FILTERED ALGEBRAIC ALGEBRAS

ALON REGEV

(Communicated by Birge Huisgen-Zimmermann)

ABSTRACT. Small and Zelmanov posed the question whether every element of a graded algebra over an uncountable field must be nilpotent, provided that the homogeneous elements are nilpotent. This question has recently been answered in the negative by Smoktunowicz. In this paper we prove that the answer is affirmative for associated graded algebras of filtered algebraic algebras. Our result is based on Amitsur's theorems on algebras over infinite fields.

1. INTRODUCTION

Throughout this paper, k is a field and A is a k -algebra.

Definition 1.1.

- (1) If A is a k -algebra, we say that A is *nil* if every element of A is nilpotent.
- (2) If A is a graded k -algebra, we say that A is *graded-nil* if every homogeneous element of A is nilpotent.

Clearly, every nil, graded algebra is graded-nil. The question of whether the converse to this statement is true has been of some interest.

Bartholdi [3] constructed a graded-nil but not nil algebra over a countable field. Small and Zelmanov [5] posed the question whether, for an uncountable field, every graded-nil algebra is nil. This was answered in the negative by A. Smoktunowicz [9], who constructed an algebra over any field which is graded-nil but not nil.

Let $A = \bigcup_{i \geq 0} F_i$ be a filtered k -algebra. Let $\text{gr } A$ be the associated graded algebra, and let $(\text{gr } A)_{\geq 1} = \frac{F_1}{F_0} \oplus \frac{F_2}{F_1} \oplus \dots$ be the positive part of $\text{gr } A$. Our main result is the following:

Theorem 1.2. *If k is uncountable and A is algebraic, then $(\text{gr } A)_{\geq 1}$ is nil.*

Thus, being graded-nil does imply being nil, for associated graded algebras of algebraic algebras over uncountable fields.

This result relates to the open question of whether there exists an affine division algebra over an uncountable field. Suppose that such an algebra exists, and suppose further that this algebra has a finite Gelfand-Kirillov dimension. By [2], it is algebraic. Then the positive part of the associated graded algebra is nil and has the same GK dimension as the original algebra. Thus the problem is reduced to the existence of a nil algebra of the same GK dimension.

Received by the editors April 24, 2009, and, in revised form, August 13, 2009, and September 14, 2009.

2010 *Mathematics Subject Classification.* Primary 16S15, 16U99.

©2010 American Mathematical Society
 Reverts to public domain 28 years from publication

The proof of Theorem 1.2 is based on Amitsur's Theorem 1.4 below.

Definition 1.3.

- (1) If A is a nil k -algebra, we say that A is *locally of bounded index (LBI)* over k if the elements of every finite-dimensional k -subspace of A have bounded index of nilpotence.
- (2) If A is an algebraic k -algebra, we say that A is *locally of bounded degree (LBD)* over k if the elements of every finite-dimensional k -subspace of A have bounded degree of algebraicity.

In [1] Amitsur proved

Theorem 1.4 ([1, Theorem 5]). *Let k be an uncountable field and let A be a k -algebra. If A is nil, then it is LBI over k . If A is algebraic over k , then it is LBD over k .*

This paper is part of the author's Ph.D. dissertation at the University of California, San Diego, written under the supervision of Lance Small, to whom he wishes to express his warm thanks.

2. THE POLYNOMIALS $p_{i_1, \dots, i_m}$

Throughout this paper k is a field and $k\langle x_1, \dots, x_m \rangle$ is the associative algebra of the noncommutative polynomials in $x_1, \dots, x_m$. Note that if $A = k\langle x_1, \dots, x_m \rangle$, then

$$(2.1) \quad A = \bigoplus_{n=0}^{\infty} A_n,$$

where A_n are the homogeneous polynomials in $x_1, \dots, x_m$ of total degree n .

The order-symmetric polynomials $p_{i_1, \dots, i_m}(x_1, \dots, x_m)$ are defined as follows, for any $i_1, \dots, i_k \in \mathbb{N}$.

Definition 2.1. We define $p_{i_1, \dots, i_m}(x_1, \dots, x_m)$ to be the sum of all the $\frac{(i_1 + \dots + i_m)!}{i_1! \dots i_m!}$ different monomials consisting of exactly i_j occurrences of x_j for each j . We take $p_{0, \dots, 0}(x_1, \dots, x_m) = 1$.

Definition 2.2. For any $n \geq 0$, any field k and any $x_1, \dots, x_m$, write

$$(2.2) \quad P_n(x_1, \dots, x_m) = \text{span}_k \{ p_{i_1, \dots, i_m}(x_1, \dots, x_m) \mid i_1 + \dots + i_m = n \}.$$

The polynomials $p_{i_1, \dots, i_m}$ can be used as follows. It is well known that for $f = \alpha_1 x_1 + \dots + \alpha_m x_m$,

$$(2.3) \quad f^n = \sum_{1 \leq j_1, \dots, j_n \leq m} (\alpha_{j_1} x_{j_1}) \cdots (\alpha_{j_n} x_{j_n}) = \sum_{1 \leq j_1, \dots, j_n \leq m} \alpha_{j_1} \cdots \alpha_{j_n} \cdot x_{j_1} \cdots x_{j_n}.$$

Lemma 2.3. *Let $\alpha_1, \dots, \alpha_m \in k$ and let $f = \alpha_1 x_1 + \dots + \alpha_m x_m$. Then*

$$(2.4) \quad f^n = \sum_{i_1 + \dots + i_m = n} \alpha_1^{i_1} \cdots \alpha_m^{i_m} \cdot p_{i_1, \dots, i_m}(x_1, \dots, x_m).$$

Proof. This follows from (2.3) and Definition 2.1, since for each monomial $x_{j_1} \cdots x_{j_n}$ that appears in $p_{i_1, \dots, i_m}$, the coefficient $\alpha_{j_1} \cdots \alpha_{j_n}$ in (2.3) is equal to $\alpha_1^{i_1} \cdots \alpha_m^{i_m}$. $\square$

Proposition 2.4. *Let $W = \text{span}_k \{ (\alpha_1 x_1 + \dots + \alpha_m x_m)^n \mid \alpha_1, \dots, \alpha_m \in k \}$. If $|k| \geq n + 1$, then $W = P_n(x_1, \dots, x_m)$.*

Proof. By Lemma 2.3, $W \subseteq P_n(x_1, \dots, x_m)$. The converse follows from [7, Lemma 1]. $\square$

Corollary 2.5. *Let A be a k -algebra with $|k| \geq n + 1$ and let $a_1, \dots, a_m \in A$. Let $W \subseteq A$ be a subspace of A . If*

$$(2.5) \quad (\alpha_1 a_1 + \dots + \alpha_m a_m)^n \in W$$

for all $\alpha_1, \dots, \alpha_m \in k$, then $P_n(a_1, \dots, a_m) \subseteq W$.

Proof. Let $W_x = W(x_1, \dots, x_m) = \text{span}_k\{(\alpha_1 x_1 + \dots + \alpha_m x_m)^n \mid \alpha_1, \dots, \alpha_m \in k\}$ and similarly $W_a = W(a_1, \dots, a_m) = \text{span}_k\{(\alpha_1 a_1 + \dots + \alpha_m a_m)^n \mid \alpha_1, \dots, \alpha_m \in k\}$, so that by assumption $W_a \subseteq W$. Since $|k| \geq n + 1$, by Proposition 2.4 $W_x = P_n(x_1, \dots, x_m)$. Replacing x_r by a_r implies that $W_a = P_n(a_1, \dots, a_m)$, which completes the proof. $\square$

Proposition 2.6. *Let A be an algebra over an infinite field k , and let $a_1, \dots, a_m \in A$. Then the subspace $U = ka_1 + \dots + ka_m \subseteq A$ is nil of bounded index at most n if and only if $P_n(a_1, \dots, a_m) = \{0\}$.*

Proof. By definition $U = ka_1 + \dots + ka_m$ is nil of bounded index at most n if and only if for all $\alpha_1, \dots, \alpha_m \in k$, $(\alpha_1 a_1 + \dots + \alpha_m a_m)^n = 0$. Thus the “if” part of the proposition follows from Lemma 2.3, and the converse is a direct consequence of Corollary 2.5 with $W = \{0\}$. $\square$

3. ALGEBRAICITY OF BOUNDED DEGREE

Recall the space $P_n(x_1, \dots, x_m)$ of Definition 2.2.

Definition 3.1. We write

$$P_{\leq r}(x_1, \dots, x_m) = \sum_{n=1}^r P_n(x_1, \dots, x_m) = \bigoplus_{n=1}^r P_n(x_1, \dots, x_m)$$

and

$$P_{\geq r}(x_1, \dots, x_m) = \sum_{n=r}^{\infty} P_n(x_1, \dots, x_m) = \bigoplus_{n=r}^{\infty} P_n(x_1, \dots, x_m).$$

Similarly, for a k -algebra A and elements $a_1, \dots, a_m \in A$, let $P_{\leq r}(a_1, \dots, a_m) = \sum_{n=1}^r P_n(a_1, \dots, a_m)$ and $P_{\geq r}(a_1, \dots, a_m) = \sum_{n=r}^{\infty} P_n(a_1, \dots, a_m)$.

Lemma 3.2. *Let $D \geq d$. Let A be a k -algebra with $|k| \geq D + 1$ and let $a_1, \dots, a_m \in A$. Assume that for all $\alpha_1, \dots, \alpha_m \in k$, $\alpha_1 a_1 + \dots + \alpha_m a_m$ is algebraic of degree at most d . Then $P_D(a_1, \dots, a_m) \subseteq P_{\leq d-1}(a_1, \dots, a_m)$.*

Proof. Write $P_j = P_j(a_1, \dots, a_m)$ for any j , and define $P_{\geq j}$ similarly. Let $u = \alpha_1 a_1 + \dots + \alpha_m a_m$. By algebraicity u^d is a linear combination of lower powers of u , therefore $u^d \in P_{\leq d-1}$. By Corollary 2.5 with $W = P_{\leq d-1}$, it follows that $P_d \subseteq P_{\leq d-1}$.

If $D + 1 \geq d + 2$, we can continue: By the same argument $P_{d+1} \subseteq P_{\leq d}$, while the previous step implies that $P_{\leq d} \subseteq P_{\leq d-1}$; hence $P_{d+1} \subseteq P_{\leq d-1}$. Continuing in this way we finally obtain $P_D \subseteq P_{\leq d-1}$. $\square$

Corollary 3.3. *Let k be infinite, let A be a k -algebra and let $a_1, \dots, a_m \in A$. If each element of $ka_1 + \dots + ka_m$ is algebraic of degree at most d , then $P_{\geq 0}(a_1, \dots, a_m) \subseteq P_{\leq d-1}(a_1, \dots, a_m)$ and, in particular, $\dim_k P_{\geq 0}(a_1, \dots, a_m) < \infty$.*

Definition 3.4. Let

$$M_{d,m} = \binom{d+m-1}{m}.$$

We note that

$$(3.1) \quad \dim_k P_n(x_1, \dots, x_m) = \binom{m+n-1}{m-1}$$

and

$$(3.2) \quad \dim_k P_{\leq n}(x_1, \dots, x_m) = \sum_{j=0}^n \dim_k P_j = \sum_{j=0}^n \binom{j+m-1}{m-1} = \binom{n+m}{m}.$$

Thus, given an algebra A and elements $a_1, \dots, a_m \in A$, we have

$$(3.3) \quad \dim_k P_{\leq d-1}(a_1, \dots, a_m) \leq M_{d,m}.$$

Lemma 3.5. *If $P_{\leq M_{d,m}}(a_1, \dots, a_m) \subseteq P_{\leq d-1}(a_1, \dots, a_m)$ for some $d \geq 1$, then the subspace $ka_1 + \dots + ka_m$ is algebraic of degree at most $M_{d,m}$.*

Proof. Let $a = \alpha_1 a_1 + \dots + \alpha_m a_m \in ka_1 + \dots + ka_m$. Then by assumption, for any $0 \leq n \leq M_{d,m}$ we have (using Lemma 2.3 with x_i replaced by a_i)

$$(3.4) \quad a^n \in P_n(a_1, \dots, a_m) \subseteq P_{\leq M_{d,m}}(a_1, \dots, a_m) \subseteq P_{\leq d-1}(a_1, \dots, a_m).$$

Thus

$$(3.5) \quad \dim_k \text{span}_k \{a^n \mid 0 \leq n \leq M_{d,m}\} \leq \dim_k P_{\leq d-1}(a_1, \dots, a_m) \leq M_{d,m}.$$

Therefore the set $\{a^n \mid 0 \leq n \leq M_{d,m}\}$ is linearly dependent over k , and hence a is algebraic of degree at most $M_{d,m}$. $\square$

The following corollaries are immediate.

Corollary 3.6. *If $P_{\geq 0}(a_1, \dots, a_m) \subseteq P_{\leq d-1}(a_1, \dots, a_m)$ for some $d \geq 1$, then $ka_1 + \dots + ka_m$ is algebraic of degree at most $M_{d,m}$.*

Corollary 3.7. *If $\dim_k P_{\geq 0}(a_1, \dots, a_m) < \infty$, then $ka_1 + \dots + ka_m$ is algebraic of bounded degree.*

We can now prove

Proposition 3.8 ([4, Lemma 16(i)]). *Let A be an algebra over an infinite field k , and let $a_1, \dots, a_m \in A$. Then the subspace $ka_1 + \dots + ka_m$ is algebraic of bounded degree if and only if $\dim_k P_{\geq 0}(a_1, \dots, a_m) < \infty$.*

Proof. One direction of the proof is given by Corollary 3.3, and the opposite direction by Corollary 3.7. $\square$

4. FILTERED ALGEBRAIC ALGEBRAS

Let A be a filtered k -algebra with the filtration $\{F_n\}_{n \geq 0}$. We aim to prove that if k is uncountable and A is algebraic, then every element of $(\text{gr } A)_{\geq 1}$ is nilpotent. In particular, being graded-nil does imply being nil, for associated graded algebras of algebraic algebras over uncountable fields.

We first note that if $a_{p_1}, \dots, a_{p_n}$ satisfy $a_{p_i} \in F_{p_i}$ for all $1 \leq i \leq n$, then

$$(4.1) \quad (a_{p_1} + F_{p_1-1})(a_{p_2} + F_{p_2-1}) \cdots (a_{p_n} + F_{p_n-1}) = a_{p_1} a_{p_2} \cdots a_{p_n} + F_{p_1 + \dots + p_n - 1}.$$

We note the following properties of the order-symmetric polynomials in graded and filtered algebras. If the k -algebra A is filtered by $\{F_n\}_{n \geq 0}$ and $a_p, \dots, a_q \in A$ satisfy $a_i \in F_i$ for all $p \leq i \leq q$, then

$$(4.2) \quad p_{i_1, \dots, i_m}(a_p, \dots, a_q) \in F_{i_1 p + \dots + i_m q},$$

where $m = q - p + 1$. Similarly, suppose $B = \bigoplus_{i \geq 0} B_i$ is a graded k -algebra. If $b_p, \dots, b_q$ are m homogeneous elements with $b_i \in B_i$, then

$$(4.3) \quad p_{i_1, \dots, i_m}(b_p, \dots, b_q) \in B_{i_1 p + \dots + i_m q}.$$

Now let A be a k -algebra filtered by $\{F_n\}_{n \geq 0}$ and let $B = \text{gr}(A)$ be its associated graded algebra. Let $a_p, \dots, a_q \in A$ satisfy $a_i \in F_i$ for each $p \leq i \leq q$, and let $b_i = a_i + F_{i-1} \in B$. We first note that by (4.1), if $p \leq p_j \leq q$ for all $1 \leq j \leq n$ then

$$(4.4) \quad b_{p_1} \cdots b_{p_n} = a_{p_1} \cdots a_{p_n} + F_{p_1 + \dots + p_n - 1}.$$

Therefore if this product contains exactly i_j occurrences of b_j for each $p \leq j \leq q$, then

$$(4.5) \quad b_{p_1} \cdots b_{p_n} = a_{p_1} \cdots a_{p_n} + F_{i_p p + \dots + i_q q - 1}.$$

Since each $p_{i_p, \dots, i_q}(b_p, \dots, b_q)$ is a sum of such products, we have

$$(4.6) \quad p_{i_p, \dots, i_q}(b_p, \dots, b_q) = p_{i_p, \dots, i_q}(a_p, \dots, a_q) + F_{i_p p + \dots + i_q q - 1}.$$

Theorem 4.1. *Let A be a filtered k -algebra, with a filtration $\{F_n\}_{n \geq 0}$. Let $B = \text{gr}(A)$ be its associated graded algebra. If A is LBD, then $B_{\geq 1}$ is LBI. In particular, if k is uncountable and A is a filtered algebraic k -algebra, then $\text{gr}(A)_{\geq 1}$ is nil (and LBI).*

Proof. Let S be a finite-dimensional subspace of B . We may assume $S = B_p \oplus \dots \oplus B_q$, with $1 \leq p \leq q$, since every finite-dimensional subspace of $B_{\geq 1}$ is contained in a subspace of this form. Let $b = b_p + \dots + b_q$ be any element of S , where $b_i \in \frac{F_i}{F_{i-1}}$ (we take $F_{-1} = \{0\}$); that is, for each $i \in \{p, \dots, q\}$, we have $b_i = a_i + F_{i-1}$, with $a_i \in F_i$. By assumption, the subspace $F_p + \dots + F_q$ is algebraic of bounded degree, say d . Therefore by Corollary 3.3, we have

$$(4.7) \quad P_{\geq 0}(a_p, \dots, a_q) \subseteq P_{\leq d-1}(a_p, \dots, a_q).$$

Now $P_{\leq d-1}(a_p, \dots, a_q)$ is spanned by elements of the form $p_{i_1, \dots, i_m}(a_p, \dots, a_q)$, with $i_1 + \dots + i_m = d - 1$. Such $i_1, \dots, i_m$ satisfy $i_1 p + \dots + i_m q \leq q(d - 1)$. Therefore by (4.2) this implies

$$(4.8) \quad P_{\geq 0}(a_p, \dots, a_q) \subseteq F_{(d-1)q}.$$

Now let

$$(4.9) \quad N = \left\lceil \frac{(d-1)q}{p} + 1 \right\rceil$$

and suppose $i_p + \dots + i_q \geq N$. Then

$$(4.10) \quad p i_p + \dots + q i_q - 1 \geq pN - 1 \geq (d-1)q.$$

Therefore

$$(4.11) \quad p_{i_p, \dots, i_q}(a_p, \dots, a_q) \in F_{(d-1)q} \subseteq F_{p i_p + \dots + q i_q - 1},$$

and so

$$(4.12) \quad p_{i_p, \dots, i_q}(b_p, \dots, b_q) = p_{i_p, \dots, i_q}(a_p, \dots, a_q) + F_{p i_p + \dots + q i_q - 1} = 0_B.$$

Thus $P_N(b_1, \dots, b_n) = \{0_B\}$, and it follows from Proposition 2.6 that $(b_p + \dots + b_q)^N = 0$. This shows that S is nil of bounded index at most N . Thus B is LBI. The final statement follows from Theorem 1.4. $\square$

Remark 4.2. We can use the bound N of (4.9) to derive a relation between the degrees of algebraicity in A and the indices of nilpotence in $\text{gr}(A)$: namely, if each F_i is algebraic of bounded degree at most d_i , then each $\frac{F_1}{F_0} \oplus \dots \oplus \frac{F_r}{F_{r-1}}$ is nil of bounded index at most $(d_r - 1)r + 1 \leq rd_r$.

5. INTEGRALITY

Definition 5.1. For a k -algebra A , let $A[x]$ be the algebra of polynomials over x with coefficients in A . We say that an element $a(x) \in A[x]$ is *integral* of degree n over $k[x]$ if $a(x)^n + p_{n-1}(x)a(x)^{n-1} + \dots + p_0(x) = 0$ for some $n \geq 1$ and polynomials $p_i(x) \in k[x]$.

Note that if $a(x)$ is integral over $k[x]$, then in particular it is algebraic over $k(x)$, the field of rational functions over k .

Lemma 5.2. Let A be a filtered k -algebra, with filtration $\{A_n\}_{n \geq 0}$. Let $R = \bigoplus_{n \geq 0} x^n A_n \subset A[x]$ be the Rees algebra. Let $a(x) = a_1x + \dots + a_mx^m \in R$, where $a_i \in A_i$, and suppose that $a(x)$ is integral over $k[x]$. Then $a(x)^N \in xR$ for some $N \geq 0$.

Proof. Suppose $a(x)$ is integral over $k[x]$ of degree n . Then for any $N \geq 0$ we have

$$(5.1) \quad a(x)^N = q_{n-1}(x)a(x)^{n-1} + \dots + q_1(x)a(x) + q_0(x)$$

for some polynomials $q_i(x) \in k[x]$. Choose $N \geq m(n-1) + 1$. The left-hand side of (5.1) is in $x^N A[x]$. The right-hand side is in $A_{m(n-1)}[x]$ (i.e., it is a polynomial in x with coefficients in the subspace $A_{m(n-1)}$). Thus $a(x)^N \in x^N A[x] \cap A_{m(n-1)}[x] \subseteq x^N A_{m(n-1)}[x] \subseteq x^N A_{N-1}[x] \subseteq \bigoplus_{n \geq 1} x^n A_{n-1} = xR$. $\square$

Corollary 5.3. Let A be a filtered algebra and suppose that every element of the Rees algebra $R = \bigoplus_{n \geq 0} x^n A_n$ is integral over $k[x]$. Then $\text{gr}(A)_{\geq 1}$ is nil.

Proof. The map $\phi : \text{gr}(A) \rightarrow \frac{R}{xR}$ defined by $\phi(a_0 + (a_1 + A_0) + \dots + (a_m + A_{m-1})) = a_0 + a_1x + \dots + a_mx^m + xR$ is a graded algebra isomorphism. Under this map, an element $(a_1 + A_0) + \dots + (a_m + A_{m-1}) \in \text{gr}(A)_{\geq 1}$ ($a_i \in A_i$) corresponds to $a_1x + \dots + a_mx^m + xR \in \frac{R}{xR}$. By Lemma 5.2 all such elements are nilpotent. $\square$

Remark 5.4. If A is a filtered k -algebra which satisfies the LBD property (for example, if k is uncountable and A is algebraic over k), then by [1, Lemma 6], the extension algebra $A \otimes_k k(x)$ is LBD over $k(x)$.

This raises the following question.

Question 5.5. If $A \otimes_k k(x)$ is LBD over $k(x)$, is every element of $A[x]$ integral over $k[x]$?

In view of Corollary 5.3 (and since $R \subset A[x]$), a positive answer would give another proof of Theorem 4.1.

REFERENCES

- [1] S.A. Amitsur. Algebras over infinite fields. *Proceedings of the American Mathematical Society*, 7(1):35–48, 1956. MR0075933 (17:822b)
- [2] S.A. Amitsur. Countably generated division algebras over nondenumerable fields. *Bulletin of the Research Council of Israel, Section F*, 7(1):39, 1957.
- [3] L. Bartholdi. Branch rings, thinned rings, tree enveloping rings. *Israel J. Math.*, 154:93–139, 2006. MR2254535 (2007k:20051)
- [4] G. Bergman. Radicals, tensor products and algebraicity. *Israel Mathematical Conference Proceedings*, 1:150–192, 1989. MR1029309 (91d:16033)
- [5] L. Small and E. Zelmanov. Private communication, 2006.
- [6] A. Regev. Remarks on a theorem of Amitsur. *Communications in Algebra* (to appear).
- [7] I. Kaplansky. On a problem of Kurosch and Jacobson. *Bull. Amer. Math. Soc.*, 52(12):1033–1035, 1946. MR0019600 (8:435a)
- [8] I. Schur. Über eine Klasse von Matrizen, die sich einer gegebenen Matrix zuordnen lassen. Dissertation (1901); reprinted in *Gesammelte Abhandlungen, Band, I*, 1–72, Springer Verlag, Heidelberg, 1973. MR0462891 (57:2858a)
- [9] A. Smoktunowicz. The Jacobson radical of rings with nilpotent homogeneous elements. *Bull. London Math. Soc.*, 40:917–928, 2008. MR2471940

DEPARTMENT OF MATHEMATICAL SCIENCES, WATSON HALL 320, NORTHERN ILLINOIS UNIVERSITY, DEKALB, ILLINOIS 60115

E-mail address: regev@math.niu.edu