HOPF CYCLIC COHOMOLOGY AND BIDERIVATIONS

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ABSTRACT. Hopf cyclic cohomology $HC_{(\delta,\sigma)}^*(\mathcal{H})$ for a Hopf algebra $\mathcal{H}$ with respect to a modular pair in involution (δ, σ) was introduced by Connes and Moscovici. By a biderivation D on a Hopf algebra $\mathcal{H}$ we shall mean a linear map that satisfies the axioms for both a derivation and a coderivation on $\mathcal{H}$. Given a biderivation D on a Hopf algebra, we define, under certain conditions, a map $L_D : HC_{(\delta,\sigma)}^*(\mathcal{H}) \rightarrow HC_{(\delta,\sigma)}^*(\mathcal{H})$. We give examples of such maps for the quantized universal enveloping algebra $\mathcal{U}_h(\mathfrak{g})$ of a simple Lie algebra $\mathfrak{g}$. When $\mathcal{H}$ is irreducible, cocommutative and equipped with a character δ such that $(\delta, 1)$ is a modular pair in involution, we define “inner biderivations” and use these to produce a left $\mathcal{H}$ -module structure on $HC_{(\delta,1)}^*(\mathcal{H})$. Finally, we show that every morphism $L_D : HC_{(\delta,1)}^*(\mathcal{H}) \rightarrow HC_{(\delta,1)}^*(\mathcal{H})$ induced by a biderivation D on such a Hopf algebra $\mathcal{H}$ can be realized as a morphism induced by an inner biderivation by embedding $\mathcal{H}$ into a larger Hopf algebra $\mathcal{H}[D]$.

INTRODUCTION

Cyclic cohomology for Hopf algebras was introduced, in the context of transverse index theory of foliations, by Connes and Moscovici in [3], [4]. For a Hopf algebra $\mathcal{H}$ over $\mathbb{C}$ equipped with a “modular character” $\delta : \mathcal{H} \rightarrow \mathbb{C}$ satisfying certain conditions, the authors of [3] construct a cocyclic module $\mathcal{H}_\delta^\# = \{\mathcal{H}^{\otimes n}\}_{n \geq 1}$. More generally, one can define a cocyclic module $\mathcal{H}_{(\delta,\sigma)}^\#$ corresponding to a Hopf algebra $\mathcal{H}$ equipped with a modular pair in involution (δ, σ) consisting of a character $\delta : \mathcal{H} \rightarrow \mathbb{C}$ and a “grouplike element” $\sigma \in \mathcal{H}$ satisfying certain conditions (see [3]). The Hopf cyclic cohomology $HC_{(\delta,\sigma)}^*(\mathcal{H})$ of $\mathcal{H}$ with respect to the pair (δ, σ) is defined to be the cohomology of this cocyclic module $\mathcal{H}_{(\delta,\sigma)}^\#$. This latter theory has been subsumed within the theory of stable anti-Yetter-Drinfeld modules developed by Hajac, Khalkhali, Rangipour and Sommerhäuser (see [6], [7]).

Given an algebra A over $\mathbb{C}$, a derivation D on A induces a map $L_D : HC_*(A) \rightarrow HC_*(A)$ on the cyclic homology $HC_*(A)$ of A (see [8, §4.1]). The map L_D plays the role of a Lie derivative in noncommutative geometry. In the present paper, we are concerned with understanding the maps induced on the Hopf cyclic cohomology by a “biderivation” of a Hopf algebra $\mathcal{H}$ under certain extra conditions. By a biderivation D on $\mathcal{H}$ we will mean a linear map $D : \mathcal{H} \rightarrow \mathcal{H}$ that satisfies both

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the axioms for being a derivation on the algebra underlying $\mathcal{H}$ and the axioms for being a coderivation on the coalgebra underlying $\mathcal{H}$. In this respect, our first result is the following:

Proposition 0.1. *Let $\mathcal{H}$ be a Hopf algebra over $\mathbb{C}$ equipped with a modular pair in involution (δ, σ) . Then, if D is a biderivation on $\mathcal{H}$ that satisfies (1.11), D induces a morphism*

$$(0.1) \quad L_D : HC_{(\delta, \sigma)}^*(\mathcal{H}) \longrightarrow HC_{(\delta, \sigma)}^*(\mathcal{H})$$

of Hopf cyclic cohomologies of $\mathcal{H}$.

Thereafter, we demonstrate biderivations D that induce maps L_D as in (0.1) on the Hopf cyclic cohomology of the quantized universal enveloping algebra $\mathcal{U}_h(\mathfrak{g})$ of a finite dimensional, complex, simple Lie algebra $\mathfrak{g}$. This example is constructed in Proposition 1.7.

Then we consider the case of a Hopf algebra $\mathcal{H}$ over $\mathbb{C}$ that is irreducible and cocommutative, i.e., the universal enveloping algebra of a Lie algebra $\mathfrak{g}$. In this case, the elements of $\mathcal{H}$ may be used to induce *inner biderivations* (see Definition 2.2) on $\mathcal{H}$ and hence we obtain the following result:

Proposition 0.2. *Let $\mathcal{H}$ be a Hopf algebra over $\mathbb{C}$ equipped with a character $\delta : \mathcal{H} \longrightarrow \mathbb{C}$ such that $(\delta, 1)$ is a modular pair in involution. Suppose, moreover, that $\mathcal{H}$ is irreducible and cocommutative. Then, the Hopf cyclic cohomology $HC_{(\delta, 1)}^*(\mathcal{H})$ carries the structure of a left $\mathcal{H}$ -module.*

When the Hopf algebra $\mathcal{H}$ is equipped with a character $\delta : \mathcal{H} \longrightarrow \mathbb{C}$ such that $(\delta, 1)$ is a modular pair in involution, we will denote the corresponding Hopf cyclic cohomology $HC_{(\delta, 1)}^*(\mathcal{H})$ simply by $HC_\delta^*(\mathcal{H})$. Finally, suppose that we have an irreducible and cocommutative Hopf algebra $\mathcal{H}$ over $\mathbb{C}$ with a biderivation D acting on it and that D induces an operator $L_D : HC_\delta^*(\mathcal{H}) \longrightarrow HC_\delta^*(\mathcal{H})$ on its Hopf cyclic cohomology. We want to show that the operator L_D can be interpreted as a map induced by an inner biderivation. Therefore, we construct a Hopf algebra $\mathcal{H}[D]$ into which $\mathcal{H}$ embeds, along with a character $\delta_D : \mathcal{H}[D] \longrightarrow \mathbb{C}$ extending $\delta : \mathcal{H} \longrightarrow \mathbb{C}$, and we have the following result:

Proposition 0.3. *Let $\mathcal{H}$ be a Hopf algebra over $\mathbb{C}$ equipped with a character $\delta : \mathcal{H} \longrightarrow \mathbb{C}$ such that $(\delta, 1)$ is a modular pair in involution. Suppose, moreover, that $\mathcal{H}$ is irreducible and cocommutative. Let D be a biderivation on $\mathcal{H}$ that satisfies (1.11). Then there exists a Hopf algebra $\mathcal{H}[D]$ along with a character $\delta_D : \mathcal{H}[D] \longrightarrow \mathbb{C}$ extending δ and an embedding $i : \mathcal{H} \longrightarrow \mathcal{H}[D]$ such that there exists an inner biderivation $(D) : \mathcal{H}[D] \longrightarrow \mathcal{H}[D]$ on $\mathcal{H}[D]$ inducing a commutative diagram*

$$(0.2) \quad \begin{array}{ccc} HC_\delta^*(\mathcal{H}) & \xrightarrow{L_D} & HC_\delta^*(\mathcal{H}) \\ i \downarrow & & i \downarrow \\ HC_{\delta_D}^*(\mathcal{H}[D]) & \xrightarrow{L_{(D)}} & HC_{\delta_D}^*(\mathcal{H}[D]) \end{array}$$

The organisation of this paper is as follows: the framework for Hopf cyclic cohomology is briefly recalled in Section 1. Thereafter, we construct the morphism L_D on Hopf cyclic cohomology due to a biderivation on a Hopf algebra described in Proposition 0.1 and give examples for quantized universal enveloping algebras $\mathcal{U}_h(\mathfrak{g})$. In Section 2, we introduce our concept of inner biderivations and exhibit

the left $\mathcal{H}$ -module structure on the Hopf cyclic cohomology mentioned in Proposition 0.2 above. Finally, in Section 3, we construct the Hopf algebra $\mathcal{H}[D]$ and prove Proposition 0.3.

1. THE OPERATOR INDUCED BY A BIDERIVATION ON HOPF CYCLIC COHOMOLOGY

Let $(\mathcal{H}, m, \Delta, u, \epsilon, S)$ be a Hopf algebra over $\mathbb{C}$ with unit map $u : \mathbb{C} \rightarrow \mathcal{H}$, counit map $\epsilon : \mathcal{H} \rightarrow \mathbb{C}$, algebra product $m : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H}$, coalgebra coproduct $\Delta : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ and antipode $S : \mathcal{H} \rightarrow \mathcal{H}$.

Throughout, we shall use the standard Sweedler notation; i.e. we write

$$(1.1) \quad \Delta(h) = \sum h_{(1)} \otimes h_{(2)} \quad \forall h \in \mathcal{H}.$$

We fix a character $\delta : \mathcal{H} \rightarrow \mathbb{C}$ of $\mathcal{H}$ (which we shall refer to as the modular character) and consider the twisted antipode $\tilde{S}$ defined as

$$(1.2) \quad \tilde{S}(h) = (\delta * S)(h) = \sum \delta(h_{(1)}) S(h_{(2)}).$$

Consider a “grouplike element” σ of $\mathcal{H}$ (i.e., an element $\sigma \in \mathcal{H}$ such that $\Delta(\sigma) = \sigma \otimes \sigma$). Suppose that the character $\delta : \mathcal{H} \rightarrow \mathbb{C}$ satisfies

$$(1.3) \quad \delta(\sigma) = 1.$$

Then, the pair (δ, σ) is said to be a modular pair. Further, if the pair (δ, σ) is such that

$$(1.4) \quad (\sigma^{-1} \tilde{S})^2 = 1,$$

the pair (δ, σ) is said to be a modular pair in involution. It follows from these definitions (see [3]) that

$$(1.5) \quad \begin{aligned} \tilde{S}(h^1 h^2) &= \tilde{S}(h^2) \tilde{S}(h^1) \\ \tilde{S}(1) &= 1 \\ \Delta \tilde{S}(h) &= \sum S(h_{(2)}) \otimes \tilde{S}(h_{(1)}) \\ \epsilon \circ \tilde{S} &= \delta. \end{aligned}$$

We now briefly recall the definition of Hopf cyclic cohomology for a Hopf algebra endowed with a modular pair in involution due to Connes and Moscovici [3]. Consider the cosimplicial module $\{\mathcal{H}^{\otimes n}\}_{n \geq 1}$ with face operators $\delta_i : \mathcal{H}^{\otimes n-1} \rightarrow \mathcal{H}^{\otimes n}$,

$$(1.6) \quad \begin{aligned} \delta_0(h^1 \otimes \dots \otimes h^{n-1}) &= 1 \otimes h^1 \otimes \dots \otimes h^{n-1} \\ \delta_i(h^1 \otimes \dots \otimes h^{n-1}) &= h^1 \otimes \dots \otimes \Delta h^i \otimes \dots \otimes h^{n-1}, \quad 1 \leq i \leq n-1 \\ \delta_n(h^1 \otimes \dots \otimes h^{n-1}) &= h^1 \otimes \dots \otimes h^{n-1} \otimes \sigma \end{aligned}$$

and degeneracy operators $\sigma_i : \mathcal{H}^{\otimes n+1} \rightarrow \mathcal{H}^{\otimes n}$,

$$(1.7) \quad \sigma_i(h^1 \otimes \dots \otimes h^{n+1}) = h^1 \otimes \dots \otimes \epsilon(h^{i+1}) \otimes \dots \otimes h^{n+1}, \quad 0 \leq i \leq n.$$

Furthermore, define the cyclic operator

$$(1.8) \quad \tau_n(h^1 \otimes \dots \otimes h^n) = (\Delta^{n-1} \tilde{S}(h^1)) \cdot h^2 \otimes \dots \otimes h^n \otimes \sigma.$$

If the pair (δ, σ) is a modular pair in involution, i.e. $(\sigma^{-1} \tilde{S})^2 = I$, [3, Theorem 1] shows that the operators τ_n on the cosimplicial module $\{\mathcal{H}^{\otimes n}\}_{n \geq 1}$ make it a cocyclic module, which we denote by $\mathcal{H}_{(\delta, \sigma)}^\#$.

Definition 1.1. Let $\mathcal{H}$ be a Hopf algebra over $\mathbb{C}$ equipped with a modular pair in involution (δ, σ) . The cohomology of the cocyclic module $\mathcal{H}_{(\delta, \sigma)}^\#$ is the Hopf cyclic cohomology $HC_{(\delta, \sigma)}^*(\mathcal{H})$ of $\mathcal{H}$ relative to the modular pair (δ, σ) .

In particular, if the Hopf algebra $\mathcal{H}$ is equipped with a character $\delta : \mathcal{H} \rightarrow \mathbb{C}$ such that $(\delta, 1)$ defines a modular pair in involution, we shall denote the Hopf cyclic cohomology of $\mathcal{H}$ relative to $(\delta, 1)$ simply by $HC_\delta^*(\mathcal{H})$ and the corresponding cocyclic module by $\mathcal{H}_\delta^\#$.

Henceforth, we shall always assume that our Hopf algebras are defined over $\mathbb{C}$ and carry a modular pair in involution (δ, σ) . We are interested in the morphisms induced on the Hopf cyclic cohomology of a Hopf algebra $\mathcal{H}$ by maps that we shall refer to as “biderivations”.

A *biderivation* on $\mathcal{H}$ shall be a linear map $D : \mathcal{H} \rightarrow \mathcal{H}$ that satisfies the axioms both for being a derivation on the underlying algebra of $\mathcal{H}$ and for being a coderivation on the underlying coalgebra of $\mathcal{H}$. Therefore, suppose that we have a linear map $D : \mathcal{H} \rightarrow \mathcal{H}$ satisfying the following conditions:

(a) D is a coderivation for the underlying coalgebra of $\mathcal{H}$; i.e. given $h \in \mathcal{H}$, we have

$$(1.9) \quad \Delta(D(h)) = (D \otimes 1 + 1 \otimes D)\Delta(h).$$

(b) D is a derivation for the underlying algebra of $\mathcal{H}$; i.e. given $h^1, h^2 \in \mathcal{H}$,

$$(1.10) \quad D(h^1 h^2) = D(h^1)h^2 + h^1 D(h^2).$$

(c) The following relations hold:

$$(1.11) \quad \epsilon \circ D = 0 \quad D \circ u = 0 \quad DS = SD \quad \delta \circ D = 0 \quad D(\sigma) = 0.$$

If we extend the operator D to any $\mathcal{H}^{\otimes n}$, $n \geq 1$, by setting

$$(1.12) \quad D(h^1 \otimes \dots \otimes h^n) = (D(h^1) \otimes \dots \otimes h^n) + \dots + (h^1 \otimes \dots \otimes D(h^n))$$

for any $h^1, h^2, \dots, h^n \in \mathcal{H}$, we can express condition (a) more compactly as

$$(1.13) \quad \Delta \circ D(h) = D \circ \Delta(h) \quad \forall h \in \mathcal{H}.$$

A linear map $D : \mathcal{H} \rightarrow \mathcal{H}$ that satisfies conditions (a) and (b) above will be referred to as a *biderivation*. We will refer to the module of all biderivations on $\mathcal{H}$ as $Bider(\mathcal{H})$. The submodule of all biderivations that also satisfy (c) will be referred to as $Bider_0^{(\delta, \sigma)}(\mathcal{H})$. The pair (δ, σ) will usually be clear from the context and hence we will denote this subset simply by $Bider_0(\mathcal{H})$.

Lemma 1.2. Let $\mathcal{H}$ be a Hopf algebra and let $D : \mathcal{H} \rightarrow \mathcal{H}$ be a biderivation on $\mathcal{H}$ as described above. Then, if additionally $D \in Bider_0(\mathcal{H})$:

- (a) The map D commutes with the face maps δ_i and the degeneracy operators σ_j of the cocyclic module $\mathcal{H}_{(\delta, \sigma)}^\#$.
- (b) D commutes with the twisted antipode; i.e. $D \circ \tilde{S}(y) = \tilde{S} \circ D(y)$ for all $y \in \mathcal{H}$.

Proof. (a) We have maps $\delta_i : \mathcal{H}^{\otimes n-1} \rightarrow \mathcal{H}^{\otimes n}$, $0 \leq i \leq n$. The result is obvious for $i = 0$ and for $i = n$ (since $D(\sigma) = 0$). For $1 \leq i \leq n-1$, we have

$$\begin{aligned}
 (1.14) \quad D(\delta_i(h^1 \otimes \dots \otimes h^{n-1})) &= D(\sum h^1 \otimes \dots \otimes h_{(1)}^i \otimes h_{(2)}^i \otimes \dots \otimes h^{n-1}) \\
 &= \sum_{j=1}^{i-1} (h^1 \otimes \dots \otimes h^{j-1} \otimes D(h^j) \otimes h^{j+1} \otimes \dots \\
 &\quad \otimes h^{i-1} \otimes h_{(1)}^i \otimes h_{(2)}^i \otimes \dots \otimes h^{n-1}) \\
 &\quad + \sum h^1 \otimes \dots \otimes h^{i-1} \otimes D(h_{(1)}^i) \otimes h_{(2)}^i \otimes \dots \otimes h^{n-1} \\
 &\quad + \sum h^1 \otimes \dots \otimes h^{i-1} \otimes h_{(1)}^i \otimes D(h_{(2)}^i) \otimes \dots \otimes h^{n-1} \\
 &\quad + \sum_{j=i+1}^{n-1} (h^1 \otimes \dots \otimes h_{(1)}^i \otimes h_{(2)}^i \otimes h^{j-1} \\
 &\quad \otimes D(h^j) \otimes h^{j+1} \otimes \dots \otimes h^{n-1}) \\
 &= \delta_i(D(h^1 \otimes \dots \otimes h^{n-1})).
 \end{aligned}$$

For the degeneracy operators, we have

$$\begin{aligned}
 (1.15) \quad D(\sigma_i(h^1 \otimes \dots \otimes h^{n+1})) &= D(h^1 \otimes \dots \otimes \epsilon(h^{i+1}) \otimes \dots \otimes h^{n+1}) \\
 &= \epsilon(h^{i+1}) \sum_{j=1}^i (h^1 \otimes \dots \otimes h^{j-1} \otimes D(h^j) \otimes h^{j+1} \otimes \dots \\
 &\quad \otimes h^i \otimes h^{i+2} \otimes \dots \otimes h^{n+1}) \\
 &\quad + \epsilon(h^{i+1}) \sum_{j=i+2}^{n+1} (h^1 \otimes \dots \otimes h^i \otimes h^{i+2} \otimes \dots \otimes h^{j-1} \\
 &\quad \otimes D(h^j) \otimes \dots \otimes h^{n+1}) \\
 &= \sigma_i(D(h^1 \otimes \dots \otimes h^{n+1})) \quad (\text{since } \epsilon \circ D = 0).
 \end{aligned}$$

(b) Let $y \in \mathcal{H}$. Then

$$\begin{aligned}
 (1.16) \quad \tilde{S}(D(y)) &= (\delta * S)(D(y)) \\
 &= \sum \delta(D(y_{(1)}))S(y_{(2)}) + \sum \delta(y_{(1)})S(Dy_{(2)}) \\
 &= \sum \delta(y_{(1)})D(S(y_{(2)})) \quad (\text{since } \delta \circ D = 0) \\
 &= D(\sum \delta(y_{(1)})S(y_{(2)})) = D(\delta * S)(y) = D \circ \tilde{S}(y).
 \end{aligned}$$

□

Lemma 1.3. Let $\mathcal{H}$ be a Hopf algebra and let $D : \mathcal{H} \rightarrow \mathcal{H}$ be a biderivation on $\mathcal{H}$ as described above. Then, if additionally $D \in \text{Bider}_0(\mathcal{H})$, D commutes with the cyclic operators τ_n on the cocyclic module $\mathcal{H}_{(\delta, \sigma)}^\# = \{\mathcal{H}^{\otimes n}\}_{n \geq 1}$.

Proof. We note that the extension of the operator D to $\mathcal{H}^{\otimes n}$ is also a derivation; i.e. given $h^1 \otimes h^2 \otimes \dots \otimes h^n, k^1 \otimes k^2 \otimes \dots \otimes k^n \in \mathcal{H}^{\otimes n}$, we have

$$\begin{aligned}
 (1.17) \quad D((h^1 \otimes h^2 \otimes \dots \otimes h^n) \cdot (k^1 \otimes k^2 \otimes \dots \otimes k^n)) \\
 = D(h^1 \otimes h^2 \otimes \dots \otimes h^n) \cdot (k^1 \otimes k^2 \otimes \dots \otimes k^n) \\
 + (h^1 \otimes h^2 \otimes \dots \otimes h^n) \cdot D(k^1 \otimes k^2 \otimes \dots \otimes k^n).
 \end{aligned}$$

By definition,

$$(1.18) \quad \tau_n(h^1 \otimes \dots \otimes h^n) = (\Delta^{n-1} \tilde{S}(h^1)) \cdot (h^2 \otimes \dots \otimes h^n \otimes \sigma).$$

We have noted before that $D \circ \Delta = \Delta \circ D$, and we have $D \circ \tilde{S} = \tilde{S} \circ D$ from Lemma 1.2 above. Hence,

$$\begin{aligned}
 (1.19) \quad D\tau_n(h^1 \otimes \dots \otimes h^n) &= D((\Delta^{n-1}\tilde{S}(h^1)) \cdot (h^2 \otimes \dots \otimes h^n \otimes \sigma)) \\
 &= D(\Delta^{n-1}\tilde{S}(h^1)) \cdot (h^2 \otimes \dots \otimes h^n \otimes \sigma) \\
 &\quad + (\Delta^{n-1}\tilde{S}(h^1)) \cdot D(h^2 \otimes \dots \otimes h^n \otimes \sigma) \\
 &= (\Delta^{n-1}\tilde{S}(Dh^1)) \cdot (h^2 \otimes \dots \otimes h^n \otimes \sigma) \\
 &\quad + \sum_{i=2}^n (\Delta^{n-1}\tilde{S}(h^1)) \cdot (h^2 \otimes \dots \otimes D(h^i) \otimes \dots \otimes h^n \otimes \sigma) \\
 &= \tau_n(D(h^1) \otimes h^2 \otimes \dots \otimes h^n) \\
 &\quad + \sum_{i=2}^n \tau_n(h^1 \otimes \dots \otimes h^{i-1} \otimes D(h^i) \otimes \dots \otimes h^n) \\
 &= \tau_n(D(h^1 \otimes \dots \otimes h^n)).
 \end{aligned}$$

□

We are now ready to define the operator $L_D : HC_{(\delta, \sigma)}^*(\mathcal{H}) \longrightarrow HC_{(\delta, \sigma)}^*(\mathcal{H})$.

Proposition 1.4. *Let $\mathcal{H}$ be a Hopf algebra equipped with a modular pair in involution (δ, σ) and let $D : \mathcal{H} \longrightarrow \mathcal{H}$ be a biderivation on $\mathcal{H}$ as described above. Then, if additionally $D \in \text{Bider}_0(\mathcal{H})$, the map D induces a morphism in Hopf cyclic cohomology:*

$$L_D : HC_{(\delta, \sigma)}^*(\mathcal{H}) \rightarrow HC_{(\delta, \sigma)}^*(\mathcal{H}).$$

Proof. This follows directly from the fact that the action of $D \in \text{Bider}_0(\mathcal{H})$ commutes with the face maps, degeneracies and cyclic operators on the cocyclic module $\mathcal{H}_{(\delta, \sigma)}^\#$ defining Hopf cyclic cohomology. □

We will now consider the commutator $[D_1, D_2]$ of two biderivations D_1 and D_2 on $\mathcal{H}$. These calculations will be useful in Section 2 when we define the left $\mathcal{H}$ -module structure on $HC_\delta^*(\mathcal{H})$ for an irreducible, cocommutative Hopf algebra $\mathcal{H}$ over $\mathbb{C}$ equipped with a character $\delta : \mathcal{H} \longrightarrow \mathbb{C}$ such that $(\delta, 1)$ is a modular pair in involution.

Lemma 1.5. *Let $\mathcal{H}$ be a Hopf algebra equipped with a modular pair in involution (δ, σ) and let $D_1, D_2 \in \text{Bider}_0(\mathcal{H})$. Then, the commutator $[D_1, D_2] \in \text{Bider}_0(\mathcal{H})$.*

Proof. Let $D_1, D_2 \in \text{Bider}_0(\mathcal{H})$. The commutator $[D_1, D_2]$ is a derivation for the algebra structure of $\mathcal{H}$. To verify that $[D_1, D_2]$ is a coderivation, we check, for $h \in \mathcal{H}$, that

$$\begin{aligned}
 \Delta(D_1 D_2(h)) &= (D_1 \otimes 1 + 1 \otimes D_1) \Delta(D_2(h)) \\
 &= (D_1 \otimes 1 + 1 \otimes D_1) (D_2 \otimes 1 + 1 \otimes D_2) (\Delta(h)) \\
 (1.20) \quad \Delta(D_2 D_1(h)) &= (D_2 \otimes 1 + 1 \otimes D_2) (D_1 \otimes 1 + 1 \otimes D_1) (\Delta(h)) \\
 \Rightarrow \Delta([D_1, D_2](h)) &= ([D_1, D_2] \otimes 1 + 1 \otimes [D_1, D_2]) (\Delta(h)).
 \end{aligned}$$

Since furthermore D_1 and D_2 lie in $\text{Bider}_0(\mathcal{H})$, it follows that $\epsilon \circ [D_1, D_2] = 0$, $[D_1, D_2] \circ u = 0$, $[D_1, D_2]S = S[D_1, D_2]$, $[D_1, D_2](\sigma) = 0$ and $\delta \circ [D_1, D_2] = 0$. Hence, $[D_1, D_2] \in \text{Bider}_0(\mathcal{H})$. □

Proposition 1.6. *Let $\mathcal{H}$ be a Hopf algebra equipped with a modular pair in involution (δ, σ) and let $D_1, D_2 \in \text{Bider}_0(\mathcal{H})$. Then,*

$$(1.21) \quad L_{[D_1, D_2]} = [L_{D_1}, L_{D_2}] : HC_{(\delta, \sigma)}^*(\mathcal{H}) \longrightarrow HC_{(\delta, \sigma)}^*(\mathcal{H}).$$

Proof. From Lemma 1.5, it follows that $[D_1, D_2] \in \text{Bider}_0(\mathcal{H})$. Hence, the operator $L_{[D_1, D_2]}$ descends to the cohomology $HC_{(\delta, \sigma)}^*(\mathcal{H})$. It follows directly from the definitions that $L_{[D_1, D_2]} = [L_{D_1}, L_{D_2}]$. $\square$

We end this section by giving examples of such maps L_D for the quantized universal enveloping algebra $\mathcal{U}_h(\mathfrak{g})$ of a finite dimensional, complex, simple Lie algebra $\mathfrak{g}$ of rank N . Let $\{\alpha_1, \dots, \alpha_N\}$ be a system of simple roots and let $(,)$ denote the canonical bilinear form on the root space obtained from the Killing form. Then, the Cartan matrix $A = ((a_{ij}))$ of $\mathfrak{g}$ has the description

$$(1.22) \quad a_{ij} = 2 \frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}, \quad i, j = 1, 2, \dots, N,$$

and we set

$$(1.23) \quad d_i = \frac{(\alpha_i, \alpha_i)}{2}, \quad i = 1, 2, \dots, N.$$

For any invertible $q \in \mathbb{C}[[h]]$ in the ring of formal power series, define the q -integers by

$$(1.24) \quad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \dots + q^{-n+3} + q^{-n+1}$$

for any integer n . Then, for any non-negative integer k , the q -binomial coefficients are defined by

$$(1.25) \quad [k]_q = \prod_{i=1}^k [i]_q, \quad \binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q! [k]_q!}.$$

The quantized universal enveloping algebra $\mathcal{U}_h(\mathfrak{g})$ can now be defined as an associative algebra over $\mathbb{C}[[h]]$, complete in h -adic topology, through the generators $\{H_i, X_{\pm i}\}$, $i = 1, 2, \dots, N$, and the relations (see [10, §7.1])

$$(1.26) \quad \begin{aligned} [H_i, H_j] &= 0 & [H_i, X_{\pm j}] &= \pm a_{ij} X_{\pm j} & [X_{+i}, X_{-j}] &= \delta_{ij} \frac{q^{H_i} - q^{-H_i}}{q_i - q_i^{-1}} \\ \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{q_i} X_{\pm i}^{1-a_{ij}-k} X_{\pm j} X_{\pm i}^k &= 0 & \forall i \neq j. \end{aligned}$$

It can be shown that there is an isomorphism $\mathcal{U}_h(\mathfrak{g}) \cong \mathcal{U}(\mathfrak{g})[[h]]$ of $\mathbb{C}[[h]]$ -modules (see [5]), where $\mathcal{U}(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$. We will treat $\mathcal{U}_h(\mathfrak{g})$ as a topological Hopf algebra, i.e., we will say that the coproduct Δ on $\mathcal{U}_h(\mathfrak{g})$ takes values in the completion $\mathcal{U}_h(\mathfrak{g}) \hat{\otimes} \mathcal{U}_h(\mathfrak{g})$ of the tensor product $\mathcal{U}_h(\mathfrak{g}) \otimes_{\mathbb{C}[[h]]} \mathcal{U}_h(\mathfrak{g})$ in h -adic topology. Then, the coproduct, the unit and the antipode are defined by

$$(1.27) \quad \begin{aligned} \Delta H_i &= H_i \otimes 1 + 1 \otimes H_i & \Delta X_{\pm i} &= X_{\pm i} \otimes q_i^{H_i/2} + q_i^{H_i/2} \otimes X_{\pm i} \\ \epsilon(H_i) &= \epsilon(X_{\pm i}) = 0 & S(H_i) &= -H_i & S(X_{\pm i}) &= -q_i^{\pm 1} X_{\pm i}. \end{aligned}$$

Given a modular pair in involution (δ, σ) for the Hopf algebra $\mathcal{U}_h(\mathfrak{g})$, the definitions at the beginning of this section can be extended directly to the topological case to define the topological cocyclic module $\mathcal{U}_h(\mathfrak{g})_{(\delta, \sigma)}^{\sharp, top} = \{\mathcal{U}_h(\mathfrak{g})^{\hat{\otimes} n}\}$. We will

use $HC_{(\delta,\sigma)}^*(\mathcal{U}_h(\mathfrak{g}))$ to denote the cohomology of the cocyclic module $\mathcal{U}_h(\mathfrak{g})_{(\delta,\sigma)}^{\sharp, \text{top}} = \{\mathcal{U}_h(\mathfrak{g})^{\otimes n}\}$.

It is well known that $\mathcal{U}_h(\mathfrak{g})$ carries a (topological) ribbon Hopf algebra structure (see [5] or [10]). Then, $\mathcal{U}_h(\mathfrak{g})$ has an underlying (topological) quasi-triangular Hopf algebra structure with R -matrix R (say). For the definition and properties of quasi-triangular Hopf algebras and ribbon algebras, the reader may see [10]. It follows that there exists an invertible element $v \in \mathcal{U}_h(\mathfrak{g})$ such that

$$(1.28) \quad S^2(x) = vxv^{-1} \quad \forall x \in \mathcal{U}_h(\mathfrak{g}) \quad \text{and} \quad vS(v) \text{ is central}$$

and a “ribbon element” θ which is central and satisfies (with notation as in [3]):

$$(1.29) \quad \theta^2 = vS(v) \quad \Delta(\theta) = (R_{21}R_{12})^{-1}(\theta \otimes \theta) \quad \epsilon(\theta) = 1 \quad S(\theta) = \theta.$$

Following the proof of [3, Proposition 3], we can get a grouplike element $\sigma = \theta^{-1}v$ with

$$(1.30) \quad \Delta(\sigma) = \sigma \otimes \sigma \quad \epsilon(\sigma) = 1 \quad S(\sigma) = \sigma^{-1}.$$

It follows from the proof of [3, Proposition 3] that (ϵ, σ) forms a modular pair in involution for the Hopf algebra $\mathcal{U}_h(\mathfrak{g})$.

Proposition 1.7. *Let $\mathfrak{g}$ be a finite dimensional, complex, simple Lie algebra having quantized universal enveloping algebra $\mathcal{U}_h(\mathfrak{g})$ with generators $\{H_i, X_{\pm i}\}$, $i = 1, 2, \dots, N$, as above. For each i , let D_i denote the linear map on $\mathcal{U}_h(\mathfrak{g})$ given by the inner derivation $D_i = [H_i, -]$ on $\mathcal{H}$. Then, $D_i \in \text{Bider}_0(\mathcal{U}_h(\mathfrak{g}))$, and hence there exists a corresponding linear operator $L_i : HC_{(\epsilon,\sigma)}^*(\mathcal{U}_h(\mathfrak{g})) \longrightarrow HC_{(\epsilon,\sigma)}^*(\mathcal{U}_h(\mathfrak{g}))$.*

Proof. Choose any element $k \in \mathcal{U}_h(\mathfrak{g})$; then, we have

$$(1.31) \quad \begin{aligned} \Delta(D_i(k)) &= \Delta(H_i k - k H_i) = \Delta(H_i) \cdot \Delta(k) - \Delta(k) \Delta(H_i) \\ &= \sum (H_i k_{(1)} \otimes k_{(2)} + k_{(1)} \otimes H_i k_{(2)}) \\ &\quad - \sum (k_{(1)} H_i \otimes k_{(2)} + k_{(1)} \otimes k_{(2)} H_i) \\ &= \sum [H_i, k_{(1)}] \otimes k_{(2)} + \sum k_{(1)} \otimes [H_i, k_{(2)}] \\ &= \sum D_i(k_{(1)}) \otimes k_{(2)} + \sum k_{(1)} \otimes D_i(k_{(2)}) \end{aligned}$$

and hence D_i is a coderivation. Since $\epsilon : \mathcal{U}_h(\mathfrak{g}) \longrightarrow \mathbb{C}$ is an $\mathbb{C}$ -algebra morphism to the field $\mathbb{C}$, $\epsilon \circ D_i(k) = \epsilon([H_i, k]) = 0$. Also, $\text{Im}(u)$ lies in the centre of $\mathcal{U}_h(\mathfrak{g})$ by definition, and hence $D_i \circ u = 0$. Moreover,

$$(1.32) \quad \begin{aligned} D_i S(k) &= H_i S(k) - S(k) H_i = -S(H_i) S(k) + S(k) S(H_i) \\ &= -S(k H_i) + S(H_i k) = S(D_i(k)), \end{aligned}$$

where we recall that S is an antihomomorphism and $S(H_i) = -H_i$. Also, we have $\delta \circ D_i(k) = \delta(H_i k - k H_i) = 0$.

In order to prove that $D_i \in \text{Bider}_0(\mathcal{U}_h(\mathfrak{g}))$, it remains to show that $D_i(\sigma) = 0$. By definition, $\sigma = \theta^{-1}v$. Since $\mathcal{U}_h(\mathfrak{g})$ is quasi-triangular, it follows that $S^2(H_i) = v H_i v^{-1}$. On the other hand, we know that $S(H_i) = -H_i$ and hence $S^2(H_i) = H_i$. Hence, H_i commutes with v . Also, the ribbon element θ is central, and hence $\sigma = \theta^{-1}v$ commutes with H_i . It follows that $D_i(\sigma) = 0$.

It now follows, exactly as in the proofs of Lemmas 1.2 and 1.3, that the linear map D_i commutes with the face maps, degeneracies and cyclic operators of the cocyclic module $\mathcal{U}_h(\mathfrak{g})_{(\epsilon,\sigma)}^{\sharp, \text{top}}$. This induces a map

$$(1.33) \quad L_i := L_{D_i} : HC_{(\epsilon,\sigma)}^*(\mathcal{U}_h(\mathfrak{g})) \longrightarrow HC_{(\epsilon,\sigma)}^*(\mathcal{U}_h(\mathfrak{g}))$$

just as in the proof of Proposition 1.4. $\square$

2. $\mathcal{H}$ -MODULE STRUCTURE ON $HC_{(\delta, \sigma)}^*(\mathcal{H})$

In this section, we shall assume that our Hopf algebra $\mathcal{H}$ is the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$. Then, from [11, Proposition 11.0.9] and [11, Theorem 13.0.1] (see also [1], [9]), this is equivalent to assuming that $\mathcal{H}$ is an irreducible, cocommutative Hopf algebra over $\mathbb{C}$ (for definitions, we refer the reader to [11]). Then, $\mathcal{H} = \mathcal{U}(\text{Prim}(\mathcal{H}))$, where $\text{Prim}(\mathcal{H})$ is the Lie algebra of primitive elements of $\mathcal{H}$ and $\mathcal{U}$ is the universal enveloping algebra functor. We shall show that the Hopf cyclic cohomology $HC_{\delta}^*(\mathcal{H})$ can be given the structure of a left $\mathcal{H}$ -module.

Lemma 2.1. *Let $\mathcal{H}$ be an irreducible, cocommutative Hopf algebra over $\mathbb{C}$ with a character $\delta : \mathcal{H} \rightarrow \mathbb{C}$ such that $(\delta, 1)$ is a modular pair in involution. Let $h \in \mathcal{H}$ be a primitive element. Then, the inner derivation $D_{(h)} = [h, _] : \mathcal{H} \rightarrow \mathcal{H}$ is a biderivation on $\mathcal{H}$. Also, $D_{(h)} \in \text{Bider}_0(\mathcal{H})$, i.e.*

$$(2.1) \quad \epsilon \circ D_{(h)} = 0 \quad D_{(h)} \circ u = 0 \quad SD_{(h)} = D_{(h)}S \quad \delta \circ D_{(h)} = 0.$$

Proof. Choose any element $k \in \mathcal{H}$; then, h being primitive, we have

$$(2.2) \quad \begin{aligned} \Delta(D_{(h)}(k)) &= \Delta(hk - kh) = \Delta(h) \cdot \Delta(k) - \Delta(k) \Delta(h) \\ &= \sum (hk_{(1)} \otimes k_{(2)} + k_{(1)} \otimes hk_{(2)}) - \sum (k_{(1)}h \otimes k_{(2)} + k_{(1)} \otimes k_{(2)}h) \\ &= \sum [h, k_{(1)}] \otimes k_{(2)} + \sum k_{(1)} \otimes [h, k_{(2)}] \\ &= \sum D_{(h)}(k_{(1)}) \otimes k_{(2)} + \sum k_{(1)} \otimes D_{(h)}(k_{(2)}) \end{aligned}$$

and hence $D_{(h)}$ is a coderivation. Since $\epsilon : \mathcal{H} \rightarrow \mathbb{C}$ is an $\mathbb{C}$ -algebra morphism to the field $\mathbb{C}$, $\epsilon \circ D_{(h)}(k) = \epsilon([h, k]) = 0$. Also, $\text{Im}(u)$ lies in the centre of $\mathcal{H}$ by definition, and hence $D_{(h)} \circ u = 0$. Moreover,

$$(2.3) \quad \begin{aligned} D_{(h)}S(k) &= hS(k) - S(k)h = -S(h)S(k) + S(k)S(h) \\ &= -S(kh) + S(hk) = S(D_{(h)}(k)), \end{aligned}$$

where we recall that S is an antihomomorphism and $S(h) = -h$ since h is a primitive element of the Hopf algebra $\mathcal{H}$, which, in this case, happens to be the universal enveloping algebra of a Lie algebra. Finally, $\delta \circ D_{(h)}(k) = \delta(hk - kh) = 0$. $\square$

Definition 2.2. Let $\mathcal{H}$ be a Hopf algebra over $\mathbb{C}$. If, for some $h \in \mathcal{H}$, the linear map induced by the commutator $D_{(h)} := [h, _] : \mathcal{H} \rightarrow \mathcal{H}$ is a biderivation, we say that $D_{(h)}$ is an inner biderivation.

We now have:

Proposition 2.3. *Let $\mathcal{H}$ be an irreducible, cocommutative Hopf algebra over $\mathbb{C}$ with a character $\delta : \mathcal{H} \rightarrow \mathbb{C}$ such that $(\delta, 1)$ is a modular pair in involution. Then the Hopf cyclic cohomology $HC_{\delta}^*(\mathcal{H})$ carries the structure of a left $\mathcal{H}$ -module.*

Proof. From Lemma 2.1 it follows that for each primitive element $h \in \text{Prim}(\mathcal{H})$, we have a biderivation $D_{(h)} \in \text{Bider}_0(\mathcal{H})$. From Proposition 1.4 it now follows that we have morphisms

$$(2.4) \quad L_{(h)} := L_{D_{(h)}} : HC_{\delta}^*(\mathcal{H}) \rightarrow HC_{\delta}^*(\mathcal{H}) \quad \forall h \in \text{Prim}(\mathcal{H})$$

which, furthermore, by Proposition 1.6 satisfy $L_{[h, h']} = [L_{(h)}, L_{(h')}]$ for any two primitive elements $h, h' \in \text{Prim}(\mathcal{H})$. Therefore, the Lie algebra action of $\text{Prim}(\mathcal{H})$ on $HC_{\delta}^*(\mathcal{H})$ by operators $L_{(h)}$ as in (2.4) endows $HC_{\delta}^*(\mathcal{H})$ with the structure of a left $\mathcal{H} \simeq \mathcal{U}(\text{Prim}(\mathcal{H}))$ -module. $\square$

3. THE HOPF ALGEBRA $\mathcal{H}[D]$

As in Section 2, we suppose throughout that $\mathcal{H}$ is a Hopf algebra over $\mathbb{C}$, equipped with a character $\delta : \mathcal{H} \rightarrow \mathbb{C}$ such that $(\delta, 1)$ is a modular pair in involution. Further, we continue to assume that $\mathcal{H}$ is irreducible and cocommutative; i.e. it is the universal enveloping algebra of its primitive elements. Let D be a biderivation on $\mathcal{H}$ which additionally satisfies condition (c) of Section 1, i.e. $D \in \text{Bider}_0(\mathcal{H})$.

Now define $\mathfrak{L} = \text{Prim}(\mathcal{H}) \oplus \mathbb{C} \cdot D$, i.e. the vector space obtained by formally adjoining D to $\text{Prim}(\mathcal{H})$. $\text{Prim}(\mathcal{H})$ is already a Lie algebra, with a bracket $[_, _]$. It is also clear that, D being a coderivation, $D(\text{Prim}(\mathcal{H})) \subseteq \text{Prim}(\mathcal{H})$. We will now extend the Lie bracket on $\text{Prim}(\mathcal{H})$ to all of $\mathfrak{L}$.

Lemma 3.1. *On the vector space $\mathfrak{L} = \text{Prim}(\mathcal{H}) \oplus \mathbb{C} \cdot D$, define the bracket $[_, _]_D$ by setting*

$$(3.1) \quad \begin{aligned} [h_1, h_2]_D &= -[h_2, h_1]_D = [h_1, h_2] && \text{if } h_1, h_2 \in \text{Prim}(\mathcal{H}) \\ [D, h]_D &= -[h, D]_D = D(h) && \text{if } h \in \text{Prim}(\mathcal{H}) \\ [D, xD]_D &= [xD, D]_D = 0 && \text{if } x \in \mathbb{C} \end{aligned}$$

and extending by bilinearity to a bracket on all of $\mathfrak{L} \times \mathfrak{L}$. Then, $[_, _]_D : \mathfrak{L} \times \mathfrak{L} \rightarrow \mathfrak{L}$ defines a Lie algebra structure on $\mathfrak{L}$.

Proof. We need to check that, given $x, y, z \in \mathfrak{L}$, $[x, [y, z]_D]_D + [y, [z, x]_D]_D + [z, [x, y]_D]_D = 0$. It suffices to check that, given $y, z \in \text{Prim}(\mathcal{H})$, $[D, [y, z]_D]_D + [y, [z, D]_D]_D + [z, [D, y]_D]_D = 0$. We have

$$(3.2) \quad \begin{aligned} [D, [y, z]_D]_D &= D(yz - zy) = D(y)z + yD(z) - D(z)y - zD(y) \\ [y, [z, D]_D]_D &= -[y, D(z)]_D = -yD(z) + D(z)y \\ [z, [D, y]_D]_D &= [z, D(y)]_D = zD(y) - D(y)z \end{aligned}$$

$$\Rightarrow [D, [y, z]_D]_D + [y, [z, D]_D]_D + [z, [D, y]_D]_D = 0.$$

□

Let us, therefore, denote by $\mathcal{H}[D]$ the universal enveloping algebra $\mathcal{U}(\mathfrak{L})$. Then $\mathcal{H}$ embeds into $\mathcal{H}[D]$, and it follows from [11, Proposition 11.0.9] that $\mathcal{H}[D]$ is an irreducible, cocommutative Hopf algebra. Extend $\delta : \mathcal{H} \rightarrow \mathbb{C}$ to $\delta_D : \mathcal{H}[D] \rightarrow \mathbb{C}$ by setting $\delta_D(h) = \delta(h)$ for $h \in \text{Prim}(\mathcal{H})$ and $\delta_D(D) = 0$. Then, there exists a morphism of Hopf cyclic cohomologies

$$(3.3) \quad i : HC_{\delta}^*(\mathcal{H}) \rightarrow HC_{\delta_D}^*(\mathcal{H}[D]).$$

Also, D is a primitive element of the Hopf algebra $\mathcal{H}[D]$. The following proposition now shows that the operator $L_D : HC_{\delta}^*(\mathcal{H}) \rightarrow HC_{\delta}^*(\mathcal{H})$ can be described as an operator induced by an inner biderivation.

Proposition 3.2. *The inner biderivation $[D, _]_D$ of $\mathcal{H}[D]$ corresponding to D induces an endomorphism of $HC_{\delta_D}^*(\mathcal{H}[D])$, which we denote by $L_{(D)}$. Then, we have a commutative diagram*

$$(3.4) \quad \begin{array}{ccc} HC_{\delta}^*(\mathcal{H}) & \xrightarrow{L_D} & HC_{\delta}^*(\mathcal{H}) \\ i \downarrow & & i \downarrow \\ HC_{\delta_D}^*(\mathcal{H}[D]) & \xrightarrow{L_{(D)}} & HC_{\delta_D}^*(\mathcal{H}[D]) \end{array}$$

Proof. Following Proposition 2.3, the inner biderivation $[D, _]_D$ of $\mathcal{H}[D]$ induces an endomorphism $L_{(D)} : HC_{\delta}^*(\mathcal{H}[D]) \longrightarrow HC_{\delta}^*(\mathcal{H}[D])$. Since $\mathcal{H} \simeq \mathcal{U}(Prim(\mathcal{H}))$, it follows that $D : \mathcal{H} \longrightarrow \mathcal{H}$ is completely determined by its action on $Prim(\mathcal{H})$. Hence, for any $h \in \mathcal{H}$:

$$(3.5) \quad i(D(h)) = [D, i(h)]_D \in \mathcal{H}[D].$$

The commutativity of (3.4) follows easily from this. $\square$

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