ON REGULARITY CRITERIA FOR WEAK SOLUTIONS TO THE MICROPOLAR FLUID EQUATIONS IN LORENTZ SPACE

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ABSTRACT. In this paper the regularity of weak solutions and blow-up criteria for smooth solutions to the micropolar fluid equations in three dimensional space are studied in the Lorentz space $L^{p,\infty}(\mathbb{R}^3)$. We obtain that if $u \in L^q(0, T; L^{p,\infty}(\mathbb{R}^3))$ for $\frac{2}{q} + \frac{3}{p} \leq 1$ with $3 < p \leq \infty$ or if $\nabla u \in L^q(0, T; L^{p,\infty}(\mathbb{R}^3))$ for $\frac{2}{q} + \frac{3}{p} \leq 2$ with $\frac{3}{2} < p \leq \infty$ or if the pressure $P \in L^q(0, T; L^{p,\infty}(\mathbb{R}^3))$ for $\frac{2}{q} + \frac{3}{p} \leq 2$ with $\frac{3}{2} < p \leq \infty$ or if $\nabla P \in L^q(0, T; L^{p,\infty}(\mathbb{R}^3))$ for $\frac{2}{q} + \frac{3}{p} \leq 3$ with $1 < p \leq \infty$, then the weak solution (u, ω) satisfying the energy inequality is a smooth solution on $[0, T)$.

1. INTRODUCTION

This paper concerns the regularity of weak solutions and blow-up criteria for smooth solutions to the micropolar fluid equations in three dimensions:

$$(1.1) \quad \begin{cases} \frac{\partial u}{\partial t} - (\mu + \chi)\Delta u + u \cdot \nabla u + \nabla P - \chi \nabla \times \omega = 0, \\ \frac{\partial \omega}{\partial t} - \gamma \Delta \omega - \kappa \nabla \operatorname{div} \omega + 2\chi \omega + u \cdot \nabla \omega - \chi \nabla \times u = 0, \\ \operatorname{div} u = 0, \\ u(x, 0) = u_0(x), \quad \omega(x, 0) = \omega_0(x), \end{cases}$$

where $u = (u_1(t, x), u_2(t, x), u_3(t, x))$ denotes the velocity of the fluid at a point $x \in \mathbb{R}^3$ and time $t \in [0, T)$, and $\omega = (\omega_1(t, x), \omega_2(t, x), \omega_3(t, x))$ and $P = P(t, x)$ denote, respectively, the micro-rotational velocity and the hydrostatic pressure. u_0 and ω_0 are the prescribed initial data for the velocity and angular velocity, with $\operatorname{div} u_0 = 0$. μ is the kinematic viscosity, χ is the vortex viscosity, and κ and γ are spin viscosities. A theory of micropolar fluids was first proposed by Eringen [8] in 1966, enabling us to consider some physical phenomena that cannot be treated by the classical Navier-Stokes equations for viscous incompressible fluids, for example, the motion of animal blood, liquid crystals and dilute aqueous polymer solutions,

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etc. The problems of existence of weak and strong solutions were treated by Galdi and Rionero [10] and Yamaguchi [30], respectively. If, further, the vortex viscosity $\chi = 0$, then the velocity u does not depend on the micro-rotation field ω , and the first equation reduces to the classic Navier-Stokes equation, which has been intensively analyzed; see, for example, the classic books by Ladyzhenskaya [16], Lions [19] or Lemarié-Rieusset [18].

There is a vast literature on the mathematical theory of micropolar fluid equations (see, for example, [17, 30, 10, 9, 7, 29, 4, 5]). The existence and uniqueness of global solutions were extensively studied by Lange [17], Galdi and Rionero [10], and Yamaguchi [30]. Recently, Ferreira and Villamizar-Roa [9] considered the existence and stability of solutions to the micropolar fluid equations in exterior domains. Villamizar-Roa and Rodríguez-Bellido [29] studied the micropolar system in a bounded domain using the semigroup approach in L^p , showing the global existence of strong solutions for small data and the asymptotic behavior and stability of the solutions. Concerning the dynamical behavior of solutions to equations (1.1) one may refer to [4, 5, 7] and the references therein.

The purpose of this paper is to study the regularity of weak solutions and breakdown criteria for smooth solutions to the micropolar fluid equations (1.1). The classic blow-up criteria for smooth solutions to the Navier-Stokes equations also hold for the micropolar fluid equations. For the Navier-Stokes equations, Serrin [25], Prodi [23] and Veiga [28] established classic Serrin-type regularity criteria for weak solutions in terms of u or its gradient ∇u . Later, many improvements and extensions were established; for example, see [15, 13, 14, 32] and the references therein. Berselli and Galdi [2] and Chae and Lee [3] obtained regularity criteria for weak solutions in terms of the pressure P or its gradient ∇P . Later, Zhou extended the criteria in terms of the pressure and its gradient to a general domain [33], and Zhou [34, 35] and Struwe [27] obtained regularity criteria for weak solutions in terms of the gradient of pressure.

Theorem 1.1. *Suppose $u \in L^\infty([0, T]; L^2(\mathbb{R}^n)) \cap L^2([0, T]; H^1(\mathbb{R}^n))$ is a Leray-Hopf weak solution to the Navier-Stokes equations, and P is the pressure. Suppose one of the following conditions is satisfied:*

- (1) $u \in L^q(0, T; L^p(\mathbb{R}^n))$ for $\frac{2}{q} + \frac{n}{p} \leq 1$ with $n < p \leq \infty$;
- (2) $\nabla u \in L^q(0, T; L^p(\mathbb{R}^n))$ for $\frac{2}{q} + \frac{n}{p} \leq 2$ with $\frac{n}{2} < p \leq \infty$;
- (3) $P \in L^q(0, T; L^p(\mathbb{R}^n))$ for $\frac{2}{q} + \frac{n}{p} \leq 2$ with $\frac{n}{2} < p \leq \infty$;
- (4) $\nabla P \in L^q(0, T; L^p(\mathbb{R}^n))$ for $\frac{2}{q} + \frac{n}{p} \leq 3$ with $\frac{n}{3} < p < \infty$.

Then u is a smooth solution on $[0, T]$.

Yuan [31] established regularity criteria, similar to the Serrin-type criteria, for weak solutions to the magneto-micropolar equations, which consist of the micropolar equations (1.1) coupled with a magnetic field b , as follows.

Theorem 1.2. *Let $(u_0, \omega_0, b_0) \in L^2(\mathbb{R}^3)$ with $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$. Assume that (u, ω, b) is a Leray-Hopf type weak solution to the magneto-micropolar equations. Suppose one of the following conditions holds:*

- (1) $u \in L^q(0, T; L^p(\mathbb{R}^3))$ for $\frac{2}{q} + \frac{3}{p} \leq 1$ with $3 < p \leq \infty$;
- (2) $\nabla u \in L^q(0, T; L^p(\mathbb{R}^3))$ for $\frac{2}{q} + \frac{3}{p} \leq 2$ with $\frac{3}{2} < p \leq \infty$.

Then (u, ω, b) is a smooth solution on $[0, T]$ with initial value (u_0, ω_0, b_0) .

It is worth noting that the regularity conditions for weak solutions to the magneto-micropolar equations are imposed only on the velocity field u , which is very important. For the magneto-hydrodynamic equations, He and Xin [12] first studied and established regularity criteria imposed only on the velocity field u or its gradient ∇u . Later, Zhou [36] improved these regularity criteria imposed only on u or its gradient ∇u , and He and Wang [11] improved them to weak L^p spaces, imposed only on u or its gradient ∇u ; Chen, Miao and Zhang [6] also improved the criteria to a more general Besov-type space on Littlewood-Paley decomposition imposed only on $\nabla \times u$. Regularity criteria for weak solutions to the system (1.1) play an important role in understanding the physical essence of micropolar fluid motion. The aim of this paper is to prove that to secure the regularity of weak solutions to (1.1), one only needs to impose conditions on the velocity field u , its gradient ∇u or the pressure of the fluid in Lorentz spaces. In particular, one needs only one of the following conditions to prove regularity of weak solutions (u, ω) on $[0, T]$:

- (1) $u \in L^q(0, T; L^{p, \infty}(\mathbb{R}^3))$ for $\frac{2}{q} + \frac{3}{p} \leq 1$ with $3 < p \leq \infty$;
- (2) $\nabla u \in L^q(0, T; L^{p, \infty}(\mathbb{R}^3))$ for $\frac{2}{q} + \frac{3}{p} \leq 2$ with $\frac{3}{2} < p \leq \infty$;
- (3) $P \in L^q(0, T; L^{p, \infty}(\mathbb{R}^3))$ for $\frac{2}{q} + \frac{3}{p} \leq 2$ with $\frac{3}{2} < p \leq \infty$;
- (4) $\nabla P \in L^q(0, T; L^{p, \infty}(\mathbb{R}^3))$ for $\frac{2}{q} + \frac{3}{p} \leq 3$ with $1 < p < \infty$.

This demonstrates that, in the regularity of weak solutions, the micro-rotational velocity ω of particles plays a less important role than the velocity u does, and the regularity of weak solutions to (1.1) is dominated by the velocity u of the fluid.

We conclude this introduction by describing the plan of the paper. We give our main results on blow-up criteria for a smooth solution to (1.1) in section 2, and as an application we prove the regularity of weak solutions. Section 3 is devoted to proving Theorems 2.1 and 2.2.

2. MAIN RESULTS

Before stating our main results we introduce some function spaces, notation and a generalized Hölder inequality. $C_{0, \sigma}^\infty(\mathbb{R}^3)$ denotes the set of all C^∞ vector functions $f(x) = (f_1(x), f_2(x), f_3(x))$ with compact support such that $\operatorname{div} f(x) = 0$. $L_\sigma^r(\mathbb{R}^3)$ is the closure of the $C_{0, \sigma}^\infty(\mathbb{R}^3)$ -functions with respect to the L^r -norm $\|\cdot\|_r$ for $1 \leq r \leq \infty$. $H_\sigma^s(\mathbb{R}^3)$ denotes the closure of $C_{0, \sigma}^\infty(\mathbb{R}^3)$ with respect to the H^s -norm $\|f\|_{H^s} = \|(1 - \Delta)^{\frac{s}{2}} f\|_2$, for $s \geq 0$.

In the following we recall Lorentz spaces. Let $(X, \mathcal{M}, \mu)$ be a non-atomic measurable space. For a complex- or real-valued μ -measurable function $f(x)$ defined on X , its distribution function is defined by

$$(2.1) \quad f_*(\sigma) = \mu\{x \in X : |f(x)| > \sigma\}, \text{ for } \sigma > 0,$$

which is non-increasing and continuous from the right. Furthermore, its non-increasing rearrangement f^* is defined by

$$(2.2) \quad f^*(t) = \inf\{s > 0 : f_*(s) \leq t\}, \text{ for } t > 0,$$

which is also non-increasing and continuous from the right and has the same distribution function as $f(x)$.

The Lorentz space $L^{p,q}$ on $(X, \mathcal{M}, \mu)$ is the collection of all real- or complex-valued μ -measurable functions $f(x)$ defined on X such that $\|f\|_{p,q} < \infty$, where

$$(2.3) \quad \|f\|_{p,q} = \begin{cases} \left(\frac{q}{p} \int_0^\infty \left(t^{\frac{1}{p}} f^*(t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}, & \text{if } 1 \leq p < \infty, \ 1 < q < \infty, \\ \sup_{t>0} t^{\frac{1}{p}} f^*(t), & \text{if } 1 \leq p \leq \infty, \ q = \infty. \end{cases}$$

If $q = \infty$, we write $L^{p,\infty}(\mathbb{R}^3)$ as $L_w^p(\mathbb{R}^3)$, which is the weak L^p space. Moreover

$$(2.4) \quad \|f\|_{p,\infty} = \sup_{t>0} t^{\frac{1}{p}} f^*(t) = \sup_{\alpha>0} \alpha (f_*(\alpha))^{\frac{1}{p}},$$

for any $f(x) \in L^{p,\infty}$. For details see [20, 21] and [26].

We also need the Hölder inequality in Lorentz spaces, which we recall as follows; for details see O'Neil [22].

Proposition 2.1. *Let $1 < p_1, p_2, r < \infty$ with*

$$\frac{1}{p_1} + \frac{1}{p_2} < 1, \quad \frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2},$$

and $1 \leq q_1, q_2, s \leq \infty$ with

$$\frac{1}{q_1} + \frac{1}{q_2} \geq \frac{1}{s}.$$

If $f \in L^{p_1, q_1}$ and $g \in L^{p_2, q_2}$, then $fg \in L^{r, s}$, and the generalized Hölder inequality

$$(2.5) \quad \|h\|_{r,s} \leq r' \|f\|_{p_1, q_1} \|g\|_{p_2, q_2}$$

holds, where r' stands for the dual of r , i.e. $\frac{1}{r} + \frac{1}{r'} = 1$.

We state the main results as follows.

Theorem 2.1. *Let $u_0(x) \in H_\sigma^1(\mathbb{R}^3)$ and $\omega_0(x) \in H^1(\mathbb{R}^3)$. Assume that the pair $u(t, x) \in C([0, T]; H_\sigma^1(\mathbb{R}^3)) \cap C((0, T); H_\sigma^2(\mathbb{R}^3))$ and $\omega(t, x) \in C([0, T]; H^1(\mathbb{R}^3)) \cap C((0, T); H^2(\mathbb{R}^3))$ is a smooth solution to the equations (1.1). Suppose u satisfies one of the following conditions:*

- (a) $u(t, x) \in L^q((0, T); L^{p,\infty}(\mathbb{R}^3))$, for $\frac{2}{q} + \frac{3}{p} \leq 1$ with $3 < p \leq \infty$;
- (b) $\nabla u(t, x) \in L^q((0, T); L^{p,\infty}(\mathbb{R}^3))$, for $\frac{2}{q} + \frac{3}{p} \leq 2$ with $\frac{3}{2} < p \leq \infty$.

Then the solution (u, ω) can be extended smoothly to $[0, T')$ for some $T' > T$.

Theorem 2.2. *Let $u_0(x) \in L_\sigma^4(\mathbb{R}^3)$ and $\omega_0(x) \in L^4(\mathbb{R}^3)$. Assume that the pair $u(t, x) \in C([0, T]; L_\sigma^4(\mathbb{R}^3)) \cap C((0, T); H_\sigma^{1,4}(\mathbb{R}^3))$ and $\omega(t, x) \in C([0, T]; L^4(\mathbb{R}^3)) \cap C((0, T); H^{1,4}(\mathbb{R}^3))$ is a smooth solution to the equations (1.1), and P is the pressure. If P satisfies the condition*

- (1) $P(t, x) \in L^q((0, T); L^{p,\infty}(\mathbb{R}^3))$, for $\frac{2}{q} + \frac{3}{p} \leq 2$ with $\frac{3}{2} < p \leq \infty$,

or the gradient of the pressure ∇P satisfies the condition

- (2) $\nabla P(t, x) \in L^q((0, T); L^{p,\infty}(\mathbb{R}^3))$, for $\frac{2}{q} + \frac{3}{p} \leq 3$ with $1 < p \leq \infty$,

then the solution (u, ω) can be extended smoothly beyond $t = T$.

We next consider criteria for regularity of weak solutions to the micropolar equations (1.1); for this purpose we first introduce the definition of a weak solution.

Definition 2.1. Let $u_0(x) \in L^2_\sigma(\mathbb{R}^3)$ and $\omega_0(x) \in L^2(\mathbb{R}^3)$. A measurable function $(u(t, x), \omega(t, x))$ is called a weak solution to the micropolar equations (1.1) on $[0, T]$ if

(a)

$$\begin{aligned} u(t, x) &\in L^\infty([0, T]; L^2_\sigma(\mathbb{R}^3)) \cap L^2([0, T]; H^1_\sigma(\mathbb{R}^3)), \\ \omega(t, x) &\in L^\infty([0, T]; L^2(\mathbb{R}^3)) \cap L^2([0, T]; H^1(\mathbb{R}^3)); \end{aligned}$$

(b)

$$\begin{aligned} \int_0^T \{-(u, \partial_\tau \varphi) + (\mu + \chi)(\nabla u, \nabla \varphi) - (u \cdot \nabla \varphi, u) + \chi(\nabla \times \varphi, \omega)\} d\tau &= (u_0, \varphi(0)), \\ \int_0^T \{-(\omega, \partial_\tau \phi) + \gamma(\nabla \omega, \nabla \phi) + \kappa(\operatorname{div} \omega, \operatorname{div} \phi) \\ &+ 2\chi(\omega, \phi) - (u \cdot \nabla \phi, \omega) + \chi(\nabla \times \phi, u)\} d\tau = (\omega_0, \phi(0)), \end{aligned}$$

for any $\varphi(t, x) \in H^1([0, T]; H^1_\sigma(\mathbb{R}^3))$ and $\phi(t, x) \in H^1([0, T]; H^1(\mathbb{R}^3))$ with $\varphi(T) = 0$ and $\phi(T) = 0$.

In [24], Rojas-Medar and Boldrini proved the global existence of weak solutions to the equations of magneto-micropolar fluid motion by the Galerkin method. The weak solutions (u, ω) also satisfy the energy inequality

(2.6)

$$\begin{aligned} \|(u, \omega)\|_2^2 + 2\mu \int_0^t \|\nabla u\|_2^2 ds + 2\gamma \int_0^t \|\nabla \omega\|_2^2 ds + 2\kappa \int_0^t \|\operatorname{div} \omega\|_2^2 ds + 2\chi \int_0^t \|\omega\|_2^2 ds \\ \leq \|(u(0), \omega(0))\|_2^2, \end{aligned}$$

for $0 < t \leq T$.

As immediate corollaries we establish the regularity criteria for weak solutions.

Corollary 2.1. Let $u_0(x) \in H^1_\sigma(\mathbb{R}^3)$ and $\omega_0(x) \in H^1(\mathbb{R}^3)$. Assume $(u(t, x), \omega(t, x))$ is a weak solution to the equations (1.1) and satisfies the energy inequality (2.6). Suppose u satisfies one of the following conditions:

- (a) $u(t, x) \in L^q((0, T); L^{p,\infty}(\mathbb{R}^3))$ for $\frac{2}{q} + \frac{3}{p} \leq 1$ with $3 < p \leq \infty$;
- (b) $\nabla u(t, x) \in L^q((0, T); L^{p,\infty}(\mathbb{R}^3))$ for $\frac{2}{q} + \frac{3}{p} \leq 2$ with $\frac{3}{2} < p \leq \infty$.

Then the solution (u, ω) is a regular solution on $(0, T]$.

Corollary 2.2. Let $u_0(x) \in L^2_\sigma \cap L^4_\sigma(\mathbb{R}^3)$ and $\omega_0(x) \in L^2 \cap L^4(\mathbb{R}^3)$. Assume that $(u(t, x), \omega(t, x))$ is a weak solution to the equations (1.1) and satisfies the energy inequality (2.6), and P is the pressure. If P satisfies the condition

- (1) $P(t, x) \in L^q((0, T); L^{p,\infty}(\mathbb{R}^3))$, for $\frac{2}{q} + \frac{3}{p} \leq 2$ with $\frac{3}{2} < p \leq \infty$,

or the gradient of the pressure ∇P satisfies the condition

- (2) $\nabla P(t, x) \in L^q((0, T); L^{p,\infty}(\mathbb{R}^3))$, for $\frac{2}{q} + \frac{3}{p} \leq 3$ with $1 < p \leq \infty$,

then the solution (u, ω) is a regular solution on $(0, T]$.

Remark 2.1. In Theorems 2.1 and 2.2, if $p = \infty$, then the space $L^{\infty,\infty}$ is identified with L^∞ .

Remark 2.2. In this paper the regularity results Theorem 2.1 and Corollary 2.1 are established for the solution of the micropolar equations (1.1). By the coupling of the velocity field and the magnetic field, the conclusions of Theorem 2.1 and Corollary 2.1 are also valid for the magneto-micropolar equations.

Remark 2.3. For the magneto-hydrodynamic equations, He and Wang [11] proved that to assure the regularity of weak solutions one needs only the condition

$$(2.7) \quad \nabla u \in L^q((0, T); L^{p,\infty}(\mathbb{R}^3)),$$

for $\frac{2}{q} + \frac{3}{p} = 2$ with $1 < q \leq 2$. In our Theorem 2.1(b) and Corollary 2.2(b) the condition $1 \leq q < \infty$ is more general than that for (2.7). Moreover, our proof, which is based on a priori estimate of the H^1 -norm of the solution, is simple, while that in [11] was based on a priori estimate of the L^p -norm for $p \geq 3$.

The proofs of Corollaries 2.1 and 2.2 are standard. For completeness we sketch the proof of Corollary 2.1 only. Since $u_0(x) \in H_\sigma^1(\mathbb{R}^3)$ and $\omega_0(x) \in H^1(\mathbb{R}^3)$, by the local existence theorem for strong solutions to the micropolar equations (1.1) there exists a unique solution $(\hat{u}, \hat{\omega})$ with $\hat{u}(t, x) \in C([0, T^*]; H_\sigma^1(\mathbb{R}^3))$ and $\hat{\omega}(t, x) \in C([0, T^*]; H^1(\mathbb{R}^3))$ on a small time interval $[0, T^*)$. Since (u, ω) is a weak solution satisfying the energy inequality (2.6), it follows from the Serrin-type uniqueness criterion [25] that $(u(t), \omega(t)) \equiv (\hat{u}(t), \hat{\omega}(t))$ on $[0, T^*)$. Thus it is sufficient to show that $T = T^*$. If not, suppose that $T^* < T$. Without loss of generality, one may assume that T^* is the maximal existence time of the strong solution $(\hat{u}, \hat{\omega})$. By condition (a) or (b) in Corollary 2.1, we have

$$\int_0^{T^*} \|\hat{u}(t)\|_{p,\infty}^q dt < \infty, \text{ for } \frac{2}{q} + \frac{3}{p} \leq 1 \text{ with } 3 < p \leq \infty$$

or

$$\int_0^{T^*} \|\nabla \hat{u}(t)\|_{p,\infty}^q dt < \infty, \text{ for } \frac{2}{q} + \frac{3}{p} \leq 2 \text{ with } \frac{3}{2} < p \leq \infty$$

because $(\hat{u}(t), \hat{\omega}(t)) \equiv (u(t), \omega(t))$. Therefore it follows from Theorem 2.1 that there exists a time $T' > T^*$ such that $(\hat{u}, \hat{\omega})$ can be extended smoothly to $[0, T')$, which contradicts the maximality of T^* . We thus complete the proof of Corollary 2.1.

In the following arguments the letter C denotes an inessential constant which may vary from line to line but does not depend on particular solutions or functions. We also use $C(\chi, \gamma, \dots)$ to denote a constant which depends on the parameters $\chi, \gamma, \dots$ and may vary from line to line.

3. PROOF OF THEOREMS 2.1 AND 2.2

In this section we prove Theorems 2.1 and 2.2 by a simple method.

Proof of Theorem 2.1. We differentiate the equations (1.1) with respect to x_i , then multiply the resulting equations by $\partial_{x_i} u$ and $\partial_{x_i} \omega$ for $i = 1, 2, 3$, integrate with respect to x and sum them up. It follows that

$$\begin{aligned} (3.1) \quad & \frac{1}{2} \frac{d}{dt} (\|(\partial_{x_i} u, \partial_{x_i} \omega)\|_2^2) + \sum_{j=1}^3 \left((\mu + \chi) \|\partial_{x_i x_j}^2 u\|_2^2 + \gamma \|\partial_{x_i x_j}^2 \omega\|_2^2 \right) \\ & + \kappa \|\operatorname{div} \partial_{x_i} \omega\|_2^2 + 2\chi \|\partial_{x_i} \omega\|_2^2 \\ & \leq |(\partial_{x_i} u \cdot \nabla u, \partial_{x_i} u)| + |(\partial_{x_i} u \cdot \nabla \omega, \partial_{x_i} \omega)| + 2\chi |(\nabla \times \partial_{x_i} u, \partial_{x_i} \omega)| \\ & = I_1 + I_2 + I_3, \end{aligned}$$

where we have used the facts that

$$(3.2) \quad (\nabla \times \partial_{x_i} u, \partial_{x_i} \omega) = (\nabla \times \partial_{x_i} \omega, \partial_{x_i} u)$$

and

$$(3.3) \quad (u \cdot \nabla \partial_{x_i} u, \partial_{x_i} u) = (u \cdot \nabla \partial_{x_i} \omega, \partial_{x_i} \omega) = 0,$$

where $(\cdot, \cdot)$ denotes the L^2 inner product on $\mathbb{R}^3$. For conciseness, the short notation

$$(3.4) \quad \|(A, B)\|_2^2 = \|A\|_2^2 + \|B\|_2^2$$

has been used and will be used in the following arguments.

(a) We estimate the terms I_1 , I_2 and I_3 . After integrations by parts and using the generalized Hölder inequality (2.5) it follows that

$$(3.5) \quad \begin{aligned} I_1 &\leq \left| \int_{\mathbb{R}^3} \partial_{x_i} u \cdot \nabla \partial_{x_i} u \cdot u(x) dx \right| + \left| \int_{\mathbb{R}^3} \partial_{x_i} \partial_{x_i} u \cdot \nabla u \cdot u(x) dx \right| \\ &\leq C(p) \|u\|_{p,\infty} \left(\|\nabla \partial_{x_i} u \partial_{x_i} u\|_{\frac{p}{p-1},1} + \|\partial_{x_i x_i} u \partial_{x_i} u\|_{\frac{p}{p-1},1} \right) \\ &\leq C(p) \|u\|_{p,\infty} \|\nabla u\|_{\frac{2p}{p-2},2} \|D^2 u\|_2. \end{aligned}$$

Applying the real interpolation (see [1])

$$L^{\frac{2p}{p-2},2}(\mathbb{R}^3) = (L^2, L^6)_{\frac{p-3}{p},2}(\mathbb{R}^3)$$

and the Sobolev embedding $L^6(\mathbb{R}^3) \hookrightarrow \dot{H}^1(\mathbb{R}^3)$ yields that

$$(3.6) \quad \|\nabla u\|_{\frac{2p}{p-2},2} \leq C \|\nabla u\|_2^{1-\frac{3}{p}} \|D^2 u\|_2^{\frac{3}{p}}.$$

Inserting the above estimate (3.6) into the estimate (3.5) for I_1 one has

$$(3.7) \quad \begin{aligned} I_1 &\leq C(p) \|u\|_{p,\infty} \|\nabla u\|_2^{1-\frac{3}{p}} \|D^2 u\|_2^{1+\frac{3}{p}} \\ &\leq \frac{\chi}{12} \|D^2 u\|_2^2 + C(p, \chi) \|u\|_{p,\infty}^{\frac{2p}{p-3}} \|\nabla u\|_2^2. \end{aligned}$$

Similarly, for I_2 one also has

$$(3.8) \quad I_2 \leq \frac{\gamma}{6} \|D^2 \omega\|_2^2 + C(p, \gamma) \|u\|_{p,\infty}^{\frac{2p}{p-3}} \|\nabla \omega\|_2^2.$$

For the term I_3 , the Hölder and Young inequalities imply that

$$(3.9) \quad I_3 \leq \frac{\chi}{2} \|\nabla \times \partial_{x_i} u\|_2^2 + 2\chi \|\nabla \omega\|_2^2.$$

Inserting the estimates (3.7)–(3.9) into the inequality (3.1) and summing over i from 1 to 3, it follows that

$$\begin{aligned} &\frac{d}{dt} (\|(\nabla u, \nabla \omega)\|_2^2) + (2\mu + \frac{1}{2}\chi) \|D^2 u\|_2^2 + \gamma \|D^2 \omega\|_2^2 + 2\kappa \|\nabla \operatorname{div} \omega\|_2^2 \\ &\leq C(p, \chi, \gamma) \|u\|_{p,\infty}^{2p/(p-3)} \|(\nabla u, \nabla \omega)\|_2^2. \end{aligned}$$

Gronwall's inequality leads to the a priori estimate

$$(3.10) \quad \|(\nabla u, \nabla \omega)\|_2^2 \leq \|(\nabla u_0, \nabla \omega_0)\|_2^2 \exp \left\{ C(p, \chi, \gamma) \int_0^t \|u(s)\|_{p,\infty}^{2p/(p-3)} ds \right\}.$$

In case (b), we estimate I_1 and I_2 in another way. Using the generalized Hölder's and Young's inequalities, we have

$$\begin{aligned}
 I_1 &\leq C \|\nabla u\|_{p,\infty} \|\nabla u\|_{\frac{p}{p-1},1} \leq C(p) \|\nabla u\|_{p,\infty} \|\nabla u\|_{\frac{2p}{p-1},2}^2 \\
 (3.11) \quad &\leq C(p) \|\nabla u\|_{p,\infty} \|\nabla u\|_2^{2-\frac{3}{p}} \|D^2 u\|_2^{\frac{3}{p}} \\
 &\leq \frac{\chi}{12} \|D^2 u\|_2^2 + C(p, \chi) \|\nabla u\|_{p,\infty}^{\frac{2p}{2p-3}} \|\nabla u\|_2^2,
 \end{aligned}$$

where use has been made of the facts

$$L^{\frac{2p}{p-1},2}(\mathbb{R}^3) = (L^2, L^6)^{\frac{2p-3}{2p},2}(\mathbb{R}^3)$$

and

$$\|\nabla u\|_{\frac{2p}{p-2},2}^2 \leq C \|\nabla u\|_2^{2-\frac{3}{p}} \|D^2 u\|_2^{\frac{3}{p}}.$$

Arguing similarly, I_2 can be estimated as follows:

$$(3.12) \quad I_2 \leq \frac{\gamma}{6} \|D^2 \omega\|^2 + C(p, \gamma) \|\nabla u\|_{p,\infty}^{\frac{2p}{2p-3}} \|\nabla \omega\|_2^2.$$

Inserting the estimates (3.11), (3.12) and (3.9) into (3.1), summing over i from 1 to 3, and applying Gronwall's inequality give the a priori estimate

$$(3.13) \quad \|(\nabla u, \nabla \omega)\|_2^2 \leq \|(\nabla u_0, \nabla \omega_0)\|_2^2 \exp \left\{ C(p, \chi, \gamma) \int_0^t \|\nabla u(s)\|_{p,\infty}^{2p/(2p-3)} ds \right\}.$$

The above estimates are also valid for $p = \infty$ provided we modify them accordingly.

Combining the a priori estimates (3.10) and (3.13) with the energy inequality (2.6) and using standard arguments for continuation of local solutions, we conclude that the solution $(u(t, x), \omega(t, x))$ can be extended beyond $t = T$ provided that $u(t, x) \in L^q(0, T; L^{p,\infty}(\mathbb{R}^3))$ for $\frac{2}{q} + \frac{3}{p} \leq 1$ with $3 < p \leq \infty$, or $\nabla u(t, x) \in L^q((0, T); L^{p,\infty}(\mathbb{R}^3))$ for $\frac{2}{q} + \frac{3}{p} \leq 2$ with $\frac{3}{2} < p \leq \infty$. The proof of Theorem 2.1 is thus complete. $\square$

Proof of Theorem 2.2. To prove the theorem we need an L^4 a priori estimate. For this purpose, we take the inner product of the first equation of (1.1) with $|u|^2 u$ and integrate by parts; it can be deduced that

$$\begin{aligned}
 (3.14) \quad &\frac{1}{4} \frac{d}{dt} \|u\|_4^4 + (\mu + \chi) \int_{\mathbb{R}^3} |\nabla u|^2 |u|^2 dx + \frac{1}{2} (\mu + \chi) \int_{\mathbb{R}^3} |\nabla |u|^2|^2 dx \\
 &\leq 2 \int_{\mathbb{R}^3} |P| |u|^2 |\nabla u| dx + 3\chi \int_{\mathbb{R}^3} |w| |u|^2 |\nabla u| dx,
 \end{aligned}$$

where we used the following relations arising from the divergence free condition $\operatorname{div} u = 0$:

$$\begin{aligned}
 \int_{\mathbb{R}^3} u \cdot \nabla u \cdot |u|^2 u dx &= \frac{1}{2} \int_{\mathbb{R}^3} u \cdot \nabla |u|^4 dx = 0, \\
 \int_{\mathbb{R}^3} \Delta u \cdot |u|^2 u dx &= - \int_{\mathbb{R}^3} |\nabla u|^2 |u|^2 dx - \frac{1}{2} \int_{\mathbb{R}^3} |\nabla |u|^2|^2 dx, \\
 \int_{\mathbb{R}^3} \nabla \times \omega \cdot |u|^2 u dx &= - \int_{\mathbb{R}^3} |u|^2 \omega \cdot \nabla \times u dx - \int_{\mathbb{R}^3} \omega \cdot \nabla |u|^2 \times u dx,
 \end{aligned}$$

and

$$|\nabla \times u| \leq |\nabla u|, \quad |\nabla |u|| \leq |\nabla u|.$$

Using an argument similar to that used in deriving the estimate (3.14) it can be obtained for the second equation of (1.1) that

$$\begin{aligned}
 (3.15) \quad & \frac{1}{4} \frac{d}{dt} \|\omega\|_4^4 + \gamma \int_{\mathbb{R}^3} |\nabla \omega|^2 |\omega|^2 dx + \frac{1}{2} \gamma \int_{\mathbb{R}^3} |\nabla |\omega|^2|^2 dx \\
 & + \kappa \int_{\mathbb{R}^3} \operatorname{div} \omega|^2 dx + 2\chi \int_{\mathbb{R}^3} |\omega|^4 dx \\
 & \leq 3\chi \int_{\mathbb{R}^3} |u| |\omega|^2 |\nabla \omega| dx.
 \end{aligned}$$

Combining estimates (3.14) and (3.15), we arrive at

$$\begin{aligned}
 (3.16) \quad & \frac{1}{4} \frac{d}{dt} (\|u\|_4^4 + \|\omega\|_4^4) + (\mu + \chi) \int_{\mathbb{R}^3} |\nabla u|^2 |u|^2 dx + \frac{1}{2} (\mu + \chi) \int_{\mathbb{R}^3} |\nabla |u|^2|^2 dx \\
 & + \gamma \int_{\mathbb{R}^3} |\nabla \omega|^2 |\omega|^2 dx + \frac{1}{2} \gamma \int_{\mathbb{R}^3} |\nabla |\omega|^2|^2 dx + \kappa \int_{\mathbb{R}^3} |\operatorname{div} \omega|^2 dx + 2\chi \int_{\mathbb{R}^3} |\omega|^4 dx \\
 & \leq 2 \int_{\mathbb{R}^3} |P| |u|^2 |\nabla u| dx + 3\chi \int_{\mathbb{R}^3} |u| |\omega|^2 |\nabla u| dx + 3\chi \int_{\mathbb{R}^3} |u| |\omega|^2 |\nabla \omega| dx \\
 & = I + II + III.
 \end{aligned}$$

Applying Hölder's and Young's inequalities for II , it follows that

$$(3.17) \quad II \leq \frac{1}{2} \chi \int_{\mathbb{R}^3} |\nabla u|^2 |u|^2 dx + C(\chi) (\|u\|_4^4 + \|\omega\|_4^4).$$

Arguing similarly to above we can derive for III that

$$(3.18) \quad III \leq \frac{1}{2} \gamma \int_{\mathbb{R}^3} |\nabla \omega|^2 |\omega|^2 dx + C(\gamma) (\|u\|_4^4 + \|\omega\|_4^4).$$

Concerning the term I , by virtue of the generalized Hölder's inequality (2.5) we have

$$\begin{aligned}
 I & \leq C(p) \| |P|^{1/2} \|_{2p, \infty} \| |P|^{1/2} |u| \|_{\frac{2p}{p-1}, 2} \| u \nabla u \|_2 \\
 & \leq C(p) \| P \|_{p, \infty}^{1/2} \| u \nabla u \|_2 \| P \|_{\frac{2p}{p-1}, 2}^{1/2} \| u \|_{\frac{4p}{p-1}, 4}.
 \end{aligned}$$

Applying the divergence operator div to the first equation of (1.1), one formally has

$$(3.19) \quad P = \sum_{i,j=1}^3 R_i R_j u_i u_j,$$

where R_j denotes the j -th Riesz operator. By the boundedness of the Riesz operator on a Lorentz space $L^{p,q}(\mathbb{R}^3)$ for $1 < p \leq q < \infty$, and upon applying the generalized Hölder's inequality (2.5) again, we obtain that

$$(3.20) \quad \|P\|_{\frac{2p}{p-1}, 2} \leq C \|u\|_{\frac{4p}{p-1}, 4}^2.$$

So the term I can be estimated as

$$(3.21) \quad I \leq C(p) \|P\|_{p, \infty}^{1/2} \|u \nabla u\|_2 \|u\|_{\frac{4p}{p-1}, 4}^2.$$

In view of the real interpolation

$$L^{\frac{4p}{p-1}, 4}(\mathbb{R}^3) = (L^4, L^{12})_{\frac{2p-3}{2p}, 4}(\mathbb{R}^3)$$

and the Sobolev's inequality we have

$$(3.22) \quad \|u\|_{\frac{4p}{p-1},4}^2 \leq C \|u\|_4^{\frac{2p-3}{p}} \|u\|_{12}^{\frac{3}{p}} \leq C \|u\|_4^{\frac{2p-3}{p}} \|u \nabla |u|\|_2^{\frac{3}{2p}}.$$

Inserting the estimate (3.22) into the estimate (3.21) for I , one estimates I as

$$(3.23) \quad I \leq C(p) \|P\|_{p,\infty}^{1/2} \|u\|_4^{\frac{2p-3}{p}} \|u \nabla u\|_2^{\frac{2p+3}{2p}} \leq C(p, \chi) \|P\|_{p,\infty}^{\frac{2p}{2p-3}} \|u\|_4^4 + \frac{1}{2} \chi \|u \nabla u\|_2^2.$$

Inserting the estimates (3.23), (3.17) and (3.18) for I , II and III into (3.16) we get that

$$\frac{1}{4} \frac{d}{dt} (\|u\|_4^4 + \|\omega\|_4^4) \leq C(p, \chi) \|P\|_{p,\infty}^{\frac{2p}{2p-3}} \|u\|_4^4 + C(\chi, \gamma) (\|u\|_4^4 + \|\omega\|_4^4).$$

Gronwall's inequality implies that

$$(3.24) \quad \|u\|_4^4 + \|\omega\|_4^4 \leq (\|u_0\|_4^4 + \|\omega_0\|_4^4) \exp \left\{ C(p, \chi, \gamma) \int_0^t (1 + \|P\|_{p,\infty}^{\frac{2p}{2p-3}}) ds \right\}.$$

In case (2), we estimate I by another method. First, I also equals $\int_{\mathbb{R}^3} \nabla P \cdot u|u|^2 dx$, so

$$\begin{aligned} I &\leq \int_{\mathbb{R}^3} |\nabla P|^{1/2} |\nabla P|^{1/2} |u|^3 dx \\ &\leq C \| |\nabla P|^{1/2} \|_{2p,\infty} \| |\nabla P|^{1/2} |u|^3 \|_{\frac{2p}{2p-1},1} \\ &\leq C(p) \|\nabla P\|_{p,\infty}^{1/2} \|\nabla P\|_2^{1/2} \|u\|_{\frac{12p}{3p-2},4}^3, \end{aligned}$$

where we have used the generalized Hölder's inequality (2.5). Noting the relation (3.19) between P and u , the Riesz operator's boundedness on L^p for $1 < p < \infty$ and the real interpolation

$$L^{\frac{12p}{3p-2},4}(\mathbb{R}^3) = (L^4, L^{12})_{\frac{p-1}{p},4}(\mathbb{R}^3),$$

the term I can be estimated as

$$\begin{aligned} I &\leq C(p) \|\nabla P\|_{p,\infty}^{1/2} \|u \nabla u\|_2^{1/2} \|u\|_4^{\frac{3(p-1)}{p}} \|u\|_{12}^{\frac{3}{p}} \\ (3.25) \quad &\leq C(p) \|\nabla P\|_{p,\infty}^{1/2} \|u \nabla u\|_2^{\frac{p+3}{2p}} \|u\|_4^{\frac{3(p-1)}{p}} \\ &\leq C(p, \chi) \|\nabla P\|_{p,\infty}^{\frac{2p}{3(p-1)}} \|u\|_4^4 + \frac{1}{2} \chi \|u \nabla u\|_2^2, \end{aligned}$$

where the Sobolev embedding $L^6(\mathbb{R}^3) \hookrightarrow \dot{H}^1(\mathbb{R}^3)$ and Young's inequality were used. Inserting the estimates (3.25), (3.17) and (3.18) for I , II and III into (3.16) one also has

$$\frac{1}{4} \frac{d}{dt} (\|u\|_4^4 + \|\omega\|_4^4) \leq C(p, \chi) \|\nabla P\|_{p,\infty}^{\frac{2p}{3(p-1)}} \|u\|_4^4 + C(\chi, \gamma) (\|u\|_4^4 + \|\omega\|_4^4).$$

Apply Gronwall's inequality again to arrive at

$$(3.26) \quad \|u\|_4^4 + \|\omega\|_4^4 \leq (\|u_0\|_4^4 + \|\omega_0\|_4^4) \exp \left\{ C(p, \chi, \gamma) \int_0^t (1 + \|\nabla P\|_{p,\infty}^{\frac{2p}{3(p-1)}}) ds \right\}.$$

The above estimates are also valid for $p = \infty$ provided we modify them accordingly.

Combining the a priori estimates (3.24) and (3.26) with the energy inequality (2.6) and using standard arguments for continuation of local solutions, we conclude that the solution $(u(t, x), \omega(t, x))$ can be extended beyond $t = T$ provided that

$P(t, x) \in L^q((0, T); L^{p,\infty}(\mathbb{R}^3))$ for $\frac{2}{q} + \frac{3}{p} \leq 2$ with $\frac{3}{2} < p \leq \infty$, or $\nabla P(t, x) \in L^q((0, T); L^{p,\infty}(\mathbb{R}^3))$ for $\frac{2}{q} + \frac{3}{p} \leq 3$ with $1 < p \leq \infty$. We thus complete the proof of Theorem 2.2. $\square$

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