THE IRREDUCIBILITY OF SYMMETRIC YAGZHEV MAPS

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ABSTRACT. Let $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a polynomial mapping in Yagzhev form, i.e.

$$F(x_1, \dots, x_n) = (x_1 + H_1(x_1, \dots, x_n), \dots, x_n + H_n(x_1, \dots, x_n)),$$

where H_i are homogeneous polynomials of degree 3. We show that if $\text{Jac}(F) \in \mathbb{C}^*$ and the Jacobian matrix of F is symmetric, then the polynomials $x_i + H_i(x_1, \dots, x_n)$ are irreducible as elements of the ring $\mathbb{C}[x_1, \dots, x_n]$.

1. INTRODUCTION

Let $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a polynomial mapping. The Jacobian Conjecture, which comes from Keller [4], states that if $\text{Jac}(F) \in \mathbb{C}^*$, then F is a polynomial automorphism. It is known that if the Jacobian Conjecture holds for a special class of mappings, then it holds in general. One of these special classes consists of mappings in the so-called Yagzhev form ([1], [3], [6]), i.e. mappings of the form

$$F(x_1, \dots, x_n) = (x_1 + H_1(x_1, \dots, x_n), \dots, x_n + H_n(x_1, \dots, x_n)),$$

where $H_1, \dots, H_n$ are homogeneous polynomials of degree 3. By convention, the zero polynomial is homogeneous of any degree. In [2] and [5] it was shown that it is enough to prove the Jacobian Conjecture for mappings in Yagzhev form with symmetric Jacobian matrix.

One of the questions related to this conjecture is the question of whether the coordinates of the Jacobian mapping (i.e. with constant Jacobian) are irreducible. Of course if the Jacobian Conjecture holds, then the answer to the above question is affirmative. In this paper we prove that if a Jacobian mapping $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ in Yagzhev form has symmetric Jacobian matrix, then all its coordinates are irreducible as elements of the ring $\mathbb{C}[x_1, \dots, x_n]$.

2. MAIN RESULT

In this section we prove our main result, Theorem 2.2.

We begin with the following lemma:

Lemma 2.1. *Let $F = (F_1, \dots, F_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a polynomial mapping in Yagzhev form for homogeneous polynomials $H_1, \dots, H_n$ of degree 3 such that*

$$F(x_1, \dots, x_n) = (x_1 + H_1(x_1, \dots, x_n), \dots, x_n + H_n(x_1, \dots, x_n)).$$

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Assume that $H_k \neq 0$. Then the polynomial F_k is reducible if and only if x_k divides H_k .

Proof. One implication is obvious. If F_k is reducible, then its homogenization is also reducible. Hence there exist homogeneous polynomials P_i of degree i , where $i = 1$ or 2 , such that

$$(*) \quad x_k x_{n+1}^2 + H_k(x_1, \dots, x_n) = P_1 P_2.$$

Let $P_1 = (\alpha x_{n+1} + L_1(x_1, \dots, x_n))$ and $P_2 = (\beta x_{n+1}^2 + L_2(x_1, \dots, x_n)x_{n+1} + Q(x_1, \dots, x_n))$, where L_1, L_2 are linear polynomials and Q is a homogeneous polynomial of degree 2. Comparing the coefficients in $(*)$ we get the following equalities:

- (a) $\alpha\beta = 0$,
- (b) $\alpha L_2 + \beta L_1 = x_k$,
- (c) $\alpha Q + L_1 L_2 = 0$,
- (d) $L_1 Q = H_k$.

If $\alpha = 0$, then from (b) we have that $L_1 = \frac{1}{\beta}x_k$, and using $(*)$ we obtain that $x_k | H_k$. On the other hand, if $\alpha \neq 0$, then $\beta = 0$ and from (b) it follows that $L_2 = \frac{1}{\alpha}x_k$. Substituting this into (c) we get

$$\alpha Q + \frac{1}{\alpha} L_1 x_k = 0.$$

The above equality shows that $x_k | Q$ and from (d) we have that $x_k | H_k$. This completes the proof. $\square$

Before going further, we need a closer characterization of mappings in Yagzhev form with symmetric Jacobian matrix. In [2] it is shown that the mapping $F = (F_1, \dots, F_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ has the above property if and only if there exists a homogeneous polynomial $P \in \mathbb{C}[x_1, \dots, x_n]$ of degree 4 such that $F_i = x_i + \frac{\partial P}{\partial x_i}$.

Now we are ready to prove the main theorem.

Theorem 2.2. *Let P be a homogeneous polynomial of degree 4 and suppose $F = (F_1, \dots, F_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is a polynomial mapping such that $F_i = x_i + \frac{\partial P}{\partial x_i}$. If $\text{Jac } F = \text{const} \neq 0$, then the polynomials $F_1, \dots, F_n$ are irreducible.*

Proof. First observe that if F is as assumed, then the condition $\text{Jac}(F) \in \mathbb{C}^*$ implies that $\text{Jac}(F) = 1$.

Without loss of generality we assume that F_n is reducible. Therefore $\frac{\partial P}{\partial x_n}$ is not zero and is divisible by x_n ; i.e. $\frac{\partial P}{\partial x_n} = x_n r(x_1, \dots, x_n)$ for some non-zero polynomial r . Hence there exist polynomials g and h such that

$$P = x_n^2 g(x_1, \dots, x_n) + h(x_1, \dots, x_{n-1}),$$

where g is a non-zero, homogeneous polynomial of degree 2 and h is a homogeneous polynomial of degree 4 in variables $x_1, \dots, x_{n-1}$.

Since P is as above and $F_n = x_n + \frac{\partial P}{\partial x_n}$, the following holds:

$$1 = \det JF|_{x_n=0} = \det \begin{pmatrix} * & 0 \\ 0 & 1 + 2g(x_1, \dots, x_{n-1}, 0) \end{pmatrix}.$$

By the Laplace formula this gives that $\det(*)$, as well as $1 + 2g(x_1, \dots, x_{n-1}, 0)$, is a non-zero constant. If $1 + 2g(x_1, \dots, x_{n-1}, 0) \neq 1$, then $g(x_1, \dots, x_{n-1}, 0)$ would be a non-zero constant, contradicting the fact that $g(0, \dots, 0) = 0$. Therefore $\det(*) = 1$

and $1 + 2g(x_1, \dots, x_{n-1}, 0) = 1$. However, this implies that $g(x_1, \dots, x_{n-1}, 0) = 0$, which means that x_n divides g . Thus

$$P = x_n^3(a_1x_1 + \dots + a_nx_n) + h(x_1, \dots, x_{n-1})$$

for some $a_1, \dots, a_n \in \mathbb{C}$. Since $JF = I + \left[\frac{\partial^2 P}{\partial x_i \partial x_j} \right]_{i,j=1,\dots,n}$, we have that the matrix $\left[\frac{\partial^2 P}{\partial x_i \partial x_j} \right]_{i,j=1,\dots,n}$ is nilpotent (see [3]). In particular its trace is equal to zero. Therefore

$$0 = A(x_1, \dots, x_{n-1}) + 6x_n(a_1x_1 + \dots + a_{n-1}x_{n-1}) + 12a_nx_n^2,$$

where $A(x_1, \dots, x_{n-1})$ is a homogeneous polynomial of degree 2 in $x_1, \dots, x_{n-1}$.

From the last equation one gets that $g(x_1, \dots, x_n) = 0$, a contradiction. This proves that F_n is irreducible and completes the proof. $\square$

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REFERENCES

1. H. Bass, E.H. Connell, D. Wright, The Jacobian conjecture: reduction of degree and formal expansion of the inverse, *Bull. Amer. Math. Soc.* 7 (1982), 287-330. MR663785 (83k:14028)
2. M. de Bondt, A. van den Essen, A reduction of the Jacobian Conjecture to the symmetric case, *Proc. Amer. Math. Soc.* 133 (8) (2005), 2201-2205. MR2138860 (2006a:14107)
3. L.M. Drużkowski, An effective approach to Keller's Jacobian conjecture, *Math. Ann.* 264 (1983), 303-313. MR714105 (85b:14015a)
4. O.-H. Keller, Ganze Cremona-Transformationen, *Monatsh. Math. Phys.* 47 (1939), 299-306. MR1550818
5. G. Meng, Legendre transform, Hessian conjecture and tree formula, *Appl. Math. Lett.* 19 (2006), 503-510. MR2221506 (2007b:14133)
6. A. Yagzhev, On Keller's problem, *Siberian Math. J.* 21 (1980), 747-754. MR592226 (82e:14020)

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