A DOUBLING MEASURE ON $\mathbb{R}^d$ CAN CHARGE A RECTIFIABLE CURVE

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ABSTRACT. For $d \geq 2$, we construct a doubling measure ν on $\mathbb{R}^d$ and a rectifiable curve Γ such that $\nu(\Gamma) > 0$.

1. INTRODUCTION

A Borel measure ν on $\mathbb{R}^d$ is said to be *doubling* if there is a constant $C_\nu < \infty$ such that for any $x \in \mathbb{R}^d$ and $0 < r < \infty$ we have

$$(1.1) \quad \nu(B(x, 2r)) \leq C_\nu(B(x, r)),$$

where $B(x, r)$ is the ball $\{y : |y - x| < r\}$. A *rectifiable curve* is a continuous map $\gamma : [0, 1] \rightarrow \mathbb{R}^d$ with

$$\text{length}(\gamma) := \sup_{0 \leq t_0 \leq \dots \leq t_n \leq 1} \sum_{j=1}^n |\gamma(t_j) - \gamma(t_{j-1})| < \infty.$$

By reparametrization, one may assume that γ is Lipschitz with constant equal to $\text{length}(\gamma)$. We will also make use of the following simple (and well-known) criterion: a compact set Γ is the image of a rectifiable curve if and only if it is connected and $\mathcal{H}^1(\Gamma) < \infty$. Indeed, one may choose γ so that $\text{length}(\gamma) \leq C\mathcal{H}^1(\Gamma)$; see, for example, [1, 2]. Here and below, $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure.

The purpose of this paper is to prove

Theorem 1.1. *Let $d \geq 2$. There exists a doubling measure ν on $\mathbb{R}^d$ and a rectifiable curve Γ such that $\nu(\Gamma) > 0$.*

We note that doubling measures cannot charge even slightly more regular curves; indeed, the authors' initial belief was that a rectifiable curve could not carry any weight. As discussed in [4, §I.8.6], doubling measures give zero weight to any smooth hyper-surface. The argument, based on Lebesgue's density theorem (for ν), adapts without difficulty to show that for any connected set Γ ,

$$\nu(\{x \in \Gamma : \liminf_{r \rightarrow 0} r^{-1} \mathcal{H}^1(B(x, r) \cap \Gamma) < \infty\}) = 0.$$

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Therefore if Γ is a rectifiable curve, then $\nu|_\Gamma$ must be singular to $\mathcal{H}^1|_\Gamma$. Similarly, no doubling measure can charge an Ahlfors regular curve.

We will prove Theorem 1.1 by explicitly constructing a measure and a rectifiable curve.

2. PROOF

2.1. The measure. Our measure ν will be the d -fold product of a doubling measure μ on $\mathbb{R}$. The latter is constructed by a simple iterative procedure that we will now describe. It may be viewed as a variant of the classic Riesz product construction and a ‘lift the middle’ idea of Kahane (cf. [3]). A very general form of this construction appears in [6].

Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be the 1-periodic function

$$h(x) = \begin{cases} 2 & : x \in [\frac{1}{3}, \frac{2}{3}) + \mathbb{Z}, \\ -1 & : \text{otherwise.} \end{cases}$$

Then given $\delta \in (0, \frac{1}{3}]$, we define μ as the weak-* limit of

$$d\mu_n := \prod_{j=0}^{n-1} [1 + (1 - 3\delta)h(3^j x)] dx.$$

When $\delta = 1/3$, μ is Lebesgue measure.

By viewing points $x \in \mathbb{R}$ in terms of their ternary (i.e., base 3) expansion, we may interpret μ as the result of a sequence of independent trials. More precisely, let $\mathcal{T}_n$ denote the collection of triadic intervals of size 3^{-n} , that is,

$$(2.1) \quad \mathcal{T}_n = \{[i3^{-n}, (i+1)3^{-n}) : i \in \mathbb{Z}\}.$$

Then the measure of a triadic interval $I = [i3^{-n}, (i+1)3^{-n})$ is related to that of its parent $\hat{I}$, the unique interval in $\mathcal{T}_{n-1}$ containing I , by

$$(2.2) \quad \mu(I) = \begin{cases} (1 - 2\delta)\mu(\hat{I}) & : i \equiv 1 \pmod{3}, \\ \delta\mu(\hat{I}) & : \text{otherwise.} \end{cases}$$

Coupled with the fact that $\mu([i, i+1)) = 1$ for $i \in \mathbb{Z}$, condition (2.2) uniquely determines μ . In particular, we note that if $j, n \geq 0$, and $0 \leq i < 3^n$ are integers, then

$$(2.3) \quad \mu([j + i3^{-n}, j + (i+1)3^{-n})) = \delta^{n-k(i)}(1 - 2\delta)^{k(i)},$$

where $k(i)$ is the number of times the digit 1 appears in the ternary expansion of i .

We claim that μ is a doubling measure on $\mathbb{R}$. First let I and J be adjacent triadic intervals of equal size. By (2.3) we have that $\mu(I)/\mu(J) \leq \frac{1-2\delta}{\delta}$. Several applications of this show that $\mu(I)/\mu(J) \leq C(\delta)$ for any pair I and J of adjacent intervals of equal size. Thus μ is doubling.

Let ν be the product measure $\mu \times \cdots \times \mu$ on $\mathbb{R}^d$. This is a doubling measure: $\nu(I_1 \times \cdots \times I_d) \leq C(\delta)^d \nu(J_1 \times \cdots \times J_d)$ for any d pairs of identical or adjacent intervals I_l, J_l that obey $|I_l| = |J_l|$. Indeed, this holds even without the requirement that $|I_l| = |I_{l'}|$ for $l \neq l'$.

2.2. The basic building blocks.

Definition 2.1. Given integer parameters $0 \leq k \leq n$, we define $K(n, k) \subset [0, 1)$ via

$$(2.4) \quad K(n, k) = \bigcup \{I \in \mathcal{T}_n : I \subseteq [0, 1) \text{ and } \mu(I) \geq \delta^k (1 - 2\delta)^{n-k}\}.$$

Equivalently, if $\delta < \frac{1}{3}$, $K(n, k)$ is the set of those $x \in [0, 1)$ whose ternary expansion contains at most k zeros or twos amongst the first n digits.

Lemma 2.2. For $2\delta n \leq k \leq \frac{2}{3}n$ and $K = K(n, k)$ defined as in (2.4), we have

$$(2.5) \quad 1 - \mu(K) \leq \exp\{-2n(\frac{k}{n} - 2\delta)^2\}$$

and

$$(2.6) \quad |K| \leq 3^{-n} e^{k[1+\log(\delta^{-1})]}.$$

Proof. Both inequalities rest on standard estimates for tail probabilities for the binomial distribution. These are proved by the usual large deviation technique of Cramér (cf. [5, Theorem 1.3.13]):

$$\begin{aligned} \sum_{m \geq an} \binom{n}{m} p^m (1-p)^{n-m} &\leq \inf_{t \geq 0} \sum_{m=0}^n \binom{n}{m} p^m (1-p)^{n-m} e^{(m-an)t} \\ &= \inf_{t \geq 0} [e^{-at} (1-p + p e^t)]^n. \end{aligned}$$

This infimum can be determined exactly, and for $0 < p \leq a < 1$ we obtain

$$\sum_{m \geq an} \binom{n}{m} p^m (1-p)^{n-m} \leq e^{-nH(a,p)},$$

where

$$H(a, p) = a \log\left(\frac{a}{p}\right) + (1-a) \log\left(\frac{1-a}{1-p}\right).$$

For (2.5) we set $a = k/n$ and $p = 2\delta$ and make use of the fact that

$$H(a, p) \geq 2(a-p)^2.$$

Indeed, H and $\partial_a H$ vanish at $a = p$, while $\partial_a^2 H = a^{-1}(1-a)^{-1} \geq 4$.

To obtain (2.6), we set $p = \frac{1}{3}$ and $a = \frac{n-k}{n}$. We simplified the answer by using

$$H(a, p) \geq \log\left(\frac{1}{p}\right) - (1-a) \left[\log\left(\frac{1-p}{p}\right) + 1 + \log\left(\frac{1}{1-a}\right) \right],$$

which amounts simply to $a \log(a) + 1 - a = -\int_a^1 \log(t) dt \geq 0$. $\square$

Remark 2.3. Choosing $\delta < 2/9$ and $k = 3\delta n$ and sending $n \rightarrow \infty$, we see by Lemma 2.2 that μ gives all its weight to a set of Hausdorff dimension $O(\delta \log(\delta^{-1}))$. The precise dimension of μ is not important to us; however, we will exploit the fact that it can be made as small as we wish by sending $\delta \downarrow 0$. Indeed, the product measure ν cannot charge a set of Hausdorff dimension one (not to mention a rectifiable curve) unless μ gives positive weight to a set of dimension d^{-1} or smaller.

By definition, $K(n, k)$ is a union of intervals from $\mathcal{T}_n$. Correspondingly, the d -fold Cartesian product $K(n, k)^d$ can be viewed as a union of triadic cubes $Q \subseteq \mathbb{R}^d$ (with side-length 3^{-n}). We denote this collection of cubes by $\mathcal{K}^d(n, k)$. By (2.6),

$$(2.7) \quad \#\mathcal{K}^d(n, k) \leq e^{kd[1+\log(\delta^{-1})]}.$$

Similarly, we write $\mathcal{G}(n, k)$ for the gaps in $K(n, k)$, that is, the bounded connected components of $\mathbb{R} \setminus K(n, k)$. As each gap has a right end-point, (2.6) gives

$$(2.8) \quad \#\mathcal{G}(n, k) \leq e^{k[1+\log(\delta^{-1})]}.$$

Note also that $|\bigcup \mathcal{G}(n, k)| \leq 1$, as $K(n, k) \subseteq [0, 1]$.

We now define a curve $\Gamma(n, k) \subset \mathbb{R}^d$ which visits each cube $Q \in \mathcal{K}^d(n, k)$. Actually, we merely construct a connected family of line segments $\Gamma(n, k)$ that do this and bound its total length. As noted in the introduction, all segments in $\Gamma(n, k)$ can be traversed by a single curve of comparable total length.

The family $\Gamma(n, k)$ is the union of skeletons of rectangular boxes, where we define the *skeleton* of a box to be

$$\text{Sk}(I_1 \times \cdots \times I_d) = \bigcup_{j=1}^d \partial I_1 \times \cdots \times \partial I_{j-1} \times I_j \times \partial I_{j+1} \times \cdots \times \partial I_d.$$

Thus $\text{Sk}(Q)$ is the union of the edges—as opposed to vertices, faces, 3-faces, etc.—of the box Q . With this notation,

$$\Gamma(n, k) = \bigcup_{Q \in \mathcal{K}^d(n, k)} \text{Sk}(Q) \cup \bigcup_{I_1, \dots, I_d \in \mathcal{G}(n, k)} \text{Sk}(I_1 \times \cdots \times I_d).$$

Note that $\Gamma(n, k)$ is connected. We now estimate the total length of this set.

Lemma 2.4 (The length of the $\Gamma(n, k)$). *Assuming $2\delta n \leq k \leq \frac{2}{3}n$,*

$$(2.9) \quad \mathcal{H}^1(\Gamma(n, k)) \leq d2^d e^{dk[1+\log(\delta^{-1})]}.$$

Proof. By (2.7) and (2.8),

$$\begin{aligned} \mathcal{H}^1(\Gamma(n, k)) &\leq \sum_{Q \in \mathcal{K}^d(n, k)} \mathcal{H}^1(\text{Sk}(Q)) + \sum_{I_1, \dots, I_d \in \mathcal{G}(n, k)} \mathcal{H}^1(\text{Sk}(I_1 \times \cdots \times I_d)) \\ &= d2^{d-1} 3^{-n} [\#\mathcal{K}^d(n, k)] + d2^{d-1} [\#\mathcal{G}(n, k)]^{d-1} \sum_{I \in \mathcal{G}(n, k)} |I| \\ &\leq d2^{d-1} 3^{-n} e^{kd[1+\log(\delta^{-1})]} + d2^{d-1} e^{k(d-1)[1+\log(\delta^{-1})]}, \end{aligned}$$

which easily yields (2.9). $\square$

2.3. The curve. Using $\Gamma(n, k)$ as a building block, we now explain the iterative construction of the full curve Γ . It depends upon a collection of parameters $\{n_j, k_j\}_{j=1}^\infty$. The guiding principle is to replace each cube in $\mathcal{K}^d(n_j, k_j)$ by rescaled/translated copies of $\mathcal{K}^d(n_{j+1}, k_{j+1})$ and $\Gamma(n_{j+1}, k_{j+1})$. See Figure 1.

To this end, we define a version $\Gamma_Q(n, k)$ of $\Gamma(n, k)$ adapted to any cube Q :

$$\Gamma_Q(n, k) = A_Q(\Gamma(n, k)),$$

where A_Q is the affine transformation that maps $[0, 1]^d$ to Q . Similarly, we inductively define

$$\mathcal{K}_0 = \{[0, 1]^d\} \text{ and } \mathcal{K}_l = \bigcup_{Q \in \mathcal{K}_{l-1}} \{A_Q(Q') : Q' \in \mathcal{K}^d(n_l, k_l)\} \text{ for } l \geq 1.$$

Thus $\mathcal{K}_l$ is the collection of cubes remaining after the l^{th} iteration in the construction of Γ . Subsequent iterations will not modify Γ outside their union,

$$K_l = \bigcup \{Q : Q \in \mathcal{K}_l\}.$$

FIGURE 1. Each cube $Q \in \mathcal{K}^2(n_1, k_1)$ will be replaced by the affine image of $\mathcal{K}^2(n_2, k_2)$. We also illustrate one of the many rectangle-skeletons in $\Gamma(n_1, k_1)$.

We note that the cubes in $\mathcal{K}_l$ have disjoint interiors, and that by (2.7),

$$(2.10) \quad \begin{aligned} \#\mathcal{K}_l &\leq [\#\mathcal{K}_{l-1}] \exp\{k_l d[1 + \log(\delta^{-1})]\} \\ &\leq \exp\{(k_1 + \cdots + k_l)d[1 + \log(\delta^{-1})]\}. \end{aligned}$$

We define

$$(2.11) \quad \Gamma = \text{Sk}([0, 1]^d) \cup \bigcup_{l=1}^{\infty} \bigcup_{Q \in \mathcal{K}_{l-1}} \Gamma_Q(n_l, k_l) \cup \bigcap_{l=1}^{\infty} K_l.$$

Note that Γ is connected. The proof of Theorem 1.1 now reduces to the following two propositions, which show that $\mathcal{H}^1(\Gamma) < \infty$ and $\nu(\Gamma) > 0$ for a certain explicit choice of parameters.

Proposition 2.5 (The length of Γ). *Let $\delta > 0$ and $n_1 \in \mathbb{Z}$ be parameters so that*

$$(2.12) \quad 18d[\delta + \delta \log(\delta^{-1})] \leq \log(3)$$

and $k_1 := 3\delta n_1 \geq 1$ is an integer. If Γ is the curve defined above with parameters $n_l = ln_1$ and $k_l = lk_1$, then

$$(2.13) \quad \mathcal{H}^1(\Gamma) \leq 3d2^d e^{3\delta n_1[\delta + \delta \log(\delta^{-1})]}.$$

Proof. By (2.7) we have

$$\mathcal{H}^1\left(\bigcap_{l=1}^{\infty} K_l\right) \leq \prod_{l=1}^{\infty} (3^{-n_l} e^{dk_l(1+\log(\delta^{-1}))}) = 0.$$

Hence by (2.9) and (2.10),

$$\begin{aligned} \mathcal{H}^1(\Gamma) &\leq d2^{d-1} + \sum_{l=1}^{\infty} \sum_{Q \in \mathcal{K}_{l-1}} d2^d 3^{-(n_1 + \cdots + n_{l-1})} \exp\{dk_l[1 + \log(\delta^{-1})]\} \\ &\leq d2^d \left[1 + \sum_{l=1}^{\infty} 3^{-(n_1 + \cdots + n_{l-1})} \exp\{(k_1 + \cdots + k_l)d[1 + \log(\delta^{-1})]\} \right]. \end{aligned}$$

Inserting the values of our parameters and performing a few elementary manipulations, we find

$$\mathcal{H}^1(\Gamma_0) \leq d2^{d-1}e^{3dn_1[\delta+\delta\log(\delta^{-1})]} \left[2 + \sum_{l=2}^{\infty} \exp\left\{-\frac{1}{4}l(l-1)\log(3)n_1\right\} \right],$$

which yields (2.13) with a few more manipulations. $\square$

Proposition 2.6 (The measure of Γ). *Let δ and $\{n_l, k_l\}_{l=1}^{\infty}$ be as in Proposition 2.5. Then*

$$(2.14) \quad \nu(\Gamma) \geq \exp\left\{-\frac{de^{-2\delta^2 n_1}}{(1-e^{-2\delta^2 n_1})^2}\right\}.$$

Proof. By the dominated convergence theorem,

$$\nu(\Gamma) \geq \lim_{l \rightarrow \infty} \nu\left(\overline{\bigcup \{Q : Q \in \mathcal{K}_l\}}\right) \geq \lim_{l \rightarrow \infty} \nu\left(\bigcup \{Q : Q \in \mathcal{K}_l\}\right).$$

(In fact, since doubling measures cannot charge straight lines, equality actually holds above, but we will not need this.) Since the cubes in $\mathcal{K}_l$ have disjoint interiors, (2.5) and induction give us

$$\nu\left(\bigcup \{Q : Q \in \mathcal{K}_l\}\right) \geq [1 - e^{-2\delta^2 n_l}]^d \sum_{Q \in \mathcal{K}_{l-1}} \nu(Q) \geq \prod_{j=1}^l [1 - e^{-2\delta^2 n_j}]^d.$$

Inserting the values of our parameters and performing a few elementary manipulations, we conclude that

$$\nu(\Gamma) \geq \exp\left\{d \sum_{j=1}^{\infty} \log(1 - Z^j)\right\} \geq \exp\left\{-d \sum_{j,k=1}^{\infty} Z^{jk}\right\} \geq \exp\left\{-\frac{dZ}{(1-Z)^2}\right\},$$

where $Z := \exp\{-2\delta^2 n_1\}$. That proves (2.14). $\square$

In closing, we note that the curve Γ can be made to capture an arbitrarily large proportion of the ν -mass of the unit cube; one merely chooses the parameter n_1 large (with δ fixed).

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