EXISTENCE OF STEADY STABLE SOLUTIONS FOR THE GINZBURG-LANDAU EQUATION IN A DOMAIN WITH NONTRIVIAL TOPOLOGY

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(Communicated by Matthew J. Gursky)

ABSTRACT. Let $N \geq 2$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain with boundary $\partial\Omega$. Let $\Gamma \subset \partial\Omega$ be closed. Our purpose in this paper is to consider the existence of stable solutions $u \in H^1(\Omega, \mathbb{C})$ of the Ginzburg-Landau equation

$$\begin{cases} -\Delta u(x) &= \lambda(w_0^2(x) - |u|^2)u & \text{in } \Omega, \\ u &= g & \text{on } \partial\Omega \setminus \Gamma, \\ \frac{\partial u}{\partial \nu} &= 0 & \text{on } \Gamma \end{cases}$$

where $\lambda > 0$, $w_0 \in C^2(\overline{\Omega}, \mathbb{R}^+)$ and $g \in C^2(\partial\Omega \setminus \Gamma)$ such that $|g(x)| = w_0(x)$ on $\partial\Omega \setminus \Gamma$.

1. INTRODUCTION

Let $N \geq 2$, $\Omega \subset \mathbb{R}^N$ be a bounded domain with boundary $\partial\Omega \in C^2$ and $\Gamma \subset \partial\Omega$ be closed. In the present paper, we consider the existence of solutions $u \in H^1(\Omega, \mathbb{C})$ of the Ginzburg-Landau equation

$$(P) \quad \begin{cases} -\Delta u &= \lambda(w_0^2(x) - |u|^2)u & \text{in } \Omega, \\ u &= g & \text{on } \partial\Omega \setminus \Gamma, \\ \frac{\partial u}{\partial \nu} &= 0 & \text{on } \Gamma, \end{cases}$$

where ν stands for the outward normal vector, $\lambda > 0$, $w_0 \in C^2(\overline{\Omega}, \mathbb{R}^+)$ such that $\frac{\partial w_0}{\partial \nu} = 0$ on Γ and $g \in C^2(\partial\Omega \setminus \Gamma, \mathbb{C})$ such that $|g(x)| = w_0(x)$ on $\partial\Omega \setminus \Gamma$.

Each solution of problem (P) is a critical point of the associated functional defined by

$$I_\lambda(u) = \int_\Omega \left(\frac{1}{2} |\nabla u|^2 + \frac{\lambda}{4} (w_0^2(x) - |u|^2)^2 \right) dx \quad \text{for } u \in V \text{ and } \lambda > 0,$$

where $V = \{u \in H^1(\Omega; \mathbb{C}) : u = g \text{ on } \partial\Omega \setminus \Gamma \text{ and } \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma\}$.

Problems of this type are related to models of phase transition problems occurring in superconductivity and superfluidity (see [1], [10], [11]). The existence, multiplicity and properties of solutions of problem (P) have been investigated by many authors under various boundary conditions (see [2] and references therein for

Received by the editors June 28, 2009, and, in revised form, July 3, 2009, and August 20, 2009.
 2000 *Mathematics Subject Classification*. Primary 35J50, 35Q80.

Key words and phrases. Ginzburg-Landau equation, nontrivial topology of domain.

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Dirichlet boundary conditions). Let $w(x) \equiv 1$ and $\partial\Omega \setminus \Gamma = \{\emptyset\}$. Then problem (P) becomes

$$(P_N) \quad \begin{cases} -\Delta u &= \lambda(1 - |u|^2)u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= 0 & \text{on } \partial\Omega. \end{cases}$$

It is known that problem (P_N) does not have a stable solution in the case where Ω is convex. In [7] and subsequent works ([5], [8] and [9]), the implications of the topology and geometry of the domain for the existence of some types of solutions of (P_N) have been investigated. In the celebrated paper [7], Jimbo, Morita and Zhai considered the effect of the topology of the domain Ω on the existence of stable solutions of (P_N) . They assumed the following condition:

(A) There exists $\theta \in C(\overline{\Omega}, \mathbb{C} \setminus \{0\})$ such that θ is not homotopic to a constant function on $\overline{\Omega}$

and established that for λ sufficiently large, problem (P_N) possesses a solution u such that

$$(1.1) \quad \text{the mapping } \overline{\Omega} \ni x \longrightarrow \frac{u(x)}{|u(x)|} \in S^1 = \{x \in \mathbb{C} : |x| = 1\} \text{ is homotopic to } \theta.$$

Our purpose in the present paper is to simplify the arguments employed in [7] (cf. also [9]) and extend the result for (P_N) to a result for the mixed boundary value problem (P). The argument employed in [7] is involved. In [7], the existence of solutions for (P_N) is proved by applying the Schauder fixed point theorem. The argument also needs an upper-lower solution method, a $C^{2,\alpha}$ -regularity estimate (with $\alpha \in (0, 1)$) and variational characterizations of solutions. The stability of the solution of (P_N) is proved by another argument.

In this paper we will find the solution of (P) using a variational method; that is, we will find a minimizer of a functional associated with the problem (P) and show that the minimizer is a solution of (P). Since the solution is obtained as a minimizer of an associated functional, the stability of the solution follows almost automatically. The argument employed here is based on $W^{2,p}$ -theory so that we do not need a $C^{2,\alpha}$ -estimate.

We impose the following condition instead of (A):

$$(A) \quad \Lambda = \left\{ \theta \in C(\overline{\Omega}, S^1) : \theta(x) = \frac{g(x)}{|g(x)|} \text{ on } \partial\Omega \setminus \Gamma \text{ and } \theta \text{ is not} \right. \\ \left. \text{homotopic to a constant-valued function in } C(\overline{\Omega}, S^1) \right\} \neq \{\emptyset\}.$$

We can now state our main result.

Theorem 1.1. *Suppose (A) holds. Then for each $\theta \in \Lambda$, there exists $\lambda_0 > 0$ such that for each $\lambda \geq \lambda_0$, problem (P) has a stable solution $u \in C(\overline{\Omega}, \mathbb{C})$ such that*

- (1) $u(x) \neq 0$ on Ω
- (2) *the mapping $\overline{\Omega} \ni x \rightarrow \frac{u(x)}{|u(x)|} \in \{z \in \mathbb{C} : |z| = 1\}$ is homotopic to θ .*

2. PROOF OF THEOREM 1.1

We denote by $\|\cdot\|$ the norm of $H := H^1(\Omega, \mathbb{C})$ defined by $\|v\|^2 = \int_{\Omega} (|\nabla v|^2 + |v|^2) dx$ for $v \in H$. For $p \geq 2$, we put $|v|_p^p = \int_{\Omega} |v|^p dx$ for $v \in L^p(\Omega, \mathbb{C})$. For $u, v \in H$, we put $\langle u, v \rangle = \int_{\Omega} u \overline{v} dx$. We denote by $B_r(x)$ the open ball in $\mathbb{R}^N$ centered at x with radius r . For $x, y \in \mathbb{R}^N$, $x \cdot y$ denotes the inner product of x and y . For each $a \in \mathbb{R}$ and a functional $F : H \rightarrow \mathbb{R}$, we denote by F^a the level

set $F^a = \{v \in H : F(v) \leq a\}$. For $\theta_1, \theta_2 \in C(\Omega, S^1)$, we write $\theta_1 \cong \theta_2$ when θ_1 and θ_2 are homotopic. In this paper, we say that a solution u of (P) is stable if u is a stable stationary solution of the initial boundary value problem

$$\begin{cases} \frac{du}{dt} = \Delta u + \lambda(w_0^2(x) - |u|^2)u & \text{in } (0, \infty) \times \Omega, \\ u = g & \text{on } (0, \infty) \times \partial\Omega \setminus \Gamma, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } (0, \infty) \times \Gamma, \\ u(0) = u_0 \in H. \end{cases}$$

Throughout the rest of this paper, we assume that condition (A) holds. Let $\theta_0 \in \Lambda$. We may assume without any loss of generality that $\theta_0 \in C^2(\bar{\Omega}, S^1)$ and $\frac{\partial \theta_0}{\partial \nu} = 0$ on Γ . We fix $x_0 \in \Omega$. For each $\lambda > 0$, define a functional $I_\lambda \in C^2(H, \mathbb{R})$ associated with problem (P) by

$$I_\lambda(u) = \int_{\Omega} \left(\frac{1}{2} |\nabla u|^2 + \frac{\lambda}{4} (w_0^2(x) - |u|^2)^2 \right) dx \quad \text{for each } u \in H.$$

For each $u \in H^1(\Omega)$ with $\inf_{x \in \Omega} |u(x)| > 0$, by putting $u(x) = w(x)e^{i\theta(x)}$, $I_\lambda(u)$ is rewritten as

$$(2.1) \quad I_\lambda(u) = \int_{\Omega} \left(\frac{1}{2} (|\nabla w|^2 + w^2 |\nabla \theta|^2) + \frac{\lambda}{4} (w_0^2(x) - |u|^2)^2 \right) dx,$$

where $w \in H^1(\mathbb{R}^N)$ is positive and $\theta \in H^1(\mathbb{R}^N)$.

Let $\widehat{\Omega}$ be the universal covering space of Ω . For each $\theta \in C^0(\Omega, S^1)$, $\widehat{\theta}$ stands for the lift of θ (cf. [4]). It is known that the universal covering space $\widehat{S^1}$ of S^1 is isomorphic to $\mathbb{R}$. We denote by p_1 and p_2 the covering projections from $\widehat{\Omega}$ to Ω and from $\mathbb{R}$ to S^1 , respectively. Then the following diagram is commutative:

$$\begin{array}{ccc} \widehat{\Omega} & \xrightarrow{\widehat{\theta}} & \mathbb{R} = \widehat{S^1} \\ p_1 \downarrow & & \downarrow p_2 \\ \Omega & \xrightarrow{\theta} & S^1 \end{array}$$

The following result due to Jimbo et al. [6, 7] is a consequence of the covering mapping property (see [4]).

Proposition 2.1 ([6]). *For any $\theta \in C(\Omega, S^1)$ such that $\theta \cong \theta_0$ (and $\theta(x_0) = 0$ in the case where $\partial\Omega \setminus \Gamma = \emptyset$), the function $\eta'(z) = \widehat{\theta}(z) - \widehat{\theta}_0(z)$ in $\widehat{\Omega}$ can be identified with an $\mathbb{R}$ -valued function in Ω ; i.e., there exists a unique $\mathbb{R}$ -valued function η in Ω such that $\eta'(z) = \eta(p_1(z))$. On the other hand, for any $\mathbb{R}$ -valued function $\eta \in C(\Omega)$ (which satisfies $\eta(x_0) = 0$ in the case where $\partial\Omega \setminus \Gamma = \emptyset$), there exists a unique $\theta \in C(\Omega, S^1)$ such that $\eta(p_1(z)) = \widehat{\theta}(z) - \widehat{\theta}_0(z)$ in $\widehat{\Omega}$ (and $\theta(x_0) = 0$ in the case where $\partial\Omega \setminus \Gamma = \emptyset$).*

Then translating the function θ to η , the functional (2.1) is rewritten as

$$I_\lambda(v) = I_\lambda(w, \eta) = \int_{\Omega} \left(\frac{1}{2} (|\nabla w|^2 + w^2 |\nabla \eta + \xi|^2 + (w_0^2 - w^2)^2) \right) dx,$$

where $v = (w, \eta) \in H^1(\Omega) \times H^1(\Omega)$ and $\xi = \nabla \theta_0$.

Here we put

$$W = H_0^1(\Omega \cup \Gamma) \text{ and } \widetilde{W} = w_0 + V.$$

(see [13] for the definition of $H_0^1(\Omega \cup \Gamma)$). We also put

$$E = W \times W \text{ and } \widetilde{E} = \widetilde{W} \times W.$$

When $u \in H_0^1(\Omega)$ and $|u(x)| > 0$ on Ω , the function $u(x) = w(x)e^{i(\eta(x)+\theta_0(x))}$ is a solution of (P) if and only if $(w, \eta) \in \tilde{E}$ satisfies

$$(2.2) \quad \begin{cases} -\Delta w + |\nabla \eta + \xi|^2 w - \lambda(w_0^2 - w^2)w = 0 & \text{in } \Omega, \\ \frac{\partial w}{\partial \nu} = 0 & \text{on } \Gamma, \end{cases}$$

and

$$(2.3) \quad \begin{cases} -\nabla(w^2(\nabla \eta + \xi)) = 0 & \text{in } \Omega, \\ \frac{\partial \eta}{\partial \nu} = 0 & \text{on } \Gamma. \end{cases}$$

One can see that each critical point $v = (w, \eta)$ of I_λ in $\tilde{W} \times W$ corresponds to a solution $u = we^{i(\eta+\theta_0)}$ of (P) which is homotopic to θ_0 .

We next define a functional $I_\infty \in C^2(W, \mathbb{R})$ by

$$I_\infty(\eta) = \frac{1}{2} \int_{\Omega} (|\nabla w_0|^2 + w_0^2 |\nabla \eta + \xi|^2) dx \quad \text{for } \eta \in W.$$

Since $w_0 \in C^2(\overline{\Omega}, \mathbb{R}^+)$, we have $\inf_{x \in \overline{\Omega}} w_0(x) > 0$. Then it is easy to see that there exists $\eta_0 \in C^2(\overline{\Omega})$ such that $I_\infty(\eta_0) = \min_{\eta \in W} I_\infty(\eta)$ (and $\eta(x_0) = 0$ in the case where $\partial\Omega \setminus \Gamma = \emptyset$). Here we put

$$M = \max \left\{ 4|\nabla w_0|_\infty^2, \left| \frac{4(\Delta w_0 - |\nabla \eta_0 + \xi|^2 w_0)}{w_0^2} \right|_p \right\}$$

and

$$c_0 = \frac{1}{2} \int_{\Omega} (|\nabla w_0|^2 + w_0^2 |\nabla \eta_0 + \xi|^2) dx.$$

Next, to find a solution of problem (P), we will make a modification to the functional I_λ . We fix $p > N$ and fix a mapping $h \in C^2([0, \infty))$ such that

$$h(t) = t \quad \text{for all } t \in [0, M],$$

$$h(t) \geq t, h'(t) \geq 1, h''(t) \geq 0 \text{ on } [0, \infty) \text{ and } 0 < h_\infty := \lim_{t \rightarrow \infty} \frac{h(t)}{t^{p/2}} < \infty.$$

Let $\lambda > 0$. We define a functional J_λ on $\tilde{E}$ by

$$J_\lambda(w, \eta) = \int_{\Omega} \left(\frac{1}{2} h(|\nabla w|^2) + \frac{1}{2} w^2 |\nabla \eta + \xi|^2 + \frac{\lambda}{4} (w_0^2 - w^2)^2 \right) dx \quad \text{for } (w, \eta) \in \tilde{E}.$$

From the definition,

$$J_\lambda(w, \eta) = I_\lambda(w, \eta) \quad \text{for } (w, \eta) \in \tilde{E} \text{ with } |\nabla w|_\infty^2 \leq M.$$

Then it follows from the definition of M that

$$c_\lambda = \inf \left\{ J_\lambda(v) : v \in \tilde{E} \right\} \leq c_0 \quad \text{for all } \lambda > 0.$$

Lemma 2.1. *There exists $\lambda_1 > 0$ such that if $\lambda \geq \lambda_1$ and $(w, \eta) \in \tilde{E}$ satisfies $J_\lambda(w, \eta) \leq 2c_0$, then*

$$(2.4) \quad w(x) > \frac{1}{\sqrt{3}} w_0 \quad \text{on } \overline{\Omega}.$$

Proof. Let $\lambda > 0$ and $u \in \tilde{E}$ be such that $J_\lambda((w, \eta)) \leq 2c_0$. Then we have that

$$|\nabla w|_p^p \leq C_0 c_0 \text{ and } \int_{\Omega} (w_0^2 - w^2)^2 dx < \frac{8c_0}{\lambda}.$$

Here $C_0 > 0$ is a constant that depends only on h . Then $w^2 \rightarrow w_0^2$ in $L^2(\Omega)$ as $\lambda \rightarrow \infty$. By the Gagliardo-Nirenberg theorem (see e.g. [12]), we have that

$$|w_0^2 - w^2|_{\infty} \leq K_1 \|w_0^2 - w^2\|_{W^{1,p}(\Omega)}^a |w_0^2 - w^2|_2^{1-a},$$

where K_1 is a positive constant and $a \in [0, 1]$ satisfies $a(\frac{1}{p} - \frac{1}{n}) + (1-a)\frac{1}{2} = 0$. Then since $\{\|w_0^2\|_{W^{1,p}(\Omega)}\}$ and $\{\|w^2\|_{W^{1,p}(\Omega)}\}$ are bounded, we find that $|w_0^2 - w^2|_{\infty} \rightarrow 0$ as $\lambda \rightarrow \infty$. Then recalling that $W^{1,p}(\Omega) \subset C^{0,1-n/p}(\overline{\Omega})$, we find that $w \in C^{0,1-n/p}(\overline{\Omega})$ and thus $|w - w_0|_{\infty} \rightarrow 0$ as $\lambda \rightarrow \infty$. Therefore the assertion follows. $\square$

Lemma 2.2. *There exists a sequence $\{(w_\lambda, \eta_\lambda)\}_{\lambda \geq \lambda_1} \subset \tilde{E}$ such that*

$$J_\lambda(w_\lambda, \eta_\lambda) = \inf \left\{ J_\lambda(v) : v \in \tilde{E} \right\} \quad \text{for } \lambda \geq \lambda_1.$$

Moreover it follows that

$$(2.5) \quad \lim_{\lambda \rightarrow \infty} \|w_\lambda - w_0\|_{W^{1,p}(\Omega)} = \lim_{\lambda \rightarrow \infty} \|\eta_\lambda - \eta_0\|_{W^{1,2}(\Omega)} = 0$$

and

$$(2.6) \quad \sup_{\lambda \geq \lambda_1} |\nabla \eta_\lambda|_{\infty} < \infty.$$

Proof. Let $\lambda \geq \lambda_1$. Then $J_\lambda(w, \eta) \rightarrow \infty$ as $\max \left\{ \|w\|_{H^{1,p}(\Omega)}, \|\eta\|_{H^1(\Omega)} \right\} \rightarrow \infty$. Then we have that there exists a bounded minimizing sequence $\{(w_n, \eta_n)\} \subset H^{1,p}(\Omega) \times H^1(\Omega)$ such that $w_n \rightharpoonup w_\lambda \in H^{1,p}(\Omega)$ weakly in $H^{1,p}(\Omega)$ and $\eta_n \rightharpoonup \eta_\lambda \in H^1(\Omega)$ weakly in $H^1(\Omega)$. Since $H^{1,p}(\Omega)$ is compactly embedded in $L^2(\Omega)$, one can see from the definition of J_λ that J_λ is weakly lower semicontinuous in $H^{1,p}(\Omega) \times H^1(\Omega)$. It then follows that $(w_\lambda, \eta_\lambda)$ is a minimizer of J_λ ; that is, there exists $\{v_\lambda\}_{\lambda \geq \lambda_1} = \{(w_\lambda, \eta_\lambda)\}_{\lambda \geq \lambda_1} \subset \tilde{E}$ which satisfies the assumption. One can see from the definition of the functional J_λ that

$$J_\lambda((w \wedge |w_0|_{\infty}, \eta)) \leq J_\lambda((w, \eta)) \quad \text{for each } (w, \eta) \in \tilde{E}.$$

Then we have that $w_\lambda(x) \leq |w_0|_{\infty}$ for all $x \in \Omega$ and $\lambda \geq \lambda_1$. Since $\lim_{\lambda \rightarrow \infty} J_\lambda(v_\lambda) \leq c_0$, we have by Lemma 2.1 that $w_\lambda \geq \frac{1}{\sqrt{3}} w_0$ on Ω . Then it follows from the definition of J_λ that $\{\|w_\lambda\|_{W^{1,p}(\Omega)}\}$ and $\{\|\eta_\lambda\|_{W^{1,2}(\Omega)}\}$ are bounded. Therefore we may assume that $w_\lambda \rightarrow w_0$ weakly in $W^{1,p}(\Omega)$, $w_\lambda \rightarrow w$ strongly in $L^\infty(\Omega)$ and $\eta_\lambda \rightarrow \eta$

weakly in $W^{1,2}(\Omega)$. Then noting that (w_0, η_0) is a minimizer of I_∞ , we have

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} \int_{\Omega} \left(\frac{1}{2} h(|\nabla w_\lambda|^2) + \frac{1}{2} w_0^2 |\nabla \eta_\lambda|^2 \right) dx &= \lim_{\lambda \rightarrow \infty} \int_{\Omega} \left(\frac{1}{2} h(|\nabla w_\lambda|^2) + \frac{1}{2} w_\lambda^2 |\nabla \eta_\lambda|^2 \right) dx \\ &\leq \lim_{\lambda \rightarrow \infty} J_\lambda(v_\lambda) \\ &\leq \int_{\Omega} \left(\frac{1}{2} h(|\nabla w_0|^2) + \frac{1}{2} w_0^2 |\nabla \eta_0|^2 \right) dx \\ &\leq \int_{\Omega} \left(\frac{1}{2} h(|\nabla w_0|^2) + \frac{1}{2} w_0^2 |\nabla \eta|^2 \right) dx \\ &\leq \lim_{\lambda \rightarrow \infty} \int_{\Omega} \left(\frac{1}{2} h(|\nabla w_0|^2) + \frac{1}{2} w_0^2 |\nabla \eta_\lambda|^2 \right) dx. \end{aligned}$$

This implies that $w_\lambda \rightarrow w_0$ strongly in $W^{1,p}(\Omega)$ and $\eta_\lambda \rightarrow \eta$ strongly in $W^{1,2}(\Omega)$. Noting that η_0 is the minimizer, we have that $\eta = \eta_0$. This completes the proof of (2.5). Let $\lambda \geq \lambda_1$ and put $w = w_\lambda$ and $\eta = \eta_\lambda$. Then since (w, η) is a minimizer of the functional J_λ on $\tilde{E}$, we find that $(w, \eta) \in \tilde{E}$ satisfies (2.2) and (2.3). Then we have

$$-w^2 \Delta \eta = \nabla w^2 \cdot (\nabla \eta + \xi) + w^2 X,$$

where $X = \operatorname{div} \xi$. Then by Theorem 3.17 of [13], we find

$$\begin{aligned} (2.7) \quad \|\eta\|_{W^{2,p}(\Omega)} &\leq C(|\nabla w^2 \cdot (\nabla \eta + \xi)|_p + |w^2 X|_p + \|\eta\|_{W^{1,2}(\Omega)}) \\ &\leq C|\nabla w^2|_p |\nabla \eta + \xi|_\infty + C_1 \\ &\leq 2C|w_0|_\infty (C_0 c_0)^{1/p} |\nabla \eta|_\infty + C_2 \end{aligned}$$

where $C > 0$ is independent of λ ,

$$\begin{aligned} C_1 &= |w_0^2 X|_p + \max \left\{ \|\eta_\lambda\|_{W^{1,2}(\Omega)} : \lambda \geq \lambda_1 \right\} \quad \text{and} \\ C_2 &= C_1 + 2C|w_0|_\infty (C_0 c_0)^{1/p} |\xi|_\infty. \end{aligned}$$

Again by the Gagliardo-Nirenberg theorem, we have that

$$(2.8) \quad |\nabla \eta|_\infty \leq C_3 \|\eta\|_{W^{2,p}(\Omega)}^a |\nabla \eta|_2^{1-a},$$

where $C_3 > 0$ is a positive constant independent of λ and $a \in (0, 1)$ is the constant defined in the proof of Lemma 2.1. Then noting that $|\nabla \eta|_2 \leq C_1$, we find by substituting (2.8) into (2.7) that

$$\|\eta\|_{W^{2,p}(\Omega)} \leq C_4 \|\eta\|_{W^{2,p}(\Omega)}^a + C_2,$$

where C_4 is independent of λ . Since $a < 1$, (2.6) follows from the inequality above. $\square$

Lemma 2.3. *Let $\lambda \geq \lambda_1$ and $v_\lambda = (w_\lambda, \eta_\lambda) \in \tilde{E}$ be a minimizer of J_λ . Then the problem*

$$(2.9) \quad -\nabla(h'(|\nabla w|^2)\nabla w) + w|\nabla \eta_\lambda + \xi|^2 - \lambda(w_0^2 - w^2)w = 0, \quad w \in \tilde{W}$$

has at most one solution.

Proof. Let $\lambda \geq \lambda_1$ and $(w_\lambda, \eta_\lambda) \in \tilde{E}$ satisfy the assumption. Suppose that $w_1, w_2 \in \tilde{E}$ are solutions of problem (2.9). We put $w_i = z_i + w_0$ for $i = 1, 2$. Then

$$-\nabla(h'(|\nabla(z_i + w_0)|^2)\nabla(z_i + w_0)) + |\nabla \eta_\lambda + \xi|^2(z_i + w_0) + \lambda z_i(z_i + 2w_0)(z_i + w_0) = 0$$

for $i = 1, 2$. Then it follows that

$$(2.10) \quad \int_{\Omega} \left(h'(|\nabla(z_1 + w_0)|^2) \nabla(z_1 + w_0) - h'(|\nabla(z_2 + w_0)|^2) \nabla(z_2 + w_0) \right) \nabla(z_1 - z_2) dx \\ + \int_{\Omega} [|\nabla \eta_{\lambda} + \xi|^2 (z_1 - z_2)^2 + (f(z_1) - f(z_2))(z_1 - z_2)] dx = 0$$

where

$$f(z) = \lambda(z^3 + 3w_0z^2 + 2w_0^2z) \quad \text{for each } z \in \mathbb{R}.$$

Since the mapping $\nabla(h'(|\nabla(\cdot + w_0)|^2) \nabla(\cdot + w_0))$ is monotone, we have that

$$(2.11) \quad \int_{\Omega} \left(h'(|\nabla(z_1 + w_0)|^2) \nabla(z_1 + w_0) - h'(|\nabla(z_2 + w_0)|^2) \nabla(z_2 + w_0) \right) \nabla(z_1 - z_2) dx \geq 0.$$

Then noting that $z_i > (\frac{1}{\sqrt{3}} - 1)w_0$ for $i = 1, 2$ by Lemma 2.1 and that

$$f'(z) = \lambda(3(z + w_0)^2 - w_0^2) > 0 \quad \text{if } z > (\frac{1}{\sqrt{3}} - 1)w_0,$$

we find from (2.10) and (2.11) that if $z_1 \neq z_2$, then

$$0 < \int_{\Omega} (f(z_1) - f(z_2))(z_1 - z_2) dx \leq 0.$$

Therefore we have $z_1 = z_2$. This completes the proof. $\square$

Proof of Theorem 1.1. Let $\lambda \geq \lambda_1$ and $u = (w_{\lambda}, \eta_{\lambda}) \in \tilde{E}$ be a minimizer of J_{λ} on $\tilde{E}$. We put

$$(2.12) \quad B_m = \left\{ z \in W^{1,\infty}(\Omega) \cap W : \|z\|_{W^{1,\infty}(\overline{\Omega})} \leq m \right\},$$

where $m > 0$ is such that $m^2 \leq |\nabla w_0|_{\infty}^2$ and

$$(2.13) \quad |z|_{\infty} \leq (1 - \frac{1}{\sqrt{3}}) \min_{x \in \overline{\Omega}} w_0(x) \quad \text{on } \Omega \quad \text{for all } z \in B_m.$$

We define a mapping $T_{\lambda} : B_m \rightarrow W$ by $T_{\lambda}z = v$ where $z \in W$ and $v \in W$ is a solution of the problem

$$(2.14) \quad -\Delta(v + w_0) + |\nabla \eta_{\lambda} + \xi|^2 (v + w_0) + \lambda v(z + 2w_0)(z + w_0) = 0.$$

Let $z \in W$ and $v = T_{\lambda}z$. Then v is the minimizer of the functional

$$\hat{J}_{\lambda,z}(v) = \int_{\Omega} (|\nabla(v + w_0)|^2 \\ + |\nabla \eta_{\lambda} + \xi|^2 (v + w_0)^2 + \lambda(z + 2w_0)(z + w_0)v^2) dx, \quad v \in W.$$

Multiplying (2.14) by v and integrating over Ω , we have

$$\int_{\Omega} (|\nabla v|^2 + (|\nabla \eta_{\lambda} + \xi|^2 + \lambda(z + 2w_0)(z + w_0))v^2) dx \\ \leq \int_{\Omega} (|\Delta w_0| + |\nabla \eta_{\lambda} + \xi|^2 |w_0|) |v| dx.$$

Then noting that (2.6) and (2.13) hold, we find that there exist $C_1, C_2 > 0$ such that

$$|\nabla v|_2^2 + \lambda C_1 |v|_2^2 \leq C_2 |v|_2,$$

where C_1, C_2 are independent of λ . Then we find that

$$(2.15) \quad |v|_2 \leq \frac{C_2 C_1^{-1}}{\lambda} \quad \text{and} \quad |\nabla v|_2 \leq \frac{C_2 C_1^{-1/2}}{\lambda^{1/2}}.$$

Similarly, by multiplying (2.14) by v^{p-1} and integrating over Ω , we have that there exists $C_3 > 0$, independent of λ , such that

$$(2.16) \quad |v|_p \leq \frac{C_3}{\lambda}.$$

Here we rewrite equation (2.14) as follows:

$$(2.17) \quad -\frac{1}{(z+2w_0)(z+w_0)}\Delta v + \frac{|\nabla\eta_\lambda + \xi|^2}{(z+2w_0)(z+w_0)}v + \lambda v = \frac{\Delta w_0 - |\nabla\eta_\lambda + \xi|^2 w_0}{(z+2w_0)(z+w_0)}.$$

We put

$$Lv = -a(z)(\Delta v + |\nabla\eta_\lambda + \xi|^2 v),$$

where

$$a(z) = \frac{1}{(z+2w_0)(z+w_0)}.$$

From (2.12) and (2.13), we have that there exist K_1, K_2 and $K_3 > 0$ such that

$$a(z)(x) \in (K_1, K_2) \quad \text{and} \quad |a(z)(x) - a(z)(y)| \leq K_3 |x - y|$$

for $x, y \in \Omega$ and $z \in B_m$.

By the definition above, the equation has the form

$$Lv + \lambda v = \frac{\Delta w_0 - |\nabla\eta_\lambda + \xi|^2 w_0}{(z+2w_0)(z+w_0)}.$$

Then by Theorem 3.17 of [13], we have that there exists $C_4 > 0$ such that

$$(2.18) \quad \|v\|_{W^{2,p}(\Omega)} \leq C_4 \left(\left\| \frac{\Delta w_0 - |\nabla\eta_\lambda + \xi|^2 w_0}{(z+2w_0)(z+w_0)} \right\|_p + \lambda |v|_p + \|v\|_{H^1(\Omega)} \right),$$

where C_4 is independent of v, z and λ . Noting that (2.13) holds, we have

$$\begin{aligned} \left\| \frac{\Delta w_0 - |\nabla\eta_\lambda + \xi|^2 w_0}{(z+2w_0)(z+w_0)} \right\|_p &\leq 3 \left\| \frac{\Delta w_0 - |\nabla\eta_\lambda + \xi|^2 w_0}{w_0^2} \right\|_p \\ &\leq 3 \left(\left\| \frac{\Delta w_0 - |\nabla\eta_0 + \xi|^2 w_0}{w_0^2} \right\|_p + \left\| \frac{\nabla(\eta_\lambda - \eta_0)(\nabla(\eta_\lambda + \eta_0) + 2\xi)w_0}{w_0^2} \right\|_p \right) \end{aligned}$$

and $\lim_{\lambda \rightarrow \infty} \|\eta_\lambda - \eta_0\|_{W^{1,2}(\Omega)} = 0$. Then by choosing λ sufficiently large we have that

$$\left\| \frac{\Delta w_0 - |\nabla\eta_\lambda + \xi|^2 w_0}{(z+2w_0)(z+w_0)} \right\|_p \leq M.$$

Then combining (2.18) with (2.15) and (2.16), we have that

$$(2.19) \quad \|v\|_{W^{2,p}(\Omega)} \leq C_4 \left(M + \lambda |v|_p + \|v\|_{H^1(\Omega)} \right) \leq 2C_4(M + C_3).$$

On the other hand, by the Gagliardo-Nirenberg theorem, we have that there exists $K_4 > 0$ such that

$$|\nabla v|_\infty \leq K_4 \|v\|_{W^{2,p}(\Omega)}^b |v|_2^{1-b},$$

where

$$\frac{1}{n} + b\left(\frac{1}{p} - \frac{2}{n}\right) + (1-b)\frac{1}{2} = 0.$$

Then combining the inequality above with (2.19), we get

$$|\nabla v|_\infty \leq K_4 \|v\|_{W^{2,p}(\Omega)}^b |v|_2^{1-b} \leq K_4 (2C_4(M + C_3))^b |v|_2^{1-b}.$$

Since $|v|_2^2 \rightarrow 0$ as $\lambda \rightarrow \infty$, we have that there exists $\lambda_0 > \lambda_1$ such that for each $\lambda \geq \lambda_0$, $|\nabla v|_\infty < m$ and $|v|_\infty \leq (1 - \frac{1}{\sqrt{3}}) \min_{x \in \bar{\Omega}} w_0(x)$. Therefore the mapping T_λ is from B_m into B_m for $\lambda \geq \lambda_0$. Since T_λ is compact by Rellich's compact embedding theorem, we find by Schauder's fixed point theorem that there exists a fixed point $z \in B_m$ of T_λ . Then by the definition of T_λ , we have by putting $w = z + w_0$ that

$$-\Delta w + |\nabla \eta_\lambda + \xi|^2 w + \lambda(w_0^2 - w^2)w = 0.$$

Then noting that

$$|\nabla w(x)|^2 \leq 2(|\nabla w_0(x)|^2 + |\nabla z(x)|^2) \leq 2(|\nabla w_0|_\infty^2 + m) \leq 4|\nabla w_0|_\infty^2 \quad \text{on } \Omega,$$

we find that $h(|\nabla w|^2) = |\nabla w|^2$ on Ω , that is, w a solution of (2.9). Since the solution is unique by Lemma 2.3, we have $w = w_\lambda$. Therefore $(w_\lambda, \eta_\lambda)$ satisfies (2.2) and (2.3).

Lastly, we prove the stability of $u = (w_\lambda, \eta_\lambda)$. We put $\widehat{I}_\lambda(z, \zeta) = I(z + w_\lambda, \zeta + \eta_\lambda)$ for $(z, \zeta) \in W \times W$. We first consider the case $\partial\Omega \setminus \Gamma \neq \emptyset$. In this case, to show the stability of u , it is sufficient to show that there exists $C > 0$ such that

$$(2.20) \quad \left\langle \widehat{I}_\lambda''(0, 0)(z, \zeta), (z, \zeta) \right\rangle \geq C(\|z\|_{W^{1,2}(\Omega)}^2 + \|\zeta\|_{W^{1,2}(\Omega)}^2) \quad \text{for all } (z, \zeta) \in W \times W,$$

where $\widehat{I}_\lambda''$ stands for the Hessian of $\widehat{I}_\lambda$ and for each $(z_1, \zeta_1), (z_2, \zeta_2) \in W \times W$ we define $\langle (z_1, \zeta_1), (z_2, \zeta_2) \rangle = \langle z_1, z_2 \rangle + \langle \zeta_1, \zeta_2 \rangle$ (see [3]).

From the definition, we find that for each $(z, \zeta) \in W \times W$,

$$\left\langle \widehat{I}_\lambda''(0, 0)(z, \zeta), (z, \zeta) \right\rangle = I_1 + I_2 + I_3,$$

where

$$I_1 = \left\langle -\Delta z + |\nabla \eta + \xi|^2 z + \lambda(3(z + w_0)^2 - w_0^2), z \right\rangle,$$

$$I_2 = \langle w^2 \nabla \zeta, \nabla \zeta \rangle \quad \text{and} \quad I_3 = \left\langle 2w |\nabla \zeta|^2 z, z \right\rangle.$$

Then by (2.4), there exists $C_1 > 0$ such that $I_1 \geq C_1 \|v\|_{W^{1,2}(\Omega)}^2$ for $v \in W$. We also have $I_3 \geq 0$. On the other hand, noting that $\partial\Omega \setminus \Gamma \neq \emptyset$, we have by the Poincaré inequality (see e.g. Theorem 1.45 of [13]) that there exists $C_2 > 0$ such that $I_2 \geq C_2 \|\zeta\|_{W^{1,2}(\Omega)}^2$. Therefore we find that (2.20) holds with $C = \min\{C_1, C_2\}$. Next suppose that $\partial\Omega \setminus \Gamma = \emptyset$; that is, (P) is a Neumann boundary value problem. In this case, $\{e^{ic}u : c \in \mathbb{R}\}$ is a one-dimensional continuum of solutions when u is a solution of (P); that is, $\{(w_\lambda, \eta_\lambda + c) : c \in \mathbb{R}\}$ is a continuum of critical points

of I_λ . Then we have that $I_2 = 0$ for $\zeta = \text{const}$ and $I_2 \geq C'_2 \|\zeta\|_{W^{1,2}(\Omega)}^2$ for all $\zeta \in W' = \{\phi \in W : \langle \varphi, 1 \rangle = 0\}$. So we have

$$\left\langle \widehat{I}_\lambda''(0, 0)(z, \zeta), (z, \zeta) \right\rangle \geq C'(\|z\|_{W^{1,2}(\Omega)}^2 + \|\zeta\|_{W^{1,2}(\Omega)}^2) \quad \text{for all } (z, \zeta) \in W \times W',$$

where $C' = \min\{C_1, C'_2\}$. This implies that u is stable (cf. [3]). $\square$

ACKNOWLEDGEMENT

The author expresses his heartfelt thanks to the referee for reading this article very carefully and for giving valuable comments.

REFERENCES

1. E. Abrahams and T. Tsuneto, *Time variation of the Ginzburg-Landau order parameter*, Phys. Rev. **152** (1966), 416–432.
2. F. Bethuel, H. Brezis, and F. Hélein, *Ginzburg-Landau vortices*, Progress in Nonlinear Differential Equations and Their Applications, 13, Birkhäuser Boston, 1994. MR1269538 (95c:58044)
3. D. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Math., 840, Springer-Verlag, Berlin-New York, 1981. MR610244 (83j:35084)
4. S.-T. Hu, *Homotopy theory*, Academic Press, New York-London, 1959. MR0106454 (21:5186)
5. S. Jimbo and Y. Morita, *Stable solutions with zeros to the Ginzburg-Landau equation with Neumann boundary condition*, J. Differential Equations **128** (1996), no. 2, 596–613. MR1398333 (97f:35202)
6. ———, *Ginzburg-Landau equation with magnetic effect in a thin domain*, Calc. Var. Partial Differential Equations **15** (2002), no. 3, 325–352. MR1938818 (2003i:35007)
7. S. Jimbo, Y. Morita, and J. Zhai, *Ginzburg-Landau equation and stable steady state solutions in a non-trivial domain*, Comm. Partial Differential Equations **20** (1995), no. 11-12, 2093–2112. MR1361731 (97i:35167)
8. S. Jimbo and J. Zhai, *Ginzburg-Landau equation with magnetic effect: non-simply-connected domains*, J. Math. Soc. Japan **50** (1998), no. 3, 663–684. MR1626354 (99m:35231)
9. ———, *Domain perturbation method and local minimizers to Ginzburg-Landau functional with magnetic effect*, Abstr. Appl. Anal. **5** (2000), no. 2, 101–112. MR1885324 (2003e:35055)
10. D. R. Nelson, *Defect-mediated phase transitions*, Phase transitions and critical phenomena, vol. 7, Academic Press, London, 1983. MR794313
11. A. C. Newell and J. A. Whitehead, *Finite bandwidth, finite amplitude convection*, J. Fluid Mech. **38** (1969), 279–303.
12. L. Nirenberg, *On elliptic partial differential equations*, Ann. Scuola Norm. Sup. Pisa **13** (1959), no. 3, 115–162. MR0109940 (22:823)
13. G. M. Troianiello, *Elliptic differential equations and obstacle problems*, Plenum Press, New York, 1987. MR1094820 (92b:35004)

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