MONOMIAL REGULAR SEQUENCES

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(Communicated by Bernd Ulrich)

ABSTRACT. We study Hilbert functions of ideals containing a regular sequence of monomials.

1. INTRODUCTION

Let $B = k[x_1, \dots, x_n]$ be a polynomial ring over a field k , and let $I \subset B$ be a homogeneous ideal. An important invariant of I is its Hilbert function, $\text{Hilb}_I^B(d) = \dim_k I_d$. We study Hilbert functions of ideals containing a regular sequence of monomials.

Macaulay [Ma] showed in 1927 that all Hilbert functions of homogeneous ideals of $B = k[x_1, \dots, x_n]$ are attained by lex ideals. Over what quotients of B is this true? Let M be a monomial ideal of B . We say that M is *Lex-Macaulay* if every Hilbert function over B/M is attained by a lex ideal of B/M . Clements and Lindström [CL] proved that $(x_1^{e_1}, \dots, x_n^{e_n})$ is Lex-Macaulay for $e_1 \leq \dots \leq e_n$. In Section 4 we classify the monomial regular sequences which are Lex-Macaulay: Theorem 4.4 says that a regular sequence of monomials is Lex-Macaulay if and only if it has the form $(x_1^{e_1}, \dots, x_{r-1}^{e_{r-1}}, x_r^{e_r-1}y)$, with $e_1 \leq \dots \leq e_r$ and $y = x_i$ for some $i \geq r$.

The main tool in our proofs is $\mathcal{A}$ -compression, which we introduce in Section 3. It generalizes the notion of x_i -compression used in [CL, MP1, MP2].

2. PRELIMINARIES

We give some definitions and notation that are used throughout.

Let k be a field and $B = k[x_1, \dots, x_n]$ be graded by $\deg x_i = 1$. A monomial $x_1^{\alpha_1} \dots x_n^{\alpha_n}$ has *exponent vector* $\alpha = (\alpha_1, \dots, \alpha_n)$ and is sometimes written $\mathbf{x}^\alpha$. We denote by $|\alpha|$ the degree of $\mathbf{x}^\alpha$, $|\alpha| = \sum_{i=1}^n \alpha_i$.

We order the variables by $x_1 > \dots > x_n$ and denote by $>_{\text{Lex}}$ the graded lexicographic order on the monomial: $\mathbf{x}^\alpha >_{\text{Lex}} \mathbf{x}^\beta$ if $|\alpha| > |\beta|$ or if $|\alpha| = |\beta|$ and for some index j we have $\alpha_j > \beta_j$ and $\alpha_i = \beta_i$ for all $i < j$. Every monomial $m \neq 1$ has a unique *predecessor* n in the graded lex order, satisfying $n <_{\text{Lex}} m$ and $n \geq_{\text{Lex}} n'$ whenever $m >_{\text{Lex}} n'$.

A *d-vector space* is a k -vector subspace of B spanned by homogeneous polynomials in degree d . A *d-monomial space* is a d -vector space spanned by monomials. The dimension of such a space M_d will be denoted $|M_d|$. If $I = (g_1, \dots, g_s)$ is a

Received by the editors July 16, 2007, and, in revised form, October 6, 2009.

2010 *Mathematics Subject Classification*. Primary 13F20.

Key words and phrases. Hilbert function, lexicographic ideals.

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monomial ideal, we denote by I_d or $(g_1, \dots, g_s)_d$ the degree d component of I . Let $\mathbf{m}$ be the monomial space generated by the variables, $\mathbf{m} = (x_1, \dots, x_n)_1$. For a d -monomial space M_d , we denote by $\mathbf{m}M_d$ the $(d+1)$ -monomial space spanned by monomials of the form $x_i m$ with $m \in M_d$.

A d -monomial space L_d is *lex* or a *lex segment* if whenever $\mathbf{x}^\alpha$ and $\mathbf{x}^\beta$ are degree d monomials with $\mathbf{x}^\alpha$ lexicographically greater than $\mathbf{x}^\beta$ and $\mathbf{x}^\beta \in L_d$, we have $\mathbf{x}^\alpha \in L_d$ as well. A monomial ideal L is *lex* if L_d is a lex segment for all d .

A d -monomial space N_d is *strongly stable* or *Borel* if whenever $i < j$ and $gx_j \in N_d$ for some monomial $g \in B_{d-1}$, we have $gx_i \in N_d$ as well. A monomial ideal N is *strongly stable* or *Borel* if N_d is Borel for all d .

A *monomial regular sequence* is a regular sequence $\{f_1, \dots, f_r\}$ consisting of monomials. Equivalently, it is a collection of monomials satisfying $\gcd(f_i, f_j) = 1$ for all $i \neq j$.

3. $\mathcal{A}$ -COMPRESSION

Compressed ideals were used heavily in [CL, MP1, MP2]. We introduce $\mathcal{A}$ -compressed ideals, which are more general.

Definition 3.1. Let $\mathcal{A}$ be a subset of $\{x_1, \dots, x_n\}$, and let M_d be a d -monomial vector space. Denote by $\bigoplus_{\mathbf{x}^\alpha \in k[\mathcal{A}^c]}$ a direct sum over exponent vectors $\alpha = (\alpha_1, \dots, \alpha_n)$ such that $\alpha_i = 0$ whenever $x_i \in \mathcal{A}$. Then M_d may be written uniquely in the form

$$M_d = \bigoplus_{\mathbf{x}^\alpha \in k[\mathcal{A}^c]} \mathbf{x}^\alpha V_\alpha,$$

where V_α is a $(d - |\alpha|)$ -monomial space in the ring $k[\mathcal{A}]$. We say that M_d is $\mathcal{A}$ -compressed if each V_α is lex in $k[\mathcal{A}]$.

If L_α is the lex monomial space in degree $(d - |\alpha|)$ with $|L_\alpha| = |V_\alpha|$, and $L_d = \bigoplus_{\mathbf{x}^\alpha \in k[\mathcal{A}^c]} \mathbf{x}^\alpha L_\alpha$, we say that L_d is the $\mathcal{A}$ -compression of M_d .

Remark. What [CL, MP1, MP2] called i -compressed ideals (or monomial spaces) are $\{x_1, \dots, \widehat{x_i}, \dots, x_n\}$ -compressed in this notation. (Here the $\widehat{x_i}$ means that x_i is omitted.) This reversal is for simplicity in several proofs below and so that $\mathcal{A}$ -compressed spaces will remain $\mathcal{A}$ -compressed after new variables are added to the ring B .

Example 3.2. If $\mathcal{A}$ is a one-element set, every monomial space is $\mathcal{A}$ -compressed.

Example 3.3. Let $M_3 = \langle a^3, a^2b, a^2c, ab^2, abc, abd, b^3, b^2c \rangle$. Then M_3 is not $\{a, b\}$ -compressed because $V_d = \langle ab \rangle$ is not lex in $k[a, b]$, but M_3 is $\mathcal{A}$ -compressed for every other two-element set $\mathcal{A}$.

Lemma 3.4. Let M_d be $\mathcal{A}$ -compressed. Then $\mathbf{m}M_d$ is $\mathcal{A}$ -compressed as well.

Definition 3.5. A monomial ideal M is $\mathcal{A}$ -compressed if it is $\mathcal{A}$ -compressed in every degree d .

It is easy to prove the following two propositions:

Proposition 3.6. A lex ideal (or lex monomial space) is $\mathcal{A}$ -compressed for every $\mathcal{A}$.

Proposition 3.7. *Suppose that $\mathcal{A} \supset \mathcal{B}$. Then every $\mathcal{A}$ -compressed ideal (or $\mathcal{A}$ -compressed monomial space) is $\mathcal{B}$ -compressed.*

Example 3.8. Borel ideals are $\mathcal{B}$ -compressed for every two-element set $\mathcal{B}$.

Due to the following lemma, $\mathcal{A}$ -compressed ideals are useful in the study of Hilbert functions:

Lemma 3.9. *Let M_d be a d -monomial vector space, and let K_d be its $\mathcal{A}$ -compression. Then $|\mathbf{m}K_d| \leq |\mathbf{m}M_d|$.*

Proof. Let $\mathbf{n}$ be the monomial space generated by the variables of $\mathcal{A}$. Denote by $\beta \uparrow \alpha$ the property that $\mathbf{x}^\beta$ divides $\mathbf{x}^\alpha$ and $\deg \mathbf{x}^\beta = \deg \mathbf{x}^\alpha - 1$.

Write $M_d = \bigoplus_{\mathbf{x}^\alpha \in k[\mathcal{A}^c]} \mathbf{x}^\alpha N_\alpha$ and $K_d = \bigoplus_{\mathbf{x}^\alpha \in k[\mathcal{A}^c]} \mathbf{x}^\alpha L_\alpha$. Now we have $\mathbf{m}M_d =$

$\bigoplus_{\mathbf{x}^\alpha \in k[\mathcal{A}^c]} \mathbf{x}^\alpha N'_\alpha$ and $\mathbf{m}K_d = \bigoplus_{\mathbf{x}^\alpha \in k[\mathcal{A}^c]} \mathbf{x}^\alpha L'_\alpha$, where

$$N'_\alpha = \mathbf{n}N_\alpha + \sum_{\beta \uparrow \alpha} N_\beta,$$

$$L'_\alpha = \mathbf{n}L_\alpha + \sum_{\beta \uparrow \alpha} L_\beta.$$

Since $|\mathbf{n}L_\alpha| \leq |\mathbf{n}N_\alpha|$ by Macaulay's Theorem and $|N_\beta| = |L_\beta|$ for all β by definition, we have

$$|L'_\alpha| = \max\{\max_{\beta \uparrow \alpha}\{|L_\beta|\}, |\mathbf{n}L_\alpha|\} \leq \max\{\max_{\beta \uparrow \alpha}\{|N_\beta|\}, |\mathbf{n}N_\alpha|\} \leq |N'_\alpha|,$$

the first equality because the L_β and $\mathbf{n}L_\alpha$ are lex segments, so their sum is the longest segment. This proves the lemma. $\square$

Example 3.10. We use compression to prove the monomial case of the Eisenbud-Green-Harris Conjecture. Eisenbud, Green, and Harris [EGH1, EGH2] made the following conjecture motivated by applications in algebraic geometry:

Conjecture 3.11 (Eisenbud-Green-Harris). *Let N be any homogeneous ideal containing a regular sequence in degrees $e_1 \leq \dots \leq e_r$. There is a lex ideal L such that N and $L + (x_1^{e_1}, \dots, x_r^{e_r})$ have the same Hilbert function.*

Using Lemma 3.9 we prove Conjecture 3.11 for ideals containing a regular sequence of monomials. This special case has been known in folklore for some time but, as near as I can tell, has never been formally written up. (In the folklore proof, the compression is replaced by a coordinate change and an initial ideal.)

Proposition 3.12. *Let $R = \{f_1, \dots, f_r\}$ be a regular sequence of monomials with $\deg f_i = e_i$ and $e_1 \leq \dots \leq e_r$. Let N be any homogeneous ideal containing R . Then there is a lex ideal L such that N and $L + (x_1^{e_1}, \dots, x_r^{e_r})$ have the same Hilbert function.*

Proof. Set $P = (x_1^{e_1}, \dots, x_r^{e_r})$. It suffices to show that, for any d -vector space N_d containing $(R)_d$, there is a lex monomial space L_d such that $|L_d + P_d| = |N_d|$ and $|\mathbf{m}L_d + P_{d+1}| \leq |\mathbf{m}N_d|$.

By renaming the variables if necessary, we may assume without loss of generality that x_i divides f_i for all i . By Gröbner basis theory, we may assume (after taking an initial ideal if necessary) that N_d is a monomial space. Set $N(0) = N_d$. For each

$i \leq r$, let $\mathcal{A}_i$ be the set of variables dividing f_i , and let $N(i)$ be the $\mathcal{A}_i$ -compression of $N(i-1)$.

We claim that whenever $d \geq e_j$, $N(i)$ contains $x_j^{e_j}$ if $j \leq i$, and contains f_j if $j > i$. Clearly, this statement is true when $i = 0$; we prove it by induction on i . Suppose $d \geq e_j$. If $i \neq j$, the monomial in question is contained in $N(i-1)$ by induction and is unaffected by $\mathcal{A}_i$ -compression. If $j = i$, then $f_i \in N(i-1)$ by induction, so $x_i^{e_i} \in N(i)$ as desired.

Furthermore, $|N(i)| = |N(i-1)|$ and $|\mathbf{m}N(i)| \leq |\mathbf{m}N(i-1)|$ by Lemma 3.9. In particular, $N(r)$ contains P_d and satisfies $|N(r)| = |N|$ and $|\mathbf{m}N(r)| \leq |\mathbf{m}N|$. By Clements-Lindström's theorem, there is a lex space L_d such that $|L_d + P_d| = |N(r)|$ and $|\mathbf{m}L_d + P_{d+1}| \leq |\mathbf{m}N(r)|$. $\square$

4. LEX-MACAULAY MONOMIAL REGULAR SEQUENCES

Throughout this section, let M be a monomial ideal of B . We recall the following definitions and results from [MP1]:

Definition 4.1 ([MP1]). An ideal M is *Lex-Macaulay* if every Hilbert function in the quotient B/M is attained by a lex ideal. Equivalently, M is Lex-Macaulay if for every d and every d -monomial vector space V_d , there exists a lex space L_d such that $|L_d + M_d| = |V_d + M_d|$ and $|\mathbf{m}L_d + M_{d+1}| \leq |\mathbf{m}V_d + M_{d+1}|$.

Theorem 4.2 ([MP1]). If M is Lex-Macaulay as an ideal of B , then $MB[y]$ is Lex-Macaulay as an ideal of $B[y]$.

Proposition 4.3 ([MP1]). If M is Lex-Macaulay and L is lex, then $L + M$ is Lex-Macaulay.

We characterize the monomial regular sequences which are Lex-Macaulay as follows.

Theorem 4.4. Let R be the ideal generated by a regular sequence of monomials. Then R is Lex-Macaulay if and only if $R = (x_1^{e_1}, \dots, x_{r-1}^{e_{r-1}}, x_r^{e_r-1}y)$, with $e_1 \leq \dots \leq e_r$ and $y = x_i$ for some $i \geq r$.

The necessary condition is given by Lemma 4.6, and the sufficient condition by Lemmas 4.7 and 4.8 below.

Remark. The sufficient condition of Theorem 4.4 is a special case of the following more general result, which was suggested by the referee. The proof is the same, modulo the necessary changes in notation. (In the statement of Proposition 4.5, R is a polarization of Q ; see for example [Fa].)

Proposition 4.5. Suppose that Q is a Lex-Macaulay monomial ideal, and let x_r be the least variable dividing a minimal monomial generator of Q . Write $Q = U + x_r V$, where x_r does not divide any minimal generator of U . Then $R = U + x_{r+1} V$ is Lex-Macaulay as well.

Lemma 4.6. Let R be the ideal of a monomial regular sequence $\{f_1, \dots, f_r\}$, with the f_i ordered so that $i < j$ if $\deg f_i < \deg f_j$ or if $\deg f_i = \deg f_j$ and $f_i >_{\text{Lex}} f_j$. Suppose that R is Lex-Macaulay. Then:

- (i) If $i \neq r$, then $f_i = x_i^{e_i}$.
- (ii) $x_r^{e_r-1}$ divides f_r .

Proof. We use induction on i . Suppose first that $i = 1$. If g is any monomial of degree $e_1 - 1$ dividing f_1 , we must have $|(x_1^{e_1-1} + R)_{e_1}| \leq |(g + R)_{e_1}| \leq |R_{e_1} + n - 1|$; i.e., $x_1^{e_1-1}$ must divide f_1 . If $r \neq 1$, put $q = e_1 + e_2 - 1$ and consider $g = x_1^{e_1-1}h$, where h is any monomial of degree $e_2 - 1$ dividing f_2 . We have $|(g + R)_q| = |R_q| + n - 2$. If $f_1 \neq x_1^{e_1}$, we would also have $|(x_1^{q-1} + R)_q| = |R_q| + n - 1$, contradicting the assumption that R is Lex-Macaulay.

In general, $R + (x_1, \dots, x_{i-1})$ is Lex-Macaulay by Proposition 4.3. By induction, $R + (x_1, \dots, x_{i-1}) = (x_1, \dots, x_{i-1}, f_i, \dots, f_r)$, so $(f_i, \dots, f_r)$ is Lex-Macaulay as an ideal of $k[x_i, \dots, x_r]$. The lemma follows from the $i = 1$ case above. $\square$

Lemma 4.7. *Suppose $r \neq n$, and let $R = (x_1^{e_1}, \dots, x_{r-1}^{e_{r-1}}, x_r^{e_r-1}x_n)$. Let N_d be a d -monomial vector space containing R_d . Then there exists a d -monomial vector space K_d containing Q_d , where $Q = (x_1^{e_1}, \dots, x_{r-1}^{e_{r-1}}, x_r^{e_r-1}x_{n-1})$ such that $|K_d| = |M_d|$ and $|\mathbf{m}K_d| \leq |\mathbf{m}M_d|$.*

Proof. Set $\mathcal{A} = \{x_{n-1}, x_n\}$, and take K_d to be the $\mathcal{A}$ -compression of M_d . Apply Lemma 3.9, and note that Q is the $\mathcal{A}$ -compression of R . $\square$

Lemma 4.8. *Suppose $r \neq n$, and let $R = (x_1^{e_1}, \dots, x_{r-1}^{e_{r-1}}, x_r^{e_r-1}x_n)$ and $Q = (x_1^{e_1}, \dots, x_{r-1}^{e_{r-1}}, x_r^{e_r-1}x_{n-1})$. Let L_d be a lex segment in B . Then there exists a lex segment T_d such that $|T_d + R_d| = |L_d + Q_d|$ and $|\mathbf{m}T_d + R_{d+1}| = |\mathbf{m}L_d + Q_{d+1}|$.*

Proof. Let T_d be the smallest lex segment such that $T_d + Q_d = L_d + Q_d$.

We claim that T_d satisfies the following property: If g is a monomial such that $gx_r^{e_r-1}x_{n-1} \in T_d$, then $gx_r^{e_r-1}x_n \in T_d$. We will prove this claim. Observe that $gx_r^{e_r-1}x_{n-1} \in Q_d$, and that $gx_r^{e_r-1}x_n$ is the predecessor of $gx_r^{e_r-1}x_{n-1}$ in the graded lex order. By construction, if $gx_r^{e_r-1}x_{n-1} \in T_d$, we must have a monomial $v \in T_d$ such that $v <_{\text{Lex}} gx_r^{e_r-1}x_{n-1}$ and $v \notin Q_d$. Then $v \leq_{\text{Lex}} gx_r^{e_r-1}x_n$ as well, and since T_d is a lex segment, we have $gx_r^{e_r-1}x_n \in T_d$, proving the claim.

Set $A = T_d + (x_1^{e_1}, \dots, x_{r-1}^{e_{r-1}})_d$ and let C and D be the monomial spaces such that $T_d + Q_d = A \oplus C$ and $T_d + R_d = A \oplus D$. Then if Γ and Δ are the sets of monomials of C and D , respectively, we have

$$\begin{aligned} \Gamma &= \{gx_r^{e_r-1}x_{n-1} \in B_d : gx_r^{e_r-1}x_{n-1} \notin A\}, \\ \Delta &= \{gx_r^{e_r-1}x_n \in B_d : gx_r^{e_r-1}x_n \notin A\} \\ &= \{gx_r^{e_r-1}x_n \in B_d : gx_r^{e_r-1}x_{n-1} \notin A\} \\ &= \{gx_r^{e_r-1}x_n \in B_d : gx_r^{e_r-1}x_{n-1} \in C\}, \end{aligned}$$

the penultimate equality following from the claim above.

In particular, multiplication by $\frac{x_n}{x_{n-1}}$ is a bijection from Γ to Δ . Thus $|\mathbf{m}C| = |\mathbf{m}D|$ and (again applying the claim) $|\mathbf{m}C \cap \mathbf{m}A| = |\mathbf{m}D \cap \mathbf{m}A|$, so we have $|\mathbf{m}(T_d + R_d)| = |\mathbf{m}(T_d + Q_d)| = |\mathbf{m}(L_d + Q_d)|$, the first equality by inclusion-exclusion, the second by construction.

Now $|\mathbf{m}T_d + R_{d+1}| = |\mathbf{m}(T_d + R_d)|$ unless R has minimal monomial generators in degree $d+1$ which are not in $\mathbf{m}T_d$; likewise $|\mathbf{m}L_d + Q_{d+1}| = |\mathbf{m}(L_d + Q_d)|$ unless Q has minimal monomial generators in degree $d+1$ which are not in $\mathbf{m}L_d$ and hence not in $\mathbf{m}T_d$. Since $x_r^d x_{n-1} \in \mathbf{m}T_d$ if and only if $x_r^d x_n \in \mathbf{m}T_d$, we obtain $|\mathbf{m}T_d + R_{d+1}| = |\mathbf{m}L_d + Q_{d+1}|$, as desired. $\square$

The preceding lemmas combine to prove Theorem 4.4.

Proof of Theorem 4.4. If R is Lex-Macaulay, apply Lemma 4.6.

Conversely, suppose $R = (x_1^{e_1}, \dots, x_{r-1}^{e_{r-1}}, x_r^{e_r-1}y)$ as in the statement of the theorem. By Theorem 4.2, we may assume $y = x_n$. If $n = r$, this is Clements-Lindström's theorem; otherwise, we induct on $n - r$. Set $Q = (x_1^{e_1}, \dots, x_{r-1}^{e_{r-1}}, x_r^{e_r-1}x_{n-1})$. Choose a degree d , and let N_d be any d -monomial space containing R_d . By Lemma 4.7, there is a d -monomial space K_d containing Q_d with $|K_d| = |N_d|$ and $|\mathbf{m}K_d| \leq |\mathbf{m}N_d|$. Q is Lex-Macaulay by induction, so there is a lex monomial space L_d containing Q_d with $|L_d| = |K_d|$ and $|\mathbf{m}L_d| \leq |\mathbf{m}K_d|$. By Lemma 4.8, there is a lex monomial space T_d containing R_d with $|T_d| = |L_d|$ and $|\mathbf{m}T_d| = |\mathbf{m}L_d|$. Thus R is Lex-Macaulay. $\square$

ACKNOWLEDGEMENTS

The author thanks David Eisenbud for asking which monomial regular sequences are Lex-Macaulay. He thanks Irena Peeva, Steve Sinnott, and Mike Stillman for helpful discussions. The author is particularly grateful to Irena Peeva for considerable help in editing this paper. Finally, he is grateful to the referee for many suggestions improving the precision and clarity of the final paper.

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