P(\overline{\varphi}e^{-ip\theta}\overline{w}^k)}(z) - P(e^{ip\theta}\varphi w^k)(0) \\ &= \begin{cases} 2(k+p+1)\widehat{\varphi}(2k+p+2)z^{k+p} & \text{if } k \geq -p \\ 2(-k-p+1)\widehat{\varphi}(-p+2)\overline{z}^{-k-p} & \text{if } k < -p \end{cases} \end{aligned}$$

and

$$\begin{aligned} T_{e^{ip\theta}\varphi}(\overline{w}^k)(z) &= P(e^{ip\theta}\varphi\overline{w}^k)(z) + \overline{P(\overline{\varphi}e^{-ip\theta}\overline{w}^k)}(z) - P(e^{ip\theta}\varphi\overline{w}^k)(0) \\ &= \begin{cases} 2(k-p+1)\widehat{\varphi}(2k-p+2)\overline{z}^{k-p} & \text{if } k \geq p \\ 2(p-k+1)\widehat{\varphi}(p+2)z^{p-k} & \text{if } k < p. \end{cases} \end{aligned}$$

This completes the proof. $\square$

Remark 2.2. Let $p, s \in \mathbb{Z}$. Then Lemma 2.1 implies that the image of $r^{|s|}e^{is\theta}$ by a quasihomogeneous Toeplitz operator of degree p is $\lambda_{p,s}r^{|p+s|}e^{i(p+s)\theta}$ for some constant $\lambda_{p,s}$.

Lemma 2.3. Let f be a bounded function on D . Then the following conditions are equivalent:

- (a) For any $k \in \mathbb{N}$, there exists $\lambda_k \in \mathbb{C}$ such that $T_f(z^k) = \lambda_k z^k$.
- (b) For any $k \in \mathbb{N}$, there exists $\lambda_k \in \mathbb{C}$ such that $T_f(\overline{z}^k) = \lambda_k \overline{z}^k$.

Proof. First suppose that (a) holds, so $T_f(z^k) = \lambda_k z^k$ for each $k \in \mathbb{N}$. Then for any $m \in \mathbb{N}$, we have

$$\begin{aligned} \langle T_f \overline{z}^k, \overline{z}^m \rangle &= \langle f z^m, z^k \rangle = \langle \lambda_m z^m, z^k \rangle \\ &= \begin{cases} \lambda_k \langle z^k, z^k \rangle & \text{if } m = k \\ 0 & \text{if } m \neq k, \end{cases} \end{aligned}$$

which implies that

$$\langle T_f \overline{z}^k, \overline{z}^m \rangle = \langle \lambda_k \overline{z}^k, \overline{z}^m \rangle.$$

Also, a similar calculation shows that

$$\langle T_f \overline{z}^k, z^m \rangle = \langle \lambda_k \overline{z}^k, z^m \rangle.$$

Since $\{\sqrt{k+1}z^k\}_{k=0}^{\infty} \cup \{\sqrt{k+1}\overline{z}^k\}_{k=1}^{\infty}$ is an orthonormal basis for the harmonic Bergman space, we have

$$T_f \overline{z}^k = \lambda_k \overline{z}^k$$

for any $k \in \mathbb{N}$.

By a similar argument, one can also show that (b) implies (a). This completes the proof. $\square$

When considering the product of two Toeplitz operators we often use a well-known fact about the Mellin convolution of their symbols. If f and g are in $L^1([0, 1], r dr)$, then their Mellin convolution is defined by

$$(f *_M g)(r) = \int_r^1 f\left(\frac{r}{t}\right)g(t)\frac{dt}{t}, \quad 0 \leq r < 1.$$

The Mellin convolution theorem states that

$$(2.1) \quad \widehat{f *_M g}(s) = \widehat{f}(s)\widehat{g}(s)$$

and that if f and g are bounded, then so is $f *_M g$.

It is important and helpful to know that the Mellin transform is uniquely determined by its value on an arithmetic sequence of integers. In fact we have the following classical theorem (see [8, p. 102]).

Theorem 2.4. *Suppose that f is a bounded analytic function on $\{z : \operatorname{Re} z > 0\}$ which vanishes at the pairwise distinct points $z_1, z_2, \dots$, where*

- i) $\inf\{|z_n|\} > 0$ and
- ii) $\sum_{n \geq 1} \operatorname{Re}\left(\frac{1}{z_n}\right) = \infty$.

Then f vanishes identically on $\{z : \operatorname{Re} z > 0\}$.

Remark 2.5. We shall use this theorem to show that if $\varphi \in L^1([0, 1], r dr)$ and if there exists a sequence $(n_k)_{k \geq 0} \subset \mathbb{N}$ such that

$$\widehat{\varphi}(n_k) = 0 \quad \text{and} \quad \sum_{k \geq 0} \frac{1}{n_k} = \infty,$$

then $\widehat{\varphi}(z) = 0$ for all $z \in \{z : \operatorname{Re} z > 2\}$ and so $\varphi = 0$.

3. PROOFS OF THE THEOREMS

We start this section with the notion of radialization (see [11]). Letting f be a bounded function on D , we define the “radialization” of f by

$$\operatorname{rad}(f)(z) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}z) dt.$$

It is clear that a function f is radial if and only if $\operatorname{rad}(f) = f$.

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. If f is a quasihomogeneous function of degree p , then (a) is a direct consequence of Remark 2.2.

Conversely, suppose (a) is true. Then for any $k \in \mathbb{N}$,

$$\begin{aligned} T_{\overline{f}}(\overline{w}^k)(z) &= \langle \overline{f}\overline{w}^k, K_z + \overline{K_z} - 1 \rangle = \overline{\langle fw^k, K_z + \overline{K_z} - 1 \rangle} = \overline{T_f(w^k)(z)} \\ &= \overline{\lambda_k} r^{|k+p|\theta} e^{-i(k+p)\theta}. \end{aligned}$$

Thus, if $p \geq 0$,

$$\begin{aligned} T_{\bar{z}^p f}(z^k) &= \langle \bar{w}^p f w^k, K_z + \overline{K_z} - 1 \rangle = \langle f w^k, w^p K_z \rangle + \langle \bar{w}^p, \bar{f} \bar{w}^k (\overline{K_z} - 1) \rangle \\ &= \sum_{j=0}^{\infty} (j+1) z^j \langle T_f(w^k), w^{j+p} \rangle + \sum_{j=1}^{\infty} (j+1) \bar{z}^j \langle \bar{w}^p, T_{\bar{f}}(\bar{w}^{j+k}) \rangle \\ &= \sum_{j=0}^{\infty} \lambda_k (j+1) z^j \langle w^{k+p}, w^{j+p} \rangle + \sum_{j=1}^{\infty} \lambda_{j+k} (j+1) \bar{z}^j \langle \bar{w}^p, \bar{w}^{j+k+p} \rangle \\ &= \frac{\lambda_k (k+1)}{k+p+1} z^k, \end{aligned}$$

and if $p < 0$,

$$\begin{aligned} T_{z^{-p} f}(z^k) &= \langle f w^{k-p}, K_z + \overline{K_z} - 1 \rangle = \langle T_f(w^{k-p}), K_z + \overline{K_z} - 1 \rangle \\ &= \langle \lambda_{k-p} w^k, K_z + \overline{K_z} - 1 \rangle \\ &= \lambda_{k-p} z^k. \end{aligned}$$

In particular, if we denote $\varphi = r^{|p|} e^{-ip\theta} f$, then it follows that

$$(3.1) \quad T_{\varphi}(z^k) = \delta_k z^k$$

for some constant $\delta_k \in \mathbb{C}$. In the following we will prove that φ is radial, which implies that f is a quasihomogeneous function of degree p .

Now, writing out the integrals and changing the order of integration, we see that for each $m \in \mathbb{N}$,

$$\begin{aligned} \langle T_{rad(\varphi)} z^k, z^m \rangle &= \frac{1}{2\pi} \left(\int_0^{2\pi} e^{i(m-k)t} dt \right) \langle T_{\varphi} z^k, z^m \rangle \\ &= \begin{cases} \langle T_{\varphi} z^k, z^m \rangle & \text{if } k = m \\ 0 & \text{if } k \neq m. \end{cases} \end{aligned}$$

On the other hand,

$$\langle T_{rad(\varphi)} z^k, \bar{z}^m \rangle = \frac{1}{2\pi} \left(\int_0^{2\pi} e^{i(k+m)t} dt \right) \langle T_{\varphi} z^k, \bar{z}^m \rangle = 0, \quad \forall m \in \mathbb{Z}_+.$$

Then by (3.1), we obtain

$$\langle T_{rad(\varphi)} z^k, z^m \rangle = \langle T_{\varphi} z^k, z^m \rangle$$

and

$$\langle T_{rad(\varphi)} z^k, \bar{z}^m \rangle = \langle T_{\varphi} z^k, \bar{z}^m \rangle$$

for all $m \in \mathbb{N}$. Thus

$$(3.2) \quad T_{rad(\varphi)} z^k = T_{\varphi} z^k.$$

Moreover, by (3.1) and Lemma 2.3, we have

$$T_{\varphi} \bar{z}^k = \delta_k \bar{z}^k, \quad \forall k \in \mathbb{N}.$$

Then by a similar discussion, we can get

$$(3.3) \quad T_{rad(\varphi)} \bar{z}^k = T_{\varphi} \bar{z}^k.$$

From (3.2) and (3.3) we deduce that $T_{rad(\varphi)} = T_{\varphi}$, which implies $\varphi(z) = rad(\varphi)(z)$. Hence φ is radial. This completes the proof. $\square$

Next we discuss the proof of Theorem 1.2.

Proof of Theorem 1.2. Suppose $f_1 = e^{ik_1\theta}\varphi_1(r)$ and $f_2 = e^{ik_2\theta}\varphi_2(r)$. Then for each $k \in \mathbb{N}$, it follows from Remark 2.2 that

$$\begin{aligned} T_{f_1}T_{f_2}(r^k e^{ik\theta}) &= T_{e^{ik_1\theta}\varphi_1}T_{e^{ik_2\theta}\varphi_2}(r^k e^{ik\theta}) \\ &= \lambda_{k_2,k}T_{e^{ik_1\theta}\varphi_1}(r^{|k+k_2|}e^{i(k+k_2)\theta}) \\ &= \lambda_{k_1,k+k_2}\lambda_{k_2,k}r^{|k+k_1+k_2|}e^{i(k+k_1+k_2)\theta} \end{aligned}$$

for some constants $\lambda_{k_2,k}$ and $\lambda_{k_1,k+k_2}$. Hence, if $T_{f_1}T_{f_2} = T_h$, then by Theorem 1.1, h is a quasihomogeneous function of degree $k_1 + k_2$. This completes the proof. $\square$

The following corollary, which characterizes the idempotent Toeplitz operators, is an immediate consequence of Theorem 1.2.

Corollary 3.1. *If f is a quasihomogeneous function of degree different from 0 and if $T_f^2 = T_f$, then $f = 0$.*

We now turn to the proof of Theorem 1.3.

Proof of Theorem 1.3. Assume $T_{e^{ik_1\theta}r^{l_1}}T_{e^{ik_2\theta}r^{l_2}}$ is a Toeplitz operator. By Theorem 1.2, this operator is of the form $T_{e^{i(k_1+k_2)\theta}\psi(r)}$. Then for any $k \in \mathbb{N}$ such that $k + k_2 \geq 0$ and $k + k_1 + k_2 \geq 0$, the equality $T_{e^{ik_1\theta}r^{l_1}}T_{e^{ik_2\theta}r^{l_2}}(z^k) = T_{e^{i(k_1+k_2)\theta}\psi}(z^k)$ together with Lemma 2.1 gives

$$(3.4) \quad \widehat{r^{l_1}}(2k + 2k_2 + k_1 + 2)\widehat{r^{l_2}}(2k + k_2 + 2) = \frac{1}{2k + 2k_2 + 2}\widehat{\psi}(2k + k_2 + k_1 + 2).$$

Denote

$$k' = \min\{0, k_1, k_2, k_1 + k_2\}.$$

Then by (2.1) and (3.4), we have

$$(r^{l_1+k_1+k_2-k'} *_M r^{l_2-k'})^\wedge(2k + k_2 + k' + 2) = (r^{k_2-k'} *_M r^{k_1-k'}\psi)^\wedge(2k + k_2 + k' + 2).$$

Next, by Remark 2.5, the above equation yields that

$$r^{l_1+k_1+k_2-k'} *_M r^{l_2-k'} = r^{k_2-k'} *_M (r^{k_1-k'}\psi).$$

The above equation is equivalent to

$$\int_r^1 \psi(t)t^{k_1-k_2-1}dt = r^{l_1+k_1} \int_r^1 t^{l_2-l_1-k_1-k_2-1}dt,$$

from which we can get

$$(3.5) \quad \psi(r) = \begin{cases} \frac{l_1+k_1}{l_1+k_1-l_2+k_2}r^{l_1+k_2} - \frac{l_2-k_2}{l_1+k_1-l_2+k_2}r^{l_2-k_1} & \text{if } l_1 + k_1 \neq l_2 - k_2 \\ r^{l_1+k_2}(1 + (l_1 + k_1)\log r) & \text{if } l_1 + k_1 = l_2 - k_2. \end{cases}$$

Similarly, it follows from $T_{e^{ik_1\theta}r^{l_1}}T_{e^{ik_2\theta}r^{l_2}}(\bar{z}^k) = T_{e^{i(k_1+k_2)\theta}\psi}(\bar{z}^k)$ that

$$r^{l_1-k'} *_M r^{l_2+k_1+k_2-k'} = r^{k_1-k'} *_M (r^{k_2-k'}\psi),$$

from which we can get

$$(3.6) \quad \psi(r) = \begin{cases} \frac{l_1-k_1}{l_1-k_1-l_2-k_2}r^{l_1-k_2} - \frac{l_2+k_2}{l_1-k_1-l_2-k_2}r^{l_2+k_1} & \text{if } l_1 - k_1 \neq l_2 + k_2 \\ r^{l_1-k_2}(1 + (l_1 - k_1)\log r) & \text{if } l_1 - k_1 = l_2 + k_2. \end{cases}$$

Since $l_1, l_2 > 0$, one can easily see that (3.5) and (3.6) are the same functions if and only if one of the following holds:

- (1) $k_1 = k_2 = 0$,
- (2) $l_1 = l_2$ and $k_1 = k_2 \neq 0$,

(3) $k_1 = l_1$ and $k_2 = l_2$,

(4) $k_1 = -l_1$ and $k_2 = -l_2$.

To show that only condition (1) holds, we next prove that conditions (2), (3) and (4) are all impossible to satisfy.

First, assume $l_1 = l_2$ and $k_1 = k_2 \neq 0$. Without loss of generality, we can also assume that $k_1 > 0$ and $k_2 > 0$, for otherwise we could take the adjoints. Then it follows from (3.5) that

$$\psi(r) = \frac{l_1 + k_1}{2k_1} r^{l_1 + k_1} - \frac{l_1 - k_1}{2k_1} r^{l_1 - k_1}.$$

Now, consider the equality $T_{e^{ik_1\theta}r^{l_1}} T_{e^{ik_1\theta}r^{l_1}}(\bar{z}^{k_1}) = T_{e^{2ik_1\theta}\psi}(\bar{z}^{k_1})$. Then we get

$$\frac{4(k_1 + 1)}{(l_1 + k_1 + 2)^2} = \frac{(l_1 + k_1)(k_1 + 1)}{k_1(l_1 + 3k_1 + 2)} - \frac{(l_1 - k_1)(k_1 + 1)}{k_1(l_1 + k_1 + 2)},$$

and hence

$$k_1(k_1 + l_1) = 0,$$

which is a contradiction since $k_1 > 0$ and $l_1 > 0$.

Next, assume $k_1 = l_1$ and $k_2 = l_2$. Then $\psi(r) = r^{l_1 + l_2}$. So the equality $T_{z^{l_1}} T_{z^{l_2}}(\bar{z}^{l_2}) = T_{z^{l_1 + l_2}}(\bar{z}^{l_2})$ gives that

$$\frac{2}{2l_2 + 2} = \frac{2l_1 + 2}{2l_1 + 2l_2 + 2},$$

and hence

$$l_1 l_2 = 0,$$

which is a contradiction since $l_1 > 0$ and $l_2 > 0$.

Finally, assume $k_1 = -l_1$ and $k_2 = -l_2$. Similarly, it follows that $\psi(r) = r^{l_1 + l_2}$. Then the equality $T_{\bar{z}^{l_1}} T_{\bar{z}^{l_2}}(z^{l_2}) = T_{\bar{z}^{l_1 + l_2}}(z^{l_2})$ gives that

$$l_1 l_2 = 0,$$

also a contradiction.

Consequently, if $T_{e^{ik_1\theta}r^{l_1}} T_{e^{ik_2\theta}r^{l_2}}$ is a Toeplitz operator, then $k_1 = k_2 = 0$.

Conversely, the Toeplitz operator T_ψ , whose symbol ψ given by

$$\psi(r) = \begin{cases} \frac{l_1}{l_1 - l_2} r^{l_1} - \frac{l_2}{l_1 - l_2} r^{l_2} & \text{if } l_1 \neq l_2 \\ r^{l_1} (1 + l_1 \log r) & \text{if } l_1 = l_2 \end{cases}$$

is obtained by taking $k_1 = k_2 = 0$ in (3.5) and (3.6), takes the same values on z^k and $\bar{z}^k$ for all $k \in \mathbb{N}$ as the product $T_{r^{l_1}} T_{r^{l_2}}$. This completes the proof. $\square$

ACKNOWLEDGMENT

We thank the referee for carefully reading the paper and providing corrections and suggestions for improvements.

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