CONVEXITY OF HESSIAN INTEGRALS AND POINCARÉ TYPE INEQUALITIES

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ABSTRACT. In this paper, integrals involving both real and complex Hessian operators over bounded domains are studied. Poincaré type inequalities are proved in both cases, which generalizes an early result of Trudinger and Wang.

1. INTRODUCTION

Throughout this paper, k and n will be two fixed integers such that $0 \leq k \leq n$. For $\lambda \in \mathbb{R}^n$, let $S_k(\lambda)$ be the normalized k^{th} elementary symmetric polynomial of λ , i.e.

$$S_k(\lambda_1, \dots, \lambda_n) = \frac{k!(n-k)!}{n!} \sum_{i_1 < i_2 < \dots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}.$$

Let $\mathcal{H}(n)$ denote the space of Hermitian matrices of size n . For $A \in \mathcal{H}(n)$, let $\lambda(A) \in \mathbb{R}^n$ denote the eigenvalue of A and

$$S_k(A) = S_k(\lambda(A)).$$

Even though we use S_k to denote functions defined on both $\mathbb{R}^n$ and $\mathcal{H}(n)$, its meaning should be clear from the context. In either case, we use $\tilde{S}_k$ to denote the complete polarization of S_k . Let

$$\Gamma_k = \{\lambda \in \mathbb{R}^n \mid S_j(\lambda) \geq 0, j = 1, \dots, k\}$$

be the k -positive cone in $\mathbb{R}^n$.

Let Ω be a smooth bounded domain in $\mathbb{C}^n$. For a real-valued function $u \in \mathcal{C}^2(\Omega)$, we define the complex k -Hessian of u by

$$H_k[u] = S_k(\text{Hess}_{\mathbb{C}}(u)),$$

where $\text{Hess}_{\mathbb{C}}(u)$ is the complex Hessian matrix of u . Let

$$\mathcal{P}_k(\Omega) = \{u \in \mathcal{C}^2(\bar{\Omega}) \mid \lambda(\text{Hess}_{\mathbb{C}}(u)) \in \Gamma_k\}$$

be the space of k -plurisubharmonic (k -psh) functions and let $\mathcal{P}_k^0(\Omega)$ be the subspace of $\mathcal{P}_k(\Omega)$ containing functions vanishing on $\partial\Omega$. It is well known [Gård59] that for $v_j \in \mathcal{P}_k(\Omega)$, $j = 1, \dots, k$,

$$\tilde{H}_k[v_1, \dots, v_k] = \tilde{S}_k(\text{Hess}_{\mathbb{C}}(v_1), \dots, \lambda(\text{Hess}_{\mathbb{C}}(v_k))) \geq 0.$$

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For $u, u_0, \dots, u_k \in \mathcal{C}^2(\bar{\Omega})$, denote

$$F_k[u_0, \dots, u_k] = \int_{\Omega} (-u_0) \tilde{H}_k[u_1, \dots, u_k]$$

and

$$I_k[u] = F_k[u, \dots, u] = \int_{\Omega} (-u) H_k[u].$$

Theorem 1. *With the notation as above:*

(a) *For any $u_0, \dots, u_k \in \mathcal{P}_k^0(\Omega)$,*

$$(1) \quad F_k[u_0, \dots, u_k] \leq \prod_{j=0}^k (I_k[u_j])^{\frac{1}{k+1}}.$$

(b) *For a fixed integer m such that $0 \leq m < k$ and fixed $v_1, \dots, v_m \in \mathcal{P}_k(\Omega)$, consider the functional*

$$\mathcal{P}_k^0(\Omega) \ni u \mapsto I_{k,v_1,\dots,v_m}[u] = \int_{\Omega} (-u) \tilde{H}_k[u, \dots, u, v_1, \dots, v_m].$$

Then $I_{k,v_1,\dots,v_m}^{1/(k-m+1)}$ is convex on $\mathcal{P}_0^k$ in the sense that for any $u, v \in \mathcal{P}_k^0(\Omega)$,

$$(2) \quad I_{k,v_1,\dots,v_m}^{\frac{1}{k-m+1}}[u+v] \leq I_{k,v_1,\dots,v_m}^{\frac{1}{k-m+1}}[u] + I_{k,v_1,\dots,v_m}^{\frac{1}{k-m+1}}[v].$$

Recall that a smooth bounded domain Ω is strongly k -pseudoconvex if $\partial\Omega$ is connected and the $(k-1)^{\text{th}}$ elementary symmetric polynomial of the eigenvalues for the Levi form of $\partial\Omega$ is strictly positive on $\partial\Omega$. For a strongly k -pseudoconvex domain, we have the following Poincaré type inequalities.

Theorem 2. *Suppose Ω is strongly k -pseudoconvex. Then for $0 \leq m < k$, there exists a constant C such that for any $u \in \mathcal{P}_k^0(\Omega)$,*

$$(3) \quad \left(I_m[u] \right)^{\frac{1}{m+1}} \leq C \left(I_k[u] \right)^{\frac{1}{k+1}},$$

where the constant C depends on m, k and Ω only.

Some special cases of Theorem 1 were well known. For example, Cegrell and Persson proved in [CP97] that if both u and v are psh functions on a domain with vanishing boundary data, then for an integer j such that $0 \leq j \leq n$,

$$\int_{\Omega} (-u) (dd^c u)^j (dd^c v)^{n-j} \leq \left(\int_{\Omega} (-u) (dd^c u)^n \right)^{\frac{j+1}{n+1}} \left(\int_{\Omega} (-v) (dd^c v)^n \right)^{\frac{n-j}{n+1}}.$$

This is just a special case of equation (1) with $u_0 = \dots = u_j = u$ and $u_{j+1} = \dots = u_n = v$.

Since the proof of Theorem 1 uses only the divergence structure of the Hessian operator H_k , it can also be applied to the study of real Hessian operators and integrals. Let D be a smooth bounded domain in $\mathbb{R}^n$. For a real-valued function $u_j \in \mathcal{C}^2(\bar{D})$, $j = 0, 1, \dots, k$ and $u \in \mathcal{C}^2(\bar{\Omega})$, we define

$$G_k[u_0, \dots, u_k] = \int_D (-u_0) \tilde{S}_k[\text{Hess}(u_1), \dots, \text{Hess}(u_k)]$$

and

$$J_k[u] = G_k[u, \dots, u],$$

where $\text{Hess}(u)$ is the Hessian matrix of u .

Theorem 3. For any k -convex function $u_0, \dots, u_k$ with $u_0 = \dots = u_k = 0$ on ∂D ,

$$(4) \quad G_k[u_0, \dots, u_k] \leq \prod_{j=0}^n \left(J_k[u_j] \right)^{\frac{1}{k+1}}.$$

Theorem 4. Suppose D is a smooth bounded domain in $\mathbb{R}^n$ such that $S_{k-1}(II) > 0$ on ∂D , where II is the second fundamental form of ∂D . Then for an integer m with $0 \leq m < k$, there exists a constant C depending on m, k and D only such that

$$(5) \quad \left(J_m[u] \right)^{\frac{1}{m+1}} \leq C \left(J_k[u] \right)^{\frac{1}{k+1}}$$

for any k -convex function u with $u = 0$ on ∂D .

Theorem 4 was originally proved by Trudinger and Wang in [TW98] using the parabolic method. Compared with their proof, our proof is quicker and more elementary and can be adopted to study the complex Hessian integral. On the other hand, Trudinger and Wang's proof contains more information. They showed that the best constant C in equation (5) is attained by the solution of the Dirichlet problem $H_k[u] = H_m[u]$ and $u = 0$ on ∂D .

Hessian integrals such as J_k were first studied by Bakelman [Bak83] and Tso [Tso90] in the setting of a real Monge-Ampère operator. In [Wan94], Wang studied the real Hessian integrals for general k and established some Sobolev type inequalities. For more information on the real Hessian integral, see [Wan09] and the references therein.

For simplicity, we always assume that the functions studied in this paper are at least twice differentiable. By an appropriate approximation, it is possible to extend the results to the case of real and complex Hessian measures for nonsmooth functions; see [TW02, Bł05] for more details.

2. COMPLEX HESSIAN INTEGRALS

Let ω be the Kähler form of the standard Euclidean metric of $\mathbb{C}^n$, i.e.

$$\omega = \sqrt{-1} \sum_{j=1}^n dz^j \wedge d\bar{z}^j.$$

It is easy to check that for $u, u_0, \dots, u_k \in \mathcal{P}_k(\Omega)$,

$$H_k[u]\omega^n = (\sqrt{-1}\partial\bar{\partial}u)^k \wedge \omega^{n-k}$$

and

$$\tilde{H}_k[u_1, \dots, u_k]\omega^n = (\sqrt{-1}\partial\bar{\partial}u_1) \wedge \dots \wedge (\sqrt{-1}\partial\bar{\partial}u_k) \wedge \omega^{n-k}.$$

Hence

$$I_k[u] = \int_{\Omega} (-u)(\sqrt{-1}\partial\bar{\partial}u)^k \wedge \omega^{n-k}$$

and

$$(6) \quad F_k[u_0, \dots, u_k] = \int_{\Omega} (-u_0)(\sqrt{-1}\partial\bar{\partial}u_1) \wedge \dots \wedge (\sqrt{-1}\partial\bar{\partial}u_k) \wedge \omega^{n-k}.$$

Obviously the F_k in equation (6) is symmetric in $u_1, \dots, u_k$. If $u_j \in \mathcal{P}_k^0(\Omega)$ for $j = 0, \dots, k$, then by integration by parts,

$$\begin{aligned} F_k[u_0, \dots, u_k] &= \int_{\Omega} \sqrt{-1} \partial u_0 \wedge \bar{\partial} u_1 \wedge \dots \wedge (\sqrt{-1} \partial \bar{\partial} u_k) \wedge \omega^{n-k} \\ &= \int_{\Omega} (-u_1) (\sqrt{-1} \partial \bar{\partial} u_0) \wedge \dots \wedge (\sqrt{-1} \partial \bar{\partial} u_k) \wedge \omega^{n-k} \\ &= F_k[u_1, u_0, \dots, u_k]. \end{aligned}$$

Therefore $F_k[u_0, \dots, u_k]$ is symmetric in its arguments if they are in $\mathcal{P}_k^0(\Omega)$.

Lemma 5. For any $u_0, u_1 \in \mathcal{P}_k^0(\Omega)$ and any $v_2, \dots, v_k \in \mathcal{P}_k(\Omega)$,

$$(7) \quad F_k[u_0, u_1, v_2, \dots, v_k]^2 \leq F_k[u_0, u_0, v_2, \dots, v_k] F_k[u_1, u_1, v_2, \dots, v_k].$$

Proof. For $t \in \mathbb{R}$, consider the quadratic function

$$\begin{aligned} q(t) &= F_k[u_0 + tu_1, u_0 + tu_1, v_2, \dots, v_k] \\ &= F[u_1, u_1, v_2, \dots, v_k] t^2 + 2F[u_0, u_1, v_2, \dots, v_k] t + F[u_0, u_0, v_2, \dots, v_k]. \end{aligned}$$

Since

$$\begin{aligned} q(t) &= \int_{\Omega} -(u_0 + tu_1) (\sqrt{-1} \partial \bar{\partial} u_0 + tu_1) \wedge (\sqrt{-1} \partial \bar{\partial} v_2) \wedge (\sqrt{-1} d\bar{\partial} v_k) \wedge \omega^{n-k} \\ &= \int_{\Omega} \sqrt{-1} \partial (u_0 + tu_1) \wedge \bar{\partial} (u_0 + tu_1) \wedge (\sqrt{-1} \partial \bar{\partial} v_2) \wedge (\sqrt{-1} d\bar{\partial} v_k) \wedge \omega^{n-k} \\ &\geq 0, \end{aligned}$$

so

$$F_k[u_0, u_1, v_2, \dots, v_k]^2 \leq F_k[u_0, u_0, v_2, \dots, v_k] F_k[u_1, u_1, v_2, \dots, v_k]. \quad \square$$

With Lemma 5, both statements in Theorem 1 follow from the following algebraic lemma.

Lemma 6. Let S be a set and f be a nonnegative symmetric function defined on $S^k = S \times \dots \times S$. Assume that for any $x_1, \dots, x_k \in S$,

$$f(x_1, x_2, x_3, \dots, x_k)^2 \leq f(x_1, x_1, x_3, \dots, x_k) f(x_2, x_2, x_3, \dots, x_k).$$

Then it follows that

$$f(x_1, x_2, \dots, x_k)^k \leq \prod_{j=1}^k f(x_j, \dots, x_j).$$

Moreover if S is a cone in some linear spaces and f is linear in each of its arguments, then

$$f(x + y, \dots, x + y)^{1/k} \leq f(x, \dots, x)^{1/k} + f(y, \dots, y)^{1/k}$$

for any x and y in S .

The proof of this lemma can be found in [Hör94].

To prove Theorem 2, we need another simple algebraic lemma for symmetric functions.

Lemma 7. *Let $\mu \in \Gamma_k \subset \mathbb{R}^n$ with $S_k(\mu) > 0$. Then there exists C depending only on $\text{dist}(\mu, \partial\Gamma_k)$ such that for a positive integer $m < k$ and any $\lambda \in \Gamma_k$,*

$$S_m(\lambda) \leq C \tilde{S}_k(\lambda, \dots, \lambda, \mu, \dots, \mu),$$

where λ and μ appear m and $k - m$ times in $\tilde{S}_k$ respectively.

Proof. Since $S_k(\mu) > 0$, μ is in the interior of Γ_k . Therefore there exists $\varepsilon > 0$ such that

$$\mu - \varepsilon \mathbf{e} \in \Gamma_k,$$

where $\mathbf{e} = (1, \dots, 1) \in \mathbb{R}^n$. Hence by Gårding's Inequality,

$$\begin{aligned} \tilde{S}_k^{\frac{1}{k-m}}(\lambda, \dots, \lambda, \mu, \dots, \mu) &\geq \tilde{S}_k^{\frac{1}{k-m}}(\lambda, \dots, \lambda, \mu - \varepsilon \mathbf{e}, \dots, \mu - \varepsilon \mathbf{e}) \\ (8) \quad &+ \tilde{S}_k^{\frac{1}{k-m}}(\lambda, \dots, \lambda, \varepsilon \mathbf{e}, \dots, \varepsilon \mathbf{e}) \\ &\geq \tilde{S}_k^{\frac{1}{k-m}}(\lambda, \dots, \lambda, \varepsilon \mathbf{e}, \dots, \varepsilon \mathbf{e}) = \varepsilon S_m^{\frac{1}{k-m}}(\lambda). \end{aligned}$$

Therefore

$$S_m(\lambda) \leq \frac{1}{\varepsilon^{k-m}} \tilde{S}_k(\lambda, \dots, \lambda, \mu, \dots, \mu). \quad \square$$

Proof of Theorem 2. Since Ω is strongly k -pseudoconvex, it is known [Vin88, Li04] that there exists a unique $v \in \mathcal{P}_k^0(\Omega)$ such that $H_k[v] = 1$. Hence there exists $\varepsilon > 0$ depending on v only such that $\text{Hess}_{\mathbb{C}}(v) - \varepsilon I_n \in \Gamma_k \subset \mathcal{H}(n)$ pointwise in Ω , where I_n is the identity matrix of size n . Similar to the proof of Lemma 7, there exists C such that

$$H_m[u] = S_m[\text{Hess}_{\mathbb{C}}(u)] \leq C \tilde{S}_k[\text{Hess}_{\mathbb{C}}(u), \dots, \text{Hess}_{\mathbb{C}}(u), \text{Hess}_{\mathbb{C}}(v), \dots, \text{Hess}_{\mathbb{C}}(v)],$$

where $\text{Hess}_{\mathbb{C}}(u)$ and $\text{Hess}_{\mathbb{C}}(v)$ appear m and $k - m$ times in $\tilde{S}_k$. Therefore

$$\begin{aligned} I_m[u] &= \int_{\Omega} (-u) H_m[u] \\ &\leq C \int_{\Omega} (-u) \tilde{S}_k[\text{Hess}_{\mathbb{C}}(u), \dots, \text{Hess}_{\mathbb{C}}(u), \text{Hess}_{\mathbb{C}}(v), \dots, \text{Hess}_{\mathbb{C}}(v)] \\ &= C F_k[u, \dots, u, v, \dots, v] \\ &\leq C (I_k[u])^{\frac{m+1}{k+1}} (I_k[v])^{\frac{1}{k+1}}. \end{aligned}$$

So

$$(I_m[u])^{\frac{1}{m+1}} \leq \tilde{C} (I_k[u])^{\frac{l+1}{k+1}}. \quad \square$$

3. REAL HESSIAN INTEGRALS

To prove Theorem 3 and Theorem 4, we only need to show that the functional G_k is symmetric in its arguments when they vanish on the boundary and the analogous Lemma 5 for G_k . They both follow from the divergence structure of polarized real Hessian operators.

Let $A^{(j)}, j = 1, 2, \dots, k$ be symmetric $n \times n$ matrices. Then the polarized symmetric function $\tilde{S}_k$ is given by

$$\tilde{S}_k(A^{(1)}, \dots, A^{(k)}) = \frac{1}{k!} \sum_{\substack{i_1, \dots, i_k \\ j_1, \dots, j_k}} \delta_{j_1 \dots j_k}^{i_1 \dots i_k} A_{i_1 j_1}^{(1)} \dots A_{i_k j_k}^{(k)},$$

where $\delta_{j_1 \dots j_k}^{i_1 \dots i_k}$ is the generalized Kronecker delta symbol. Denote

$$\tilde{S}_{k-1}^{ij}(A^{(1)}, \dots, A^{(k-1)}) = \frac{\partial}{\partial A_{ij}^{(k)}} \tilde{S}_k(A^{(1)}, \dots, A^{(k)}).$$

Then

$$\tilde{S}_k(A^{(1)}, \dots, A^{(k)}) = A_{ij}^{(k)} \tilde{S}_{k-1}^{ij}(A^{(1)}, \dots, A^{(k-1)})$$

and

$$\tilde{S}_{k-1}^{ij}(A^{(1)}, \dots, A^{(k-1)}) = \frac{1}{k!} \sum_{\substack{i_1, \dots, i_{k-1} \\ j_1, \dots, j_{k-1}}} \delta_{j_1 \dots j_{k-1} j}^{i_1 \dots i_{k-1} i} A_{i_1 j_1}^{(1)} \dots A_{i_{k-1} j_{k-1}}^{(k-1)}.$$

Lemma 8. *Let $u_1, \dots, u_{k-1} \in \mathcal{C}^2(D)$. Then for any index $i \in \{1, \dots, n\}$,*

$$\sum_{j=1}^n \frac{\partial}{\partial x_j} \tilde{S}_{k-1}^{ij}(\text{Hess}(u_1), \dots, \text{Hess}(u_{k-1})) = 0.$$

Proof. Denote $\text{Hess}(u_m) = A^{(m)}$, $m = 1, \dots, k-1$. Then

$$\begin{aligned} (9) \quad \sum_j \frac{\partial \tilde{S}_{k-1}^{ij}}{\partial x_j} &= \frac{1}{k!} \sum_j \frac{\partial}{\partial x_j} \left(\sum_{\substack{i_1, \dots, i_{k-1} \\ j_1, \dots, j_{k-1}}} \delta_{j_1 \dots j_{k-1} j}^{i_1 \dots i_{k-1} i} A_{i_1 j_1}^{(1)} \dots A_{i_{k-1} j_{k-1}}^{(k-1)} \right) \\ &= \frac{1}{k!} \sum_{\substack{i_1, \dots, i_{k-1} \\ j_1, \dots, j_{k-1}, j}} \delta_{j_1 \dots j_{k-1} j}^{i_1 \dots i_{k-1} i} \sum_{m=1}^{k-1} A_{i_1 j_1}^{(1)} \dots A_{i_{m-1} j_{m-1}}^{(m-1)} A_{i_m j_m, j}^{(m)} A_{i_{m+1} j_{m+1}}^{(m+1)} \dots A_{i_{k-1} j_{k-1}}^{(k-1)}. \end{aligned}$$

Notice that j_m and j are anti-symmetric in the Kronecker symbol, while symmetric in $A_{i_m j_m, j}^{(m)}$, so the sum in equation (9) must be zero. $\square$

Once Lemma 8 is proved, then for any k -convex functions $v_2, \dots, v_k$, and $\mathcal{C}^2$ functions u_0 and u_1 with $u_0 = u_1 = 0$ on ∂D ,

$$\begin{aligned} G_k[u_0, u_1, v_2, \dots, v_k] &= \int_D -u_0 \tilde{S}_k[(u_1)_{ij}, (v_2)_{ij}, \dots, (v_k)_{ij}] \\ &= \int_D -u_0 (u_1)_{ij} \tilde{S}_{k-1}^{ij} \\ &= \int_D (u_0)_i (u_1)_j \tilde{S}_{k-1}^{ij}. \end{aligned}$$

So G_k is symmetric in its arguments if they vanish on the boundary, and moreover

$$G_k[u, u, v_2, \dots, v_k] = \int_D u_i u_j \tilde{S}_{k-1}^{ij} \geq 0$$

for any k -convex $v_2, \dots, v_k$, and $\mathcal{C}^2$ function u with $u = 0$ on ∂D . Then the same argument as in the proof of Lemma 5 can be used to show the following lemma, which implies Theorem 3 and Theorem 4.

Lemma 9. *For any k -convex functions $v_2, \dots, v_k$, and any $\mathcal{C}^2$ functions u_0 and u_1 , if $u_0 = u_1 = 0$ on ∂D , then*

$$G_k[u_0, u_1, v_2, \dots, v_k]^2 \leq G_k[u_0, u_0, v_2, \dots, v_k] G_k[u_1, u_1, v_2, \dots, v_k].$$

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