DIAMOND, GCH AND WEAK SQUARE

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ABSTRACT. Shelah proved recently that if $\kappa > \omega$ and $S \subseteq \kappa^+$ is a stationary set of ordinals of cofinality different from $\text{cf}(\kappa)$, then $2^\kappa = \kappa^+$ implies $\diamond_{\kappa^+}(S)$. We show that for singular κ , an elaboration on his argument allows us to derive $\diamond_{\kappa^+}(T)$ from $2^\kappa = \kappa^+ + \square_\kappa^*$ where $T = \{\delta < \kappa^+ \mid \text{cf}(\delta) = \text{cf}(\kappa)\}$. This gives a strong restriction on the existence of saturated ideals on κ^+ .

INTRODUCTION

It is a well-known fact that $\diamond_\lambda$ implies $2^{<\lambda} = \lambda$. In many situations the converse is also true. Jensen [3] proved that CH does not imply $\diamond$, so when looking for the converse one has to focus on $\lambda > \omega_1$. Let

$$S_\mu^\lambda = \{\delta < \lambda \mid \text{cf}(\delta) = \mu\}$$

and

$$T_\kappa = S_{\text{cf}(\kappa)}^{\kappa^+}.$$

Gregory observed that GCH below ω_2 implies $\diamond_{\omega_2}(S_{\omega_2}^{\omega_2})$. A sequence of improvements on his result, mainly by Gregory [5], Jensen (unpublished) and Shelah [9], resulted in the following theorem, whose proof can be found in [2].

Theorem 0.1 (Gregory, Jensen, Shelah). *If $2^{<\kappa} = \kappa$ and $2^\kappa = \kappa^+$, then $\diamond_{\kappa^+}(S)$ holds whenever $S \subseteq \kappa^+$ is a stationary set of ordinals of cofinality different from $\text{cf}(\kappa)$. If κ is singular and additionally $\square_\kappa$ holds, then $\diamond_{\kappa^+}(T_\kappa)$.*

Shelah also proved that for regular κ , the condition $2^\kappa = \kappa^+ + \square_\kappa$ is not sufficient to guarantee $\diamond_{\kappa^+}(T_\kappa)$, so the absolute ZFC result is possible only for singular κ .

The question remained whether the localized GCH, i.e. the equality $2^\kappa = \kappa^+$, alone implies $\diamond_{\kappa^+}$. Shelah proved this to be true for sufficiently large κ , and recently [10] found an argument that proves it for every uncountable cardinal κ ; see Komjáth's paper [7] for a simplified proof and an elaboration on Shelah's argument.

Theorem 0.2 (Shelah). *Let $\kappa > \omega$ and $2^\kappa = \kappa^+$. Then $\diamond_{\kappa^+}(S)$ holds for every stationary $S \subseteq \kappa^+$ that is disjoint from T_κ .*

This note combines arguments from the proof of Theorem 0.1 with Shelah's argument for Theorem 0.2 to give a proof of $\diamond_{\kappa^+}(S)$ for $S \subseteq T_\kappa$.

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Theorem 0.3 (Main Theorem). *Assume κ is a singular cardinal and $T \subseteq T_\kappa$ is stationary with stationarily many reflection points. Then*

$$2^\kappa = \kappa^+ + \square_\kappa^* \implies \diamond_{\kappa^+}(T).$$

It was proved by Cummings, Foreman and Magidor [1] that for singular κ , the principle $\square_\kappa^*$ is consistent with the requirement that every stationary $T \subseteq T_\kappa$ has stationarily many reflection points. Consequently, in their model we have $\diamond_{\kappa^+}(S)$ for all stationary $S \subseteq \kappa^+$.

Corollary 0.4. *Assume κ is a singular cardinal. Then*

$$2^\kappa = \kappa^+ + \square_\kappa^* \implies \diamond_\kappa(T_\kappa).$$

Both the above theorem and its corollary provide a very strong restriction on the existence of saturated ideals on κ^+ and provide a close link between the study of such ideals and PCF theory.

Rinot [8] recently extended the result of this paper to the situation where weak square is replaced by a variant of the approachability property and also showed that, relative to the existence of a supercompact cardinal, it is consistent that $\square_\kappa^*$ fails but $\diamond_{\kappa^+}(S)$ holds for every stationary $S \subseteq \kappa^+$. Also, by a recent result of Gitik and Rinot [4], the failure of $\diamond_{\aleph_{\omega+1}}(S)$ is consistent with $\text{GCH} + \text{AP}_{\aleph_\omega}$, again relative to a supercompact cardinal.

1. THE ARGUMENT

We begin with splitting Shelah's argument into two steps. We first isolate a combinatorial statement that alone implies the existence of a $\diamond_\lambda(S)$ -sequence in ZFC; we denote this statement by $\bigcirc_\lambda(S)$. This statement is implicit in the arguments in Shelah [10] and Komjáth [7]. It turns out that the implication $\bigcirc_\lambda(S) \implies \diamond_\lambda(S)$ is true regardless of whether λ is a successor cardinal or not. The second step is a proof that $\bigcirc_\lambda(S)$ holds, which relies on the localized GCH if $\lambda = \kappa^+$ and S concentrates on points of cofinality distinct from $\text{cf}(\kappa)$, giving the original Shelah's result, and on the weak square if κ is singular and S concentrates on points of cofinality $\text{cf}(\kappa)$, giving the result in Theorem 0.3. Our approach owes a lot to Komjáth's exposition in [7].

Definition 1.1. Let λ be a regular cardinal and $S \subseteq \lambda$. We say that the pair $\langle x_\xi \mid \xi < \lambda \rangle, \langle A_\delta \mid \delta \in S \rangle$ witnesses $\bigcirc_\lambda(S)$ iff the following three conditions are met.

- (a) $\langle x_\xi \mid \xi < \lambda \rangle$ is an enumeration of $[\lambda]^{<\lambda}$.
- (b) $A_\delta \subseteq \delta$ and $\text{card}(A_\delta) < \text{card}(\delta)$ whenever $\delta \in S$.
- (c) For every $Z \subseteq \lambda$ there is a stationary $S' \subseteq S$ such that for every $\delta \in S'$ there are unboundedly many $\alpha < \delta$ for which there is $\beta < \delta$ satisfying $\alpha, \beta \in A_\delta$ and $Z \cap \alpha = x_\beta$.

We say that $\bigcirc_\lambda(S)$ holds iff there are $\langle x_\xi \rangle_\xi$ and $\langle A_\delta \rangle_\delta$ as above.

Notice that $\bigcirc_\lambda(S)$ postulates the existence of an enumeration of $[\lambda]^{<\lambda}$ of length λ , so it imposes some constraints on the behaviour of the exponential function below λ . In particular, if $\lambda = \kappa^+$, then $\bigcirc_\lambda(S)$ implies $2^\kappa = \kappa^+$. Notice also that (b) in the above definition stipulates that the cardinality of A_δ be strictly smaller than that of δ , which together with (c) implies that without loss of generality S can be viewed as a set of singular ordinals. Of course, this has a non-trivial meaning only when λ is inaccessible. Finally observe that if there is a pair $\langle x_\xi \mid \xi < \lambda \rangle$,

$\langle A_\delta \mid \delta \in S \rangle$ witnessing $\bigcirc_\lambda(S)$, then for every enumeration $\langle x'_\xi \mid \xi < \lambda \rangle$ there is a sequence $\langle A'_\delta \mid \delta \in S \rangle$ such that the pair $\langle x'_\xi \rangle_\xi, \langle A'_\delta \rangle_\delta$ witnesses $\bigcirc_\lambda(S)$. To see this, pick any $f : \lambda \rightarrow \lambda$ such that $x_\beta = x'_{f(\beta)}$ for all $\beta < \lambda$ and let $A'_\delta = A_\delta \cup f[A_\delta]$ for all $\delta \in S$ satisfying $f[\delta] \subseteq \delta$.

Lemma 1.2. *Let λ be a regular cardinal, $S \subseteq \lambda$ and $\bigcirc_\lambda(S)$ hold. Then there is a pair $\langle x_\xi \mid \xi < \lambda \rangle, \langle A_\delta \mid \delta \in S \rangle$ satisfying the following:*

- (a) $\langle x_\xi \mid \xi < \lambda \rangle$ is an enumeration of $[\lambda \times \lambda]^{<\lambda}$.
- (b) $A_\delta \subseteq \delta$ and $\text{card}(A_\delta) < \text{card}(\delta)$ whenever $\delta \in S$.
- (c) For every $Z \subseteq \lambda \times \lambda$ there is a stationary $S' \subseteq S$ such that for every $\delta \in S'$ there are unboundedly many $\alpha < \delta$ for which there is $\beta < \delta$ satisfying $\alpha, \beta \in A_\delta$ and $Z \cap (\alpha \times \alpha) = x_\beta \cap (\alpha \times \alpha)$.

Proof. Pick a pair $\langle y_\xi \mid \xi < \lambda \rangle, \langle B_\delta \mid \delta \in S \rangle$ witnessing $\bigcirc_\lambda(S)$. Let $f : \lambda \times \lambda \rightarrow \lambda$ be a bijection and $C_f = \{\delta < \lambda \mid f[\delta \times \delta] = \delta\}$.

For each $\delta \in S$ pick C_δ to be a subset of $\lim(C_f) \cap \delta$ of size strictly smaller than $\text{card}(\delta)$ that is cofinal in δ if such a set exists; let $C_\delta = \emptyset$ otherwise. Letting $x_\beta = f^{-1}[y_\beta]$ and $A_\delta = B_\delta \cup C_\delta$, we obtain a pair $\langle x_\xi \mid \xi < \lambda \rangle, \langle A_\delta \mid \delta \in S \rangle$ as in the conclusion of the lemma. To see this, it suffices to verify clause (c) in the statement of the lemma.

Given any $Z \subseteq \lambda \times \lambda$, let $S' \subseteq S$ be the stationary set obtained by applying $\bigcirc_\lambda(S)$ to $f[Z]$. Let $\delta \in S' \cap \lim(C_f)$. If $\bar{\alpha} < \delta$, pick $\alpha \in C_\delta$ such that $\bar{\alpha} \leq \alpha$. Since S' satisfies (c) in Definition 1.1 with $f[Z], y_\beta$ and B_δ in place of Z, x_β and A_δ , there are $\alpha', \beta \in B_\delta$ such that $\alpha \leq \alpha'$ and $f[Z] \cap \alpha' = y_\beta$. Then $f[Z] \cap \alpha = y_\beta \cap \alpha$ and the conclusion follows immediately from the fact that $\alpha \in C_f$. $\square$

With the statement $\bigcirc_\lambda(S)$ in hand, one can reformulate the first step in Shelah's argument as the following proposition. It reduces the proof of $\diamond_\lambda(S)$ to the proof of $\bigcirc_\lambda(S)$ and works even for cardinals λ that are not successors, which is slightly more than Shelah originally proved. The second step in Shelah's argument can then be viewed as a proof of $\bigcirc_{\kappa^+}(S)$ from the localized GCH. We will show how to obtain $\bigcirc_{\kappa^+}(S)$ from the additional assumption that $\square_\kappa^*$ holds in situations where the localized GCH does not seem to suffice.

Proposition 1.3. *Let λ be regular and $S \subseteq \lambda$ be stationary. Then*

$$\bigcirc_\lambda(S) \implies \diamond_\lambda(S).$$

Proof. Let $\langle x_\xi \mid \xi < \lambda \rangle, \langle A_\delta \mid \delta \in S \rangle$ be a pair satisfying the conclusion of Lemma 1.2. For $x \subseteq \lambda \times \lambda$ we write $(x)_\xi$ to denote $\{\zeta < \lambda \mid \langle \xi, \zeta \rangle \in x\}$. Consider sequences $\langle X_\xi, C_\xi \mid \xi < \theta \rangle$ of length $\theta \leq \lambda$ such that $X_\xi \subseteq \lambda$, C_ξ is closed unbounded in λ and, letting

$$V_\xi^\delta = \{\langle \alpha, \beta \rangle \in A_\delta \times A_\delta \mid (\forall \eta < \xi)(X_\eta \cap \alpha = (x_\beta)_\eta \cap \alpha)\},$$

for every $\xi < \theta$ and $\delta \in S \cap C_\xi$ either $\text{dom}(V_{\xi+1}^\delta)$ is bounded in δ or else $V_\xi^\delta \supsetneq V_{\xi+1}^\delta$. Notice that the non-strict inclusion $V_\xi^\delta \supseteq V_{\xi'}^\delta$ holds anyway whenever $\xi \leq \xi'$.

The crucial observation is that any sequence $\langle X_\xi, C_\xi \mid \xi < \theta \rangle$ as above has length strictly below λ . Assume for a contradiction that this fails, that is, there is such a sequence with $\theta = \lambda$. Let S' come from the application of Lemma 1.2 to the pair $\langle x_\xi \rangle_\xi, \langle A_\delta \rangle_\delta$ and the set

$$Z = \{\langle \xi, \zeta \rangle \mid \zeta \in X_\xi\},$$

and let $\delta \in S' \cap \Delta\{C_\xi \mid \xi < \lambda\}$ be such that $\delta > \kappa$ if $\lambda = \kappa^+$ and δ is a cardinal if λ is inaccessible. We have arbitrarily large $\alpha < \delta$ for which there exists $\beta < \delta$ such that $\alpha, \beta \in A_\delta$ and $Z \cap (\alpha \times \alpha) = x_\beta \cap (\alpha \times \alpha)$, so for each $\xi < \delta$ the set $\text{dom}(V_\xi^\delta)$ is unbounded in δ . Since $\delta \in S \cap C_\xi$ whenever $\xi < \delta$, from the properties of the sequence $\langle X_\xi, C_\xi \mid \xi < \theta \rangle$ we obtain $V_\xi^\delta \supsetneq V_{\xi'}^\delta$ whenever $\xi < \xi' < \delta$. This is a contradiction, as $V_\xi^\delta \subseteq A_\delta \times A_\delta$ and $\text{card}(A_\delta) < \text{card}(\delta)$.

Pick a sequence $\langle X_\xi, C_\xi \mid \xi < \theta \rangle$ as in the previous paragraph which has no proper extension. Then $\theta < \lambda$. Letting

$$D_\delta = \bigcup \{(x_\beta)_\theta \cap \alpha \mid \langle \alpha, \beta \rangle \in V_\theta^\delta\},$$

the sequence $\langle D_\delta \mid \delta \in S \rangle$ is a $\diamond_\lambda(S)$ -sequence. To see this, pick an arbitrary $X \subseteq \lambda$ and a closed unbounded $C \subseteq \lambda$. There exists some $\delta \in S \cap C$ such that $\text{dom}(V_\theta^\delta)$ is unbounded in δ and $X \cap \alpha = (x_\beta)_\theta \cap \alpha$ for all $\langle \alpha, \beta \rangle \in V_\theta^\delta$, as otherwise we could extend the sequence $\langle X_\xi, C_\xi \mid \xi < \theta \rangle$ by letting $X_\theta = X$ and $C_\theta = C$, in contradiction with its maximality. But then $X \cap \delta = D_\delta$. $\square$

We now focus on proofs of $\bigcirc_\lambda(S)$. The point of introducing $\bigcirc_\lambda(S)$ is that it is often easier to give a direct proof of $\bigcirc_\lambda(S)$ than a direct proof of $\diamond_\lambda(S)$. This is clear from Shelah's argument in [10], which in our notation is a proof of $\bigcirc_{\kappa^+}(S)$. As Proposition 1.3 also holds for inaccessible λ , our hope was that Shelah's argument could be used for proofs of $\diamond_\lambda(S)$ for inaccessible λ . It seems, however, that for inaccessible λ the proofs of $\diamond_\lambda(S)$ may require more new ideas. For instance, the proofs of $\diamond_\lambda(S_\varepsilon^\lambda)$ for a Mahlo cardinal λ in [6] and [11] can be easily modified to give proofs of $\bigcirc_\lambda(S_\varepsilon^\lambda)$, but introducing a $\bigcirc_\lambda(S_\varepsilon^\lambda)$ -sequence into the construction does not seem to enable any strengthening of the results or a simplification of the construction in [11]. For inaccessible λ that are not Mahlo it is not clear either whether an argument using $\bigcirc_\lambda(S_\varepsilon^\lambda)$ might work. It is certainly clear that constructions of a $\bigcirc_\lambda(S_\varepsilon^\lambda)$ -sequence from "below" as in Propositions 1.4 and 1.5 will not work, essentially for the same reason why constructions of $\diamond_\lambda(S_\varepsilon^\lambda)$ from "below" cannot work, as described in [11]. Analogously to [11], given any fixed $\bigcirc_\lambda(S_\varepsilon^\lambda)$ -witness $\langle x_\beta \mid \beta < \lambda \rangle$, $\langle A_\delta \mid \delta < \lambda \rangle$, there is a $< \lambda$ -distributive forcing that "kills" such a witness. On the other hand, any construction of a $\bigcirc_\lambda(S_\varepsilon^\lambda)$ -witness from "below" would give rise to the same witness in the ground model and in the generic extension.

Let us turn to the proof of $\bigcirc_{\kappa^+}(S)$. As already mentioned above, the next proposition can be viewed as the first step in Shelah's argument. We include it, as it is a starting point for our variation using weak square.

Proposition 1.4. *Assume $S \subseteq \kappa^+$ is stationary and disjoint from T_κ . Then*

$$2^\kappa = \kappa^+ \implies \bigcirc_{\kappa^+}(S).$$

Proof. Pick an arbitrary enumeration $\langle y_\xi \mid \xi < \kappa^+ \rangle$ of $[\kappa^+]^{\leq \kappa}$. The existence of such an enumeration is guaranteed by the localized GCH. Let $g : \varepsilon \times \kappa^+ \rightarrow \kappa^+$ be a bijection where $\varepsilon = \text{cf}(\kappa)$. For each $\delta \in S$ pick an increasing (with respect to the inclusion) sequence of sets $\langle A_\iota^\delta \mid \iota < \varepsilon \rangle$ such that $|A_\iota^\delta| < \kappa$ for all $\iota < \varepsilon$ and $\bigcup_{\iota < \varepsilon} A_\iota^\delta = \delta$.

We show that there is an $\iota < \varepsilon$ such that for every $Z \subseteq \kappa^+$ there are stationarily many ordinals $\delta \in S$ satisfying:

$$(1.1) \quad \begin{array}{l} \text{For unboundedly many } \alpha < \delta \text{ there are } \beta < \delta \text{ such that} \\ \alpha, \beta \in A_\iota^\delta \text{ and } Z \cap \alpha = (g^{-1}[y_\beta])_\iota. \end{array}$$

It follows that upon letting $A_\delta = A_\iota^\delta$ and $x_\beta = (g^{-1}[y_\beta])_\iota$, the pair $\langle x_\beta \mid \beta < \kappa^+ \rangle$, $\langle A_\iota^\delta \mid \delta \in S \rangle$ witnesses $\bigcirc_{\kappa^+}(S)$.¹

Assume for a contradiction that there is no ι as in the previous paragraph. Then for every $\iota < \varepsilon$ there is a set $Z_\iota \subseteq \kappa^+$ such that (1.1) holds only on a non-stationary subset of S . Let $Z = \{\langle \iota, \xi \rangle \mid \xi \in Z_\iota\}$ and $Z' = g[Z]$. The set S' consisting of all $\delta \in S$ such that

- $g[\varepsilon \times \alpha] = \alpha$ for cofinally many $\alpha < \delta$ and
- $(\forall \alpha < \delta)(\exists \beta < \delta)(Z' \cap \alpha = y_\beta)$

is stationary in κ^+ . For each $\delta \in S'$ pick a cofinal strictly increasing sequence $\langle \alpha_\eta^\delta \mid \eta < \text{cf}(\delta) \rangle$ such that $g[\varepsilon \times \alpha_\eta] = \alpha_\eta$ for each $\eta < \text{cf}(\delta)$, and for each $\eta < \text{cf}(\delta)$ pick $\beta_\eta < \delta$ such that $Z' \cap \alpha_\eta = y_{\beta_\eta}$. This is possible by the above arrangements for elements of S' .

If $\delta \in S'$, then there is an $\iota(\delta) < \varepsilon$ such that $\alpha_\eta, \beta_\eta \in A_{\iota(\delta)}^\delta$ for cofinally many $\eta < \text{cf}(\delta)$. This follows immediately if $\text{cf}(\delta) < \varepsilon$, as the assignment

$$\eta \mapsto \text{the least } \iota \text{ such that } \alpha_\eta, \beta_\eta \in A_\iota^\delta$$

cannot be cofinal in ε , so that in fact $\alpha_\eta, \beta_\eta \in A_{\iota(\delta)}^\delta$ for all $\eta < \text{cf}(\delta)$. If $\text{cf}(\delta) > \varepsilon$ this follows from the pigeonhole principle, namely that the inverse image of some A_ι^δ under this assignment must have size $\text{cf}(\delta)$. Applying the pigeonhole principle to the assignment $\delta \mapsto \iota(\delta)$, we obtain a stationary $S'' \subseteq S'$ and a $\iota < \varepsilon$ such that $\iota(\delta) = \iota$ for all $\delta \in S''$.

Pick $\delta \in S''$. By the above arrangements, there are cofinally many $\alpha < \delta$ for which there are $\beta < \delta$ such that $\alpha, \beta \in A_\iota^\delta$ and $Z' \cap \alpha = y_\beta$. Moreover, the ordinals α can be chosen so that $g[\varepsilon \times \alpha] = \alpha$. It follows that

$$Z \cap (\varepsilon \times \alpha) = g^{-1}[Z' \cap \alpha] = g^{-1}[y_\beta],$$

so $Z_\iota \cap \alpha = (g^{-1}[y_\beta])_\iota$ for all α, β as above. Since this is true of any $\delta \in S''$ we have obtained a contradiction to the fact that Z_ι is a counterexample to (1.1). $\square$

The following proposition shows how to apply a standard construction that uses $\square_\kappa^*$ to prove $\bigcirc_{\kappa^+}(T)$.

Proposition 1.5. *Assume κ is singular and $T \subseteq T_\kappa$ is a stationary subset of κ^+ with stationarily many reflection points. Then*

$$2^\kappa = \kappa^+ + \square_\kappa^* \implies \bigcirc_{\kappa^+}(T).$$

Proof. We elaborate on the argument from the proof of Proposition 1.4. Let $\varepsilon = \text{cf}(\kappa)$. Fix the following objects:

- Sequences $\langle y_\xi \mid \xi < \kappa^+ \rangle$, $\langle A_\iota^\delta \mid \iota < \varepsilon \rangle$ and a bijection $g : \varepsilon \times \kappa^+ \rightarrow \kappa^+$ as in the proof of Proposition 1.4.
- For each ordinal $\delta < \kappa^+$ an injection $h_\delta : \delta \rightarrow \kappa$.
- A $\square_\kappa^*$ -sequence $\langle \mathcal{C}_\delta \mid \delta \in \lim(\kappa, \kappa^+) \rangle$. For each δ fix an enumeration $\langle c_\zeta^\delta \mid \zeta < \kappa \rangle$ of the set $\mathcal{C}_\delta$.

¹See the proof of Proposition 1.3 for the notation $(u)_\eta$.

- An increasing (with respect to the inclusion) sequence of sets $\langle B_\iota \mid \iota < \varepsilon \rangle$ such that $|B_\iota| < \kappa$ for each ι and $\bigcup_{\iota < \varepsilon} B_\iota = \kappa \times \kappa$.

For each $\delta \in \lim(\kappa, \kappa^+)$ and $\zeta < \kappa$ define a function $f_\zeta^\delta : \delta \rightarrow \kappa \times \kappa$ and a sequence of sets $\langle A_{\zeta, \iota}^\delta \mid \iota < \varepsilon \rangle$ as follows:

$$f_\zeta^\delta(\xi) = \langle \eta, h_\gamma(\xi) \rangle$$

where γ is the least element of c_ζ^δ strictly above ξ and $\eta = \text{otp}(c_\zeta^\delta \cap \gamma) - 1$,² and

$$A_{\zeta, \iota}^\delta = (f_\zeta^\delta)^{-1}[B_\iota].$$

Notice that each f_ζ^δ is an injection. By our choice of the sets B_ι we then have $|A_{\zeta, \iota}^\delta| < \kappa$ and $\bigcup_{\iota < \varepsilon} A_{\zeta, \iota}^\delta = \delta$. We also have the following coherency property for the sets $A_{\zeta, \iota}^\delta$: If $\bar{\delta}$ is a limit point of c_ζ^δ , then there is an ordinal $\bar{\zeta} < \kappa$ such that

$$(1.2) \quad A_{\zeta, \iota}^\delta \cap \bar{\delta} = A_{\bar{\zeta}, \iota}^{\bar{\delta}}.$$

To see this notice first that if $\bar{\delta}$ is a limit point of c_ζ^δ , then there is $\bar{\zeta} < \kappa$ such that $c_\zeta^\delta \cap \bar{\delta} = c_{\bar{\zeta}}^{\bar{\delta}}$, and from the definition of f_ζ^δ we immediately conclude that $f_{\bar{\zeta}}^{\bar{\delta}} = f_\zeta^\delta \upharpoonright \bar{\delta}$. The rest follows immediately from the definition of $A_{\zeta, \iota}^\delta$.

Fix an increasing sequence $\langle \kappa_\iota \mid \iota < \varepsilon \rangle$ that is cofinal in κ . For each $\delta \in T$ and $\iota < \varepsilon$ set $A_\iota^\delta = \bigcup_{\zeta < \kappa_\iota} A_{\zeta, \iota}^\delta$. Notice that $|A_\iota^\delta| < \kappa$, as $|A_{\zeta, \iota}^\delta| \leq |B_\iota|$ for all $\zeta < \kappa_\iota$. Following the ideas from the proof of Proposition 1.4 we prove that there is an $\iota < \varepsilon$ such that for every $Z \subseteq \kappa^+$ there are stationarily many $\delta \in T$ satisfying:

$$(1.3) \quad \begin{array}{l} \text{For unboundedly many } \alpha < \delta \text{ there are } \beta < \delta \text{ such that} \\ \alpha, \beta \in A_\iota^\delta \text{ and } Z \cap \alpha = (g^{-1}[y_\beta])_\iota. \end{array}$$

It follows that upon letting $x_\beta = (g^{-1}[y_\beta])_\iota$ and $A_\delta = A_\iota^\delta$ where ι is as above, the pair $\langle x_\beta \mid \beta < \kappa^+ \rangle$, $\langle A_\delta \mid \delta \in T \rangle$ witnesses $\bigcirc_{\kappa^+}(T)$.

Assume for a contradiction that no ι as above exists. As in the proof of Proposition 1.4 pick a counterexample Z_ι for each $\iota < \varepsilon$, and let $Z = \{ \langle \iota, \xi \rangle \in \varepsilon \times \kappa^+ \mid \xi \in Z_\iota \}$ and $Z' = g[Z]$. Let C be a closed unbounded subset of κ^+ . By our assumption on T , there is a reflection point δ' of T such that:

- δ' is a limit point of C .
- $g[\varepsilon \times \alpha] = \alpha$ for cofinally many $\alpha < \delta'$.
- $(\forall \alpha < \delta')(\exists \beta < \delta')(Z' \cap \alpha = y_\beta)$.

As δ' is a reflection point of T , necessarily $\text{cf}(\delta') > \varepsilon$. Pick an increasing sequence $\langle \alpha_\eta \mid \eta < \text{cf}(\delta') \rangle$ cofinal in δ' such that $g[\varepsilon \times \alpha_\eta] = \alpha_\eta$ for each $\eta < \text{cf}(\delta')$. To each $\eta < \text{cf}(\delta')$ assign some $\beta_\eta < \delta'$ satisfying $Z' \cap \alpha_\eta = x_{\beta_\eta}$. It is convenient to pick β_η to be least possible. Since $\text{cf}(\delta') > \varepsilon$, using the pigeonhole principle we conclude that there is some $\iota' < \varepsilon$ such that $\alpha_\eta, \beta_\eta \in A_{0, \iota'}^{\delta'}$ for cofinally many $\eta < \text{cf}(\delta')$. Let $\delta \in T \cap C \cap \lim(c_0^{\delta'})$ be a limit point of $\{\alpha_\eta \mid \alpha_\eta, \beta_\eta \in A_{0, \iota'}^{\delta'}\}$. Such a δ exists by our choice of δ' and ι' and by the fact that $T \cap \delta'$ is stationary in δ' . Let $\xi < \kappa$ be such that $A_{\xi, \iota'}^\delta = A_{0, \iota'}^{\delta'} \cap \delta$ and let $\iota(\delta) > \iota'$ be such that $\kappa_{\iota(\delta)} > \xi$. The existence of such a ξ follows from (1.2). Then $A_{0, \iota'}^{\delta'} \cap \delta \subseteq A_{\iota(\delta)}^\delta$, as $B_{\iota'} \subseteq B_{\iota(\delta)}$.

The previous paragraph proves that there is a stationary $T' \subseteq T$ such that for every $\delta \in T'$ there is an $\iota(\delta) < \varepsilon$ such that for cofinally many $\alpha < \delta$ there are $\beta < \delta$ such that $\alpha, \beta \in A_{\iota(\delta)}^\delta$, $Z' \cap \alpha = y_\beta$ and $g[\varepsilon \times \alpha] = \alpha$. The rest of the proof literally

²Notice that $\text{otp}(c_\zeta^\delta \cap \gamma)$ is a successor ordinal if γ is as above.

follows the proof of Proposition 1.4. We first find a stationary $T'' \subseteq T'$ on which the $\iota(\delta)$ stabilize; let ι be the stabilized value. Then we unfold Z' and y_β using g and conclude that for α, β as above we have $Z_\iota \cap \alpha = (g^{-1}[y_\beta])_\iota$. This yields a contradiction with the fact that Z_ι is a counterexample to (1.3). $\square$

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