UPPER BOUNDS FOR SERIES INVOLVING MODERATE AND SMALL DEVIATIONS

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ABSTRACT. Let $X, X_1, X_2, \dots$ be i.i.d. random variables with $0 < EX^2 = \sigma^2 < \infty$ and $EX = 0$, and set $S_n = X_1 + \dots + X_n$. We prove Paley-type inequalities for series involving probabilities of moderate deviations $P(|S_n| \geq \lambda\sqrt{n \log n})$, $\lambda > 0$, and probabilities of small deviations $P(|S_n| \geq \lambda\sqrt{n \log \log n})$, $\lambda > \sigma\sqrt{2}$.

1. INTRODUCTION

Let $X, X_1, X_2, \dots$ be i.i.d. random variables with $0 < EX^2 = \sigma^2 < \infty$ and $EX = 0$, and set $S_n = X_1 + \dots + X_n$. Directed toward obtaining convergence rates for probabilities of moderate deviations $P(|S_n| \geq \lambda\sqrt{n \log n})$, $\lambda > 0$, and probabilities of small deviations $P(|S_n| \geq \lambda\sqrt{n \log \log n})$, $\lambda > \sigma\sqrt{2}$, Davis [3], [2] proved that $S_1(\lambda) = \sum_{n \geq 2} \frac{\log n}{n} P(|S_n| \geq \lambda\sqrt{n \log n}) < \infty$, $\lambda > 0$, and $S_2(\lambda) = \sum_{n \geq 2} \frac{1}{n} P(|S_n| \geq \lambda\sqrt{n \log \log n}) < \infty$, $\lambda > \sigma\sqrt{2}$. Previously Katz [9] showed that if $p \geq 2$ and $\alpha > 1/2$, then

$$(1) \quad \sum_{n \geq 1} n^{p\alpha-2} P(|S_n| \geq \lambda n^\alpha) < \infty, \quad \lambda > 0 \iff E|X|^p < \infty.$$

The case $p = 2$ and $\alpha = 1$, namely

$$(2) \quad \sum_{n \geq 1} P(|S_n| \geq \lambda n) < \infty, \quad \lambda > 0 \iff EX^2 < \infty,$$

had been settled by Hsu and Robbins [8] and Erdős [4], [5]. Chow and Lai [1] strengthened (1) in the form of a Paley-type inequality by showing that there exists an absolute constant $C > 0$, depending only on p and α , such that

$$(3) \quad \sum_{n \geq 1} n^{p\alpha-2} P(|S_n| \geq \lambda n^\alpha) \leq C(E|X|^p + (EX^2)^{(p\alpha-1)/(2\alpha-1)}), \quad \lambda > 0.$$

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As for (2), Pruss [13] obtained even sharper bounds; namely, he proved that there are absolute constants $C_1, C_2 > 0$ such that

$$(4) \quad C_1 \lambda^{-2} E[X^2 I\{|X| \geq \lambda\}] \leq \sum_{n \geq 1} P(|S_n| \geq \lambda n) \leq C_2 \lambda^{-2} E[X^2 I\{|X| \geq \lambda\}], \quad \lambda > 0.$$

Motivated by these results and the precise asymptotics $\lim_{\lambda \searrow 0} \lambda^4 S_1(\lambda) = 3\sigma^4/2$ and

$\lim_{\lambda \searrow \sigma\sqrt{2}} \sqrt{\lambda^2 - 2\sigma^2} S_2(\lambda) = \sigma\sqrt{2}$, due to Gut and Spătaru [6], [7], the purpose of this paper is to get upper bounds similar to (3) for the series $S_1(\lambda)$ and $S_2(\lambda)$.

Throughout C shall denote absolute positive constants (independent of the distribution of X), possibly varying from place to place, and Φ shall stand for the standard normal distribution function.

2. UPPER BOUND FOR THE SERIES $S_1(\lambda)$

Let $\psi(x) = \sqrt{x \log x}$, $x \geq 1$, and let $\varphi(x)$, $x \geq 0$, denote the inverse function of ψ . Notice that

$$(5) \quad x^2 = \varphi(x) \log \varphi(x), \quad x \geq 0.$$

For $n \geq 2$, put $X_{n,i} = X_i I\{|X_i| < \psi(n)\}$, $1 \leq i \leq n$, $U_n = X_{n,1} + \cdots + X_{n,n}$ and $\sigma_n^2 = EX_{n,1}^2$. Observe that $\sigma_n \nearrow \sigma > 0$. It will be clear from the proof of Theorem 1 that we may and do assume that $\sigma_n > 0$ for all $n \geq 2$. If this assumption fails for $2 \leq n \leq n_0$, then $U_n = 0$ a.s. for $2 \leq n \leq n_0$.

Theorem 1. *We have*

$$(6) \quad \sum_{n \geq 2} \frac{\log n}{n} P(|S_n| \geq \lambda \psi(n)) \leq C(\lambda^{-2} EX^2 + \lambda^{-4} (EX^2)^2), \quad \lambda > 0.$$

Proof.

Step 1. Suppose first that X is symmetric. Plainly,

$$\begin{aligned} P(|S_n| \geq \psi(n)) &\leq P(S_n \neq U_n) + P(|U_n| \geq \psi(n)) \\ &\leq nP(|X| \geq \psi(n)) + P(|U_n| \geq \psi(n)). \end{aligned}$$

Consequently, we get

$$(7) \quad \begin{aligned} \sum_{n \geq 2} \frac{\log n}{n} P(|S_n| \geq \psi(n)) &\leq \sum_{n \geq 2} (\log n) P(|X| \geq \psi(n)) \\ &\quad + \sum_{n \geq 2} \frac{\log n}{n} \left| P(|U_n| \geq \psi(n)) - 2\Phi\left(-\frac{\psi(n)}{\sigma_n \sqrt{n}}\right) \right| + 2 \sum_{n \geq 2} \frac{\log n}{n} \Phi\left(-\frac{\psi(n)}{\sigma_n \sqrt{n}}\right). \end{aligned}$$

We have

$$(8) \quad \begin{aligned} \sum_{n \geq 2} (\log n) P(|X| \geq \psi(n)) &= \sum_{n \geq 2} (\log n) \sum_{k \geq n} P(\psi(k) \leq |X| < \psi(k+1)) \\ &= \sum_{k \geq 2} \left(\sum_{n=2}^k \log n \right) P(\psi(k) \leq |X| < \psi(k+1)) \\ &\leq \sum_{k \geq 2} \psi(k)^2 P(\psi(k)^2 \leq X^2 < \psi(k+1)^2) \leq EX^2. \end{aligned}$$

By the non-uniform estimate of the remainder term in the central limit theorem, due to Nagaev [11] (see also Petrov [12, p. 125]),

$$\begin{aligned} \left| P(|U_n| \geq \psi(n)) - 2\Phi\left(-\frac{\psi(n)}{\sigma_n\sqrt{n}}\right) \right| &\leq C \frac{E|X_{n,1}|^3}{\sqrt{n}\sigma_n^3} \left(\frac{\sigma_n\sqrt{n}}{\psi(n)} \right)^3 \\ &= Cn^{-1/2}(\log n)^{-3/2}E|X_{n,1}|^3. \end{aligned}$$

Therefore, since $\sum_{n \geq m} n^{-\alpha}(\log n)^{-\beta} \leq Cm^{-\alpha+1}(\log m)^{-\beta}$, $m \geq 2$, whenever $\alpha \geq 1$ and $\beta > 0$, applying Fubini's theorem and using (5), we obtain

$$\begin{aligned} \sum_{n \geq 2} \frac{\log n}{n} \left| P(|U_n| \geq \psi(n)) - 2\Phi\left(-\frac{\psi(n)}{\sigma_n\sqrt{n}}\right) \right| &\leq C \sum_{n \geq 2} n^{-3/2}(\log n)^{-1/2}E|X_{n,1}|^3 \\ &= C \sum_{n \geq 2} n^{-3/2}(\log n)^{-1/2}E[|X|^3 I\{\varphi(|X|) < n\}] \\ (9) \quad &= CE \left[|X|^3 I\{|X| > 0\} \sum_{n > \varphi(|X|)} n^{-3/2}(\log n)^{-1/2} \right] \\ &\leq CE \left[|X|^3 I\{|X| > 0\} (\varphi(|X|) \log \varphi(|X|))^{-1/2} \right] = CEX^2. \end{aligned}$$

As $\sigma_n \leq \sigma$ and the function $x \mapsto \frac{\log x}{x}$ is decreasing for $x \geq e$, we see that

$$\sum_{n \geq 2} \frac{\log n}{n} \Phi\left(-\frac{\psi(n)}{\sigma_n\sqrt{n}}\right) \leq \sum_{n \geq 2} \frac{\log n}{n} \Phi(-\sqrt{\log n}/\sigma) \leq C \int_1^\infty \frac{\log x}{x} \Phi(-\sqrt{\log x}/\sigma) dx.$$

By putting $y = \sqrt{\log x}/\sigma$, we then get

$$(10) \quad \sum_{n \geq 2} \frac{\log n}{n} \Phi\left(-\frac{\psi(n)}{\sigma_n\sqrt{n}}\right) \leq C\sigma^4 \int_0^\infty 2y^3 \Phi(-y) dy = C(EX^2)^2.$$

Now (6) with $\lambda = 1$, and so with any $\lambda > 0$, follows from (7)-(10).

Step 2. Consider now the general case, and utilize the usual desymmetrization procedure. Let $X', X'_1, X'_2, \dots$ be an independent copy of $X, X_1, X_2, \dots$, and put $X^* = X - X', X_n^* = X_n - X'_n$ and $S_n^* = X_1^* + \dots + X_n^*$. Since $X^*, X_1^*, X_2^*, \dots$ are i.i.d. symmetric random variables, on account of Step 1, we have

$$(11) \quad \sum_{n \geq 2} \frac{\log n}{n} P(|S_n^*| \geq \lambda\psi(n)) \leq C(\lambda^{-2}EX^{*2} + \lambda^{-4}(EX^{*2})^2), \quad \lambda > 0.$$

If $\mu(S_n)$ denotes a median of S_n , then $|\mu(S_n)| = |\mu(S_n) - ES_n| \leq \sigma\sqrt{2n}$ (see Loève [10, p. 256]). Hence, by the weak symmetrization inequality (see Loève [10, p. 257]) and (11), we may write

$$\begin{aligned} \sum_{n \geq 2} \frac{\log n}{n} P(|S_n| \geq 3\sigma\psi(n)) &\leq \sum_{n \geq 2} \frac{\log n}{n} P(|S_n| \geq \sigma\psi(n) + \sigma\sqrt{2n}) \\ (12) \quad &\leq \sum_{n \geq 2} \frac{\log n}{n} P(|S_n - \mu(S_n)| \geq \sigma\psi(n)) \leq 2 \sum_{n \geq 2} \frac{\log n}{n} P(|S_n^*| \geq \sigma\psi(n)) \\ &\leq C(\sigma^{-2}EX^{*2} + \sigma^{-4}(EX^{*2})^2). \end{aligned}$$

Because $EX^{*2} = 2EX^2$, (12) implies that

$$(13) \quad \sum_{n \geq 2} \frac{\log n}{n} P(|S_n| \geq 3\lambda\sigma\psi(n)) \leq C(\lambda^{-2}\sigma^{-2}EX^2 + \lambda^{-4}\sigma^{-4}(EX^2)^2), \lambda > 0.$$

Replacing $3\lambda\sigma$ by λ in (13) leads to (6). $\square$

Remark 1. It follows from (6) that $\int_{\lambda_0}^{\infty} S_1(\lambda)d\lambda < \infty$, $\lambda_0 > 0$. Thus $EX^2 < \infty$ and $EX = 0$ ensure not only the convergence of the series $\sum_{n \geq 2} \frac{\log n}{n} P(|S_n| \geq \lambda\psi(n))$ for any $\lambda > 0$, but also the convergence of the double series $\sum_{k \geq 1} \sum_{n \geq 2} \frac{\log n}{n} P(|S_n| \geq k\psi(n))$.

Remark 2. On account of (6), we see that $S_1(\lambda) = O(\lambda^{-2})$ as $\lambda \rightarrow \infty$.

3. UPPER BOUND FOR THE SERIES $S_2(\lambda)$

In this section define $\psi(x) = \sqrt{x \log \log x}$, $x \geq e$, and let $\varphi(x)$, $x \geq 0$, denote the inverse function of ψ . Notice that

$$(14) \quad x^2 = \varphi(x) \log \varphi(x), \quad x \geq 0.$$

For $n \geq e^e$, set $X_{n,i} = X_i I\{|X_i| < \psi(n)\}$, $1 \leq i \leq n$, $U_n = X_{n,1} + \cdots + X_{n,n}$ and $\sigma_n^2 = EX_{n,1}^2$. As above we may and do assume that $\sigma_n > 0$ for all $n \geq e^e$.

Theorem 2. *We have*

$$(15) \quad \sum_{n \geq e^e} \frac{1}{n} P(|S_n| \geq (\lambda + 2\sigma)\psi(n)) \leq C\lambda^{-2}EX^2 + 8\lambda^{-2}EX^2 \left(-1 + \frac{\lambda}{\sqrt{\lambda^2 - 4EX^2}}\right), \lambda > 2\sigma.$$

Proof.

Step 1. Suppose first that X is symmetric. As above, we get

$$(16) \quad \begin{aligned} \sum_{n \geq e^e} \frac{1}{n} P(|S_n| \geq \psi(n)) &\leq \sum_{n \geq e^e} P(|X| \geq \psi(n)) \\ &+ \sum_{n \geq e^e} \frac{1}{n} \left| P(|U_n| \geq \psi(n)) - 2\Phi\left(-\frac{\psi(n)}{\sigma_n\sqrt{n}}\right) \right| + 2 \sum_{n \geq e^e} \frac{1}{n} \Phi\left(-\frac{\psi(n)}{\sigma_n\sqrt{n}}\right). \end{aligned}$$

Since $\psi(n) \geq \sqrt{n}$ for $n \geq e^e$, we have

$$(17) \quad \sum_{n \geq e^e} P(|X| \geq \psi(n)) \leq \sum_{n \geq e^e} P(|X| \geq \sqrt{n}) \leq EX^2.$$

Arguing as in Section 2, we obtain

$$\left| P(|U_n| \geq \psi(n)) - 2\Phi\left(-\frac{\psi(n)}{\sigma_n\sqrt{n}}\right) \right| \leq Cn^{-1/2}(\log \log n)^{-3/2} E|X_{n,1}|^3.$$

Therefore, as $\sum_{n \geq m} \psi(n)^{-3} \leq C m^{-1/2} (\log \log m)^{-3/2}$, $m \geq e^e$, by Fubini's theorem and (14), we obtain

$$\begin{aligned}
 (18) \quad & \sum_{n \geq e^e} \frac{1}{n} \left| P(|U_n| \geq \psi(n)) - 2\Phi\left(-\frac{\psi(n)}{\sigma_n \sqrt{n}}\right) \right| \leq C \sum_{n \geq e^e} \psi(n)^{-3} E[|X|^3 I\{\varphi(|X|) < n\}] \\
 & = CE \left[|X|^3 \sum_{n > e^e \vee \varphi(|X|)} \psi(n)^{-3} \right] \\
 & \leq CE[|X|^3 (e^e \vee \varphi(|X|))^{-1/2} (\log \log (e^e \vee \varphi(|X|)))^{-3/2}] \\
 & = CE[X^2 (\varphi(|X|) \log \log \varphi(|X|))^{1/2} (e^e \vee \varphi(|X|))^{-1/2} (\log \log (e^e \vee \varphi(|X|)))^{-3/2}] \\
 & \leq CEX^2.
 \end{aligned}$$

For the purpose of the next estimation we assume that $1 > \sigma\sqrt{2}$. By putting $y = \sqrt{\log \log x}/\sigma$ and then by partial integration, we have

$$\begin{aligned}
 (19) \quad & 2 \sum_{n \geq e^e} \frac{1}{n} \Phi\left(-\frac{\psi(n)}{\sigma_n \sqrt{n}}\right) \leq 2 \sum_{n \geq e^e} \frac{1}{n} \Phi(-\sqrt{\log \log n}/\sigma) \leq 2 \int_{[e^e]}^{\infty} \frac{1}{x} \Phi(-\sqrt{\log \log x}/\sigma) dx \\
 & \leq 4\sigma^2 \int_0^{\infty} y \Phi(-y) e^{\sigma^2 y^2} dy = 4\sigma^2 \left(-\frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-(1-2\sigma^2)y^2/2} dy\right) \\
 & = 2\sigma^2 \left(-1 + \frac{1}{\sqrt{1-2\sigma^2}}\right).
 \end{aligned}$$

From (16)-(19) it follows that

$$(20) \quad \sum_{n \geq e^e} \frac{1}{n} P(|S_n| \geq \lambda \psi(n)) \leq C\lambda^{-2} EX^2 + 2\lambda^{-2} EX^2 \left(-1 + \frac{\lambda}{\sqrt{\lambda^2 - 2EX^2}}\right), \quad \lambda > \sigma\sqrt{2}.$$

Step 2. Consider now the general case, and let $\gamma > 2$. By paralleling the reasoning in Step 1 of Section 2 and applying (20), we get

$$\begin{aligned}
 (21) \quad & \sum_{n \geq e^e} \frac{1}{n} P(|S_n| \geq (\gamma + 2)\sigma\psi(n)) \\
 & \leq C\gamma^{-2}\sigma^{-2} EX^{*2} + 4\gamma^{-2}\sigma^{-2} EX^{*2} \left(-1 + \frac{\gamma\sigma}{\sqrt{\gamma^2\sigma^2 - 2EX^{*2}}}\right),
 \end{aligned}$$

where X^* is the symmetrized X . As $EX^{*2} = 2EX^2$, setting $\gamma\sigma = \lambda$ in (21) leads to (15). $\square$

Remark 3. Remarks 1 and 2 of Section 2 have obvious counterparts regarding the series $S_2(\lambda)$.

Remark 4. It is reasonable to conjecture that the upper bound in (20) holds true in general, not only for symmetric X . The fact that (15) is weaker than (20) is due to the desymmetrization technique employed.

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