THE DEFECT INDEX OF SINGULAR SYMMETRIC LINEAR DIFFERENCE EQUATIONS WITH REAL COEFFICIENTS

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ABSTRACT. This paper is concerned with the defect index of singular symmetric linear difference equations of order $2n$ with real coefficients and one singular endpoint. We show that their defect index d satisfies the inequalities $n \leq d \leq 2n$ and that all values of d in this range are realized. This parallels the well known result of Glazman for differential equations established about 1950. In addition, several criteria of the limit point and strong limit point cases are established.

1. INTRODUCTION

In this paper we are interested in the defect index of the following singular symmetric linear difference equation with real coefficients:

$$(1.1) \quad \sum_{j=0}^n (-1)^j \Delta^j [p_j(t) \nabla^j y(t)] = \lambda w(t) y(t), \quad t \in I,$$

where Δ and ∇ are forward and backward difference operators, respectively; i.e., $\Delta y(t) = y(t+1) - y(t)$ and $\nabla y(t) = y(t) - y(t-1)$; $I := [0, +\infty) = \{t\}_{t=0}^{+\infty}$; $w(t) > 0$ and $p_n(t) \neq 0$ for $t \in I$; and λ is a complex spectral parameter.

By letting $u(t) = (u_1(t), u_2(t), \dots, u_{2n}(t))^T$ with

$$(1.2) \quad u_j(t) = \Delta^{j-1} y(t-j), \quad u_{n+j}(t) = \sum_{k=j}^n (-1)^{k-j} \Delta^{k-j} (p_k(t) \nabla^k y(t))$$

for $1 \leq j \leq n$, equation (1.1) can be converted into the following singular linear discrete Hamiltonian system

$$(1.3) \quad J \Delta u(t) = (\lambda W(t) + P(t)) R(u)(t), \quad t \in I,$$

where

$$(1.4) \quad W(t) = \text{diag}\{w(t), 0, \dots, 0\}, \quad P(t) = \begin{pmatrix} -C(t) & A^*(t) \\ A(t) & B(t) \end{pmatrix}$$

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with

$$A(t) = \begin{pmatrix} 0 & I_{n-1} \\ 0 & 0 \end{pmatrix}, \quad B(t) = \text{diag}\{0, \dots, 0, p_n^{-1}(t)\},$$

$$C(t) = \text{diag}\{p_0(t), p_1(t), \dots, p_{n-1}(t)\},$$

I_{n-1} is the $(n-1) \times (n-1)$ unit matrix, and the right partial shift operator

$$(1.5) \quad R(u)(t) = (u_1(t+1), \dots, u_n(t+1), u_{n+1}(t), \dots, u_{2n}(t))^T.$$

According to the classical von Neumann theory (cf. [11, 15]) and its generalization [5], a symmetric operator or a non-densely defined Hermitian operator has a self-adjoint extension if and only if its positive and negative defect indices are equal and its self-adjoint extension domains have a close relationship with its defect index. So it is very important to determine the defect indices of both differential equations and difference equations in the study of self-adjoint extensions for differential and difference equations.

The problem on the defect index d of the singular symmetric linear differential equation with real coefficients

$$(1.6) \quad \sum_{j=0}^n (-1)^j [p_j(x)y^{(j)}(x)]^{(j)} = \lambda w(x)y(x), \quad x \in (a, b),$$

has been studied for a long time. It is well known that d is equal to the number of linearly independent square integrable solutions of (1.6) for each $\lambda \in \mathbb{C} \setminus \mathbb{R}$, where $\mathbb{C}$ and $\mathbb{R}$ denote the sets of the complex and real numbers, respectively. In particular, I. M. Glazman [8] showed that the defect index d of equation (1.6) defined on $(0, +\infty)$, where $x = 0$ is a regular endpoint, satisfied the inequalities $n \leq d \leq 2n$ and all values of d in this range. Many other results on the defect index d of equation (1.6) were summarized in [7] and [11, Chapter V].

For the discrete case, F. V. Atkinson first studied the number of linearly independent square summable solutions of second-order symmetric linear difference equations [1]. Subsequently, his work was further developed (cf. [2, 3, 9, 10], [12]–[14]). Recently, some Weyl-Titchmarsh fundamental theory of singular linear discrete Hamiltonian systems was established [4, 12]. In the present paper, we study the defect index of equation (1.1) by applying the Weyl-Titchmarsh fundamental theory of singular linear discrete Hamiltonian systems and asymptotic behaviors of solutions of perturbed difference equations.

The rest of the paper is organized as follows. In Section 2, two equivalent forms of equation (1.1) are formulated, and a result on asymptotic behaviors of solutions of difference equations and some useful lemmas are stated. In Section 3, we pay attention to the defect index of equation (1.1) and its equivalent forms. Theorem 3.1 and Example 3.2 show that the defect index d of equation (1.1) satisfies the inequalities $n \leq d \leq 2n$ and all values of d in this range are realized. This parallels the above Glazman's result for differential equations. In addition, several criteria of the limit point and strong limit point cases are established.

2. PRELIMINARIES

This section is divided into four subsections. In Subsections 2.1 and 2.2, a classification of limit cases of equation (1.1) and two equivalent forms of equation (1.1) are introduced, respectively. In Subsection 2.3, some sufficient and necessary

conditions of the limit point case and the strong limit point case of equation (1.1) are given. Finally, in Subsection 2.4, asymptotic behaviors of solutions of an equivalent form to equation (1.1) are studied.

2.1. Classification of limit cases. By $\mathbb{C}$ denote the set of the complex numbers, and by $\bar{z}$ and u^* denote the conjugate of z and the complex conjugate transpose of u , respectively.

We now introduce the space

$$l_w^2[0, +\infty) := \left\{ y = \{y(t)\}_{t=-n}^{+\infty} \subset \mathbb{C} : \sum_{t=0}^{+\infty} w(t)|y(t)|^2 < +\infty \right\}$$

with inner product

$$\langle x, y \rangle_w := \sum_{t=0}^{+\infty} w(t)\bar{y}(t)x(t),$$

where the weight function $w(t) > 0$ on I . For $x, y \in l_w^2[0, +\infty)$, x is said to be equal to y if $\|x - y\|_w = 0$, where $\|\cdot\|_w := (\langle \cdot, \cdot \rangle_w)^{1/2}$. In this sense, $l_w^2[0, +\infty)$ is a Hilbert space with the inner product $\langle \cdot, \cdot \rangle_w$. In the special case of $w(t) \equiv 1$, $l_w^2[0, +\infty)$ is briefly denoted by $l^2[0, +\infty)$.

Similarly to the scalar case, denote

$$L_W^2[0, +\infty) := \left\{ u = \{u(t)\}_{t=0}^{+\infty} \subset \mathbb{C}^{2n} : \sum_{t=0}^{+\infty} R^*(u)(t)W(t)R(u)(t) < +\infty \right\}$$

with the inner product

$$\langle u, v \rangle_W := \sum_{t=0}^{+\infty} R^*(v)(t)W(t)R(u)(t),$$

where the weight function $W(t)$ is a $2n \times 2n$ non-negative Hermitian matrix and $R(\cdot)$ is defined by (1.5). Then $L_W^2[0, +\infty)$ is a Hilbert space with the inner product $\langle \cdot, \cdot \rangle_W$ in the sense that $u = v$ if $\|u - v\|_W = 0$, where $\|\cdot\|_W := (\langle \cdot, \cdot \rangle_W)^{1/2}$.

As stated in the previous section, (1.1) can be converted into (1.3) by (1.2). It is evident that for any $\lambda \in \mathbb{C}$, if $y(t)$ is a solution of (1.1), then $u(t)$ defined by (1.2) is a solution of (1.3) and conversely, if $u(t)$ is a solution of (1.3), then $y(t) = u_1(t+1)$ is a solution of (1.1). Moreover, it follows from (1.2) and (1.4) that

$$\|u\|_W^2 = \sum_{t=0}^{+\infty} R^*(u)(t)W(t)R(u)(t) = \sum_{t=0}^{+\infty} w(t)|y(t)|^2 = \|y\|_w^2.$$

By $d_w(\lambda)$ and $D_W(\lambda)$ denote the number of the linearly independent solutions of equation (1.1) in $l_w^2[0, +\infty)$ and system (1.3) in $L_W^2[0, +\infty)$, respectively. Then the following result is obtained:

Lemma 2.1. $D_W(\lambda) = d_w(\lambda)$ for any $\lambda \in \mathbb{C}$.

Denote $d_+ = d_w(i)$ and $d_- = d_w(-i)$. By [12, Theorem 5.1], d_+ and d_- are equal to the positive and negative defect indices of the minimal operator generated by (1.1), respectively. The following result is directly derived from Corollary 4.1 and Theorem 5.4 in [12] and Lemma 2.1:

Lemma 2.2. $d_w(\lambda) = d_+ = d_-$ is a constant for all $\lambda \in \mathbb{C}$ with $\text{Im}\lambda \neq 0$, and $n \leq d \leq 2n$.

Definition 2.3. The constant $d := d_{\pm}$ is called the defect index of equation (1.1), and equation (1.1) is said to be in the limit- d case at $t = +\infty$. In the special case $d = n$, (1.1) is said to be in the limit point case (l.p.c.) at $t = +\infty$, and in the other special case $d = 2n$, (1.1) is said to be in the limit circle case (l.c.c.) at $t = +\infty$.

2.2. Two equivalent forms of equation (1.1). In this subsection we formulate two equivalent forms of (1.1), one of which has a weight function being equivalent to 1 and the other has a leading-term coefficient being equivalent to $(-1)^n$. For convenience, we first introduce the following useful formulae:

$$(2.1) \quad \Delta^k y(t) = \nabla^k y(t+k) = \sum_{j=0}^k (-1)^{k-j} C_k^j y(t+j),$$

$$(2.2) \quad \Delta^k(a(t)b(t)) = \sum_{j=0}^k C_k^j \Delta^j a(t) \Delta^{k-j} b(t+j),$$

where $C_k^j = k!/(j!(k-j)!)$ is the binomial coefficient. By using (2.1), we have from (1.1) that

$$(2.3) \quad \sum_{j=0}^n P_j(t+j)y(t+j) + \sum_{j=1}^n P_j(t)y(t-j) = \lambda w(t)y(t), \quad t \in I,$$

where

$$(2.4) \quad P_j(t) = (-1)^j \sum_{s=j}^n \sum_{k=0}^{s-j} C_s^k C_s^{s-j-k} p_s(t+k), \quad 0 \leq j \leq n,$$

or

$$(2.5) \quad p_n(t) = (-1)^n P_n(t),$$

$$p_j(t) = (-1)^j P_j(t) - \sum_{s=j+1}^n \sum_{k=0}^{s-j} C_s^k C_s^{s-j-k} p_s(t+k), \quad 0 \leq j \leq n-1.$$

First, setting

$$(2.6) \quad y(t) = w^{-1/2}(t)x(t)$$

and multiplying $w^{-1/2}(t)$ on both sides of (2.3), we get

$$(2.7) \quad \sum_{j=0}^n Q_j(t+j)x(t+j) + \sum_{j=1}^n Q_j(t)x(t-j) = \lambda x(t), \quad t \in I,$$

where

$$(2.8) \quad Q_j(t) = P_j(t)w^{-1/2}(t)w^{-1/2}(t-j), \quad 0 \leq j \leq n.$$

Further, (2.7) can be rewritten as

$$(2.9) \quad \sum_{j=0}^n (-1)^j \Delta^j [q_j(t) \nabla^j x(t)] = \lambda x(t), \quad t \in I,$$

where $q_j(t)$ are determined by (2.5) with $P_j(t)$ and $p_s(t)$ replaced by $Q_j(t)$ and $q_s(t)$, respectively. It is evident that $q_j(t)$, $0 \leq j \leq n$, are all real-valued and $q_n(t) \neq 0$ for all $t \in I$.

Second, setting $y(t) = g(t)z(t)$ and multiplying $g(t)$ on both sides of (2.3), where $g(t)$ is defined by

$$g(-1) = g(-2) = \cdots = g(-n) = 1, \quad g(j) = \frac{(-1)^n}{p_n(j)}, \quad 0 \leq j \leq n-1,$$

and

$$g(t) = \begin{cases} \frac{p_n(t-n)p_n(t-3n) \cdots p_n(j-n)}{p_n(t)p_n(t-2n) \cdots p_n(j)}, & t = 2nk + j, k \geq 0, n \leq j \leq 2n-1, \\ \frac{(-1)^n p_n(t-n)p_n(t-3n) \cdots p_n(j+n)}{p_n(t)p_n(t-2n) \cdots p_n(j)}, & t = 2nk + j, k \geq 1, 0 \leq j \leq n-1, \end{cases}$$

we get

$$(2.10) \quad \sum_{j=0}^n R_j(t+j)z(t+j) + \sum_{j=1}^n R_j(t)z(t-j) = \lambda w_0(t)z(t), \quad t \in I,$$

where

$$R_n(t) = (-1)^n, \quad R_j(t) = P_j(t)g(t)g(t-j), \quad 0 \leq j \leq n-1, \\ w_0(t) = w(t)g^2(t).$$

Further, (2.10) can be rewritten as

$$(2.11) \quad (-1)^n \Delta^n \nabla^n z(t) + \sum_{j=0}^{n-1} (-1)^j \Delta^j [r_j(t) \nabla^j z(t)] = \lambda w_0(t)z(t), \quad t \in I,$$

where $r_j(t)$ are determined by (2.5) with $P_j(t)$ and $p_s(t)$ replaced by $R_j(t)$ and $r_s(t)$, respectively. In this case, $r_j(t)$, $0 \leq j \leq n-1$, are all real-valued and $w_0(t) > 0$ for $t \in I$.

It is evident that for any $\lambda \in \mathbb{C}$, $y(t)$ is a solution of equation (1.1) if and only if $x(t) = w^{1/2}(t)y(t)$ is a solution of equation (2.9) and if and only if $z(t) = g^{-1}(t)y(t)$ is a solution of equation (2.11). Moreover, it follows that

$$(2.12) \quad \sum_{t=0}^{+\infty} w(t)|y(t)|^2 = \sum_{t=0}^{+\infty} |x(t)|^2 = \sum_{t=0}^{+\infty} w_0(t)|z(t)|^2.$$

By $d(\lambda)$ and $d_{w_0}(\lambda)$ denote the numbers of the linearly independent solutions of equation (2.9) in $l^2[0, +\infty)$ and of equation (2.11) in $l^2_{w_0}[0, +\infty)$, respectively. Then we have by (2.12) and Lemma 2.2 that

$$d_{w_0}(\lambda) = d(\lambda) = d_w(\lambda) \equiv d, \quad \operatorname{Im} \lambda \neq 0.$$

This means that equation (1.1) has the same defect index as those of its equivalent forms (2.9) and (2.11).

2.3. Sufficient and necessary conditions of the limit point case and the strong limit point case. The natural difference operator corresponding to equation (1.1) is defined by

$$\mathcal{L}(y)(t) := \sum_{j=0}^n (-1)^j \Delta^j [p_j(t) \nabla^j y(t)], \quad t \in I,$$

and the bilinear form $[\cdot, \cdot]$ associated with (1.1) is defined by

$$[x, y](t) := \sum_{j=1}^n \left(\sum_{k=j}^n (-1)^{k-j} \Delta^{k-j} [p_k(t) \nabla^k \bar{y}(t)] \right) \Delta^{j-1} x(t-j) \\ - \sum_{j=1}^n \Delta^{j-1} \bar{y}(t-j) \left(\sum_{k=j}^n (-1)^{k-j} \Delta^{k-j} [p_k(t) \nabla^k x(t)] \right).$$

Then, for any $x(t)$ and $y(t)$ defined in $\{t\}_{t=-n}^{+\infty}$ we have

$$(2.13) \quad \sum_{t=0}^m [\bar{y}(t)(\mathcal{L}x)(t) - (\overline{\mathcal{L}y})(t)x(t)] = [x, y](t)|_{t=0}^{m+1}.$$

Denote $\mathcal{D} := \{y \in l_w^2[0, +\infty) : w^{-1}\mathcal{L}y \in l_w^2[0, +\infty)\}$, which is the domain of the maximal operator corresponding to operator $\mathcal{L}$. This yields from (2.13) that $\lim_{t \rightarrow +\infty} [x, y](t) = c$ for any $x, y \in \mathcal{D}$, where c is a finite constant. The following lemma is a direct consequence of [12, Theorem 6.15].

Lemma 2.4. Equation (1.1) is in l.p.c. at $t = +\infty$ if and only if for all $x, y \in \mathcal{D}$,

$$(2.14) \quad \lim_{t \rightarrow +\infty} [x, y](t) = 0.$$

Lemma 2.4 gives a sufficient and necessary condition of the limit point case for equation (1.1). In [14, Definition 2.1], a concept of the strong limit point case is introduced for a singular discrete Hamiltonian system. Following their definition, we give the following concept:

Definition 2.5. Equation (1.1) is said to be in the strong limit point case (s.l.p.c.) at $t = +\infty$ if for all $x, y \in \mathcal{D}$,

$$\lim_{t \rightarrow +\infty} \sum_{j=1}^n \Delta^{j-1} \bar{x}(t-j) \left(\sum_{k=j}^n (-1)^{k-j} \Delta^{k-j} [p_k(t) \nabla^k y(t)] \right) = 0.$$

For convenience, denote

$$S(y)(t) := \sum_{j=1}^n \Delta^{j-1} \bar{y}(t-j) \left(\sum_{k=j}^n (-1)^{k-j} \Delta^{k-j} [p_k(t) \nabla^k y(t)] \right).$$

The following result is a sufficient and necessary condition of the strong limit point case for equation (1.1), which is a direct consequence of [14, Theorem 2.1].

Lemma 2.6. Equation (1.1) is in s.l.p.c. at $t = +\infty$ if and only if $\lim_{t \rightarrow +\infty} S(y)(t) = 0$ for all $y \in \mathcal{D}$.

Lemma 2.7. Assume that there exist a constant c and an integer $t_0 \geq 0$ such that $p_j(t)$ ($1 \leq j \leq n$) and $p_0(t) - cw(t)$ are either non-negative for all $t \geq t_0$ or non-positive for all $t \geq t_0$. Then $\lim_{t \rightarrow +\infty} S(y)(t)$ exists, finite or infinite, for all $y \in \mathcal{D}$.

Proof. Using the first relationship of (2.1) and (2.2), we have

$$\begin{aligned}
 \Delta S(y)(t) &= \sum_{j=1}^n \Delta^j \bar{y}(t-j) \left(\sum_{k=j}^n (-1)^{k-j} \Delta^{k-j} [p_k(t) \nabla^k y(t)] \right) \\
 &\quad + \sum_{j=1}^n \Delta^{j-1} \bar{y}(t-j+1) \left(\sum_{k=j}^n (-1)^{k-j} \Delta^{k-j+1} [p_k(t) \nabla^k y(t)] \right) \\
 &= \sum_{j=1}^n \Delta^j \bar{y}(t-j) \left(\sum_{k=j}^n (-1)^{k-j} \Delta^{k-j} [p_k(t) \nabla^k y(t)] \right) \\
 &\quad - \sum_{j=0}^{n-1} \Delta^j \bar{y}(t-j) \left(\sum_{k=j+1}^n (-1)^{k-j} \Delta^{k-j} [p_k(t) \nabla^k y(t)] \right) \\
 &= \sum_{j=1}^n p_j(t) |\Delta^j y(t-j)|^2 + \bar{y}(t) \left(\sum_{k=1}^n (-1)^{k+1} \Delta^k [p_k(t) \nabla^k y(t)] \right) \\
 &= \sum_{j=1}^n p_j(t) |\Delta^j y(t-j)|^2 + \bar{y}(t) (p_0(t) y(t) - (\mathcal{L}y)(t)) \\
 &= \sum_{j=0}^n p_j(t) |\Delta^j y(t-j)|^2 - \bar{y}(t) (\mathcal{L}y)(t).
 \end{aligned}$$

Summing up the above relation from t_0 to m , we get

$$\begin{aligned}
 S(y)(m+1) &= S(y)(t_0) + c \sum_{t=t_0}^m w(t) |y(t)|^2 - \sum_{t=t_0}^m \bar{y}(t) (\mathcal{L}y)(t) \\
 (2.15) \quad &\quad + \sum_{t=t_0}^m (p_0(t) - cw(t)) |y(t)|^2 + \sum_{j=1}^n \sum_{t=t_0}^m p_j(t) |\Delta^j y(t-j)|^2.
 \end{aligned}$$

For any $y \in \mathcal{D}$, it follows from Cauchy's inequality that the second and third items on the right side of (2.15) converge. Since $p_j(t)$ ($1 \leq j \leq n$) and $p_0(t) - cw(t)$ are either non-negative for all $t \geq t_0$ or non-positive for all $t \geq t_0$, the last two items on the right side of (2.15) are monotonic. Therefore, $\lim_{t \rightarrow +\infty} S(y)(t)$ exists, finite or infinite. The proof is complete. $\square$

2.4. Asymptotic behaviors of solutions. At the end of this section, we consider asymptotic behaviors of solutions of equation (2.11).

Lemma 2.8. *Assume that*

$$(2.16) \quad \sum_{t=0}^{+\infty} t^{2n-1} w_0(t) < +\infty, \quad \sum_{t=0}^{+\infty} t^{2n-1} |r_j(t)| < +\infty, \quad 0 \leq j \leq n-1.$$

Then, for any $\lambda \in \mathbb{C}$, equation (2.11) has a fundamental set of solutions $z_j(t)$, $1 \leq j \leq 2n$, with the asymptotic form as $t \rightarrow +\infty$,

$$(2.17) \quad z_j(t) = (1 + o(1)) t^{j-1}.$$

Proof. By a result of Coffman [6, p. 43], if the first condition in (2.16) holds and

$$(2.18) \quad \sum_{t=0}^{+\infty} t^{2n-1} |R_j(t)| < +\infty, \quad 0 \leq j \leq n-1,$$

then for any $\lambda \in \mathbb{C}$, equation (2.10) has a fundamental set of solutions $z_j(t)$, $1 \leq j \leq 2n$, with the asymptotic form (2.17). By the relationship between $R_j(t)$ and $r_j(t)$, the second condition in (2.16) yields (2.18). Consequently, equation (2.11) has a fundamental set of solutions $z_j(t)$, $1 \leq j \leq 2n$, with the asymptotic form (2.17). This completes the proof. $\square$

3. MAIN RESULTS

In this section, several results on the defect index of equation (1.1) and its equivalent forms are given. It is first shown that the defect index d of equation (1.1) satisfies the inequality $n \leq d \leq 2n$ and all values of d in this range are realized. Next, several sufficient conditions of the limit point case for equations (2.9) and (1.1) are given. Finally, two criteria of the strong limit point case for equations (2.9) and (1.1) are established.

First, we give the following result based on asymptotic behaviors of solutions of equation (2.11):

Theorem 3.1. *Consider equation (2.11). Assume that (2.16) holds. Then*

- (1) $d = n$ if $\sum_{t=0}^{+\infty} t^{2n} w_0(t) = +\infty$;
- (2) $d = n + s$ if $\sum_{t=0}^{+\infty} t^{2n+2s-2} w_0(t) < +\infty$ and $\sum_{t=0}^{+\infty} t^{2n+2s} w_0(t) = +\infty$ for some integer $1 \leq s \leq n-1$;
- (3) $d = 2n$ if $\sum_{t=0}^{+\infty} t^{4n-2} w_0(t) < +\infty$.

Proof. We prove only that (1) holds, and the others can be shown similarly. By Lemma 2.8, if (2.16) holds, then for any $\lambda \in \mathbb{C}$, equation (2.11) has a fundamental solution $z_j(t) = (1+o(1))t^{j-1}$, $1 \leq j \leq 2n$. By the assumption in (1), it can be easily verified that $z_j(t)$, $1 \leq j \leq n$, belong to $l_{w_0}^2[0, +\infty)$, while $z_j(t)$, $n+1 \leq j \leq 2n$, do not belong to $l_{w_0}^2[0, +\infty)$. Moreover, any non-trivial linear combination of $z_j(t)$, $n+1 \leq j \leq 2n$, is not in $l_{w_0}^2[0, +\infty)$. Hence, $d = n$. The proof is complete. $\square$

Example 3.2. Consider equation (2.11) in the special case: $w_0(t) = t^\alpha$, and $r_j(t) = t^{\beta_j}$, where α, β_j , $0 \leq j \leq n-1$, are real numbers. It follows from Theorem 3.1 and Lemma 2.8 that if $\beta_j < -2n$, $1 \leq j \leq n-1$, then

$$d = \begin{cases} n, & \text{if } -2n-1 \leq \alpha < -2n; \\ n+s, & \text{if } -2n-2s-1 \leq \alpha < -2n-2s+1, 1 \leq s \leq n-1; \\ 2n, & \text{if } \alpha < -4n+1. \end{cases}$$

This example shows that the defect index d of equation (1.1) satisfies the inequality $n \leq d \leq 2n$ and all values of d in this range are realized. This parallels the above Glazman's result for differential equations.

Second, we give some sufficient conditions for the limit point case, that is, $d = n$. Similarly to the notation in Section 2, we introduce the following notation for

equation (2.9):

$$\begin{aligned}\mathcal{L}_1(y)(t) &:= \sum_{j=0}^n (-1)^j \Delta^j [q_j(t) \nabla^j y(t)], \\ [x, y]_1(t) &:= \sum_{j=1}^n \left(\sum_{k=j}^n (-1)^{k-j} \Delta^{k-j} [q_k(t) \nabla^k \bar{y}(t)] \right) \Delta^{j-1} x(t-j) \\ &\quad - \sum_{j=1}^n \Delta^{j-1} \bar{y}(t-j) \left(\sum_{k=j}^n (-1)^{k-j} \Delta^{k-j} [q_k(t) \nabla^k x(t)] \right), \\ S_1(y)(t) &:= \sum_{j=1}^n \Delta^{j-1} \bar{y}(t-j) \left(\sum_{k=j}^n (-1)^{k-j} \Delta^{k-j} [q_k(t) \nabla^k y(t)] \right), \\ \mathcal{D}_1 &:= \{y \in l^2[0, +\infty) : \mathcal{L}_1 y \in l^2[0, +\infty)\}.\end{aligned}$$

Theorem 3.3. Assume that there exist a non-negative function $\sigma(t)$ defined on I , a constant $M > 0$ and an integer $t_0 \geq 0$ such that

$$(3.1) \quad \sum_{t=0}^{+\infty} \sigma(t) = +\infty, \quad \sigma(t) |\Delta^m q_s(t)| \leq M, \quad t \geq t_0,$$

for all $1 \leq s \leq n$ and $0 \leq m \leq s-1$. Then equation (2.9) is in l.p.c. at $t = +\infty$.

Proof. By Lemma 2.4, it suffices to show that $\lim_{t \rightarrow +\infty} [x, y]_1(t) = 0$ for all $x, y \in \mathcal{D}_1$. By the discussion before Lemma 2.4, $\lim_{t \rightarrow +\infty} [x, y]_1(t)$ exists and is finite for all $x, y \in \mathcal{D}_1$. Suppose that there exist $x, y \in \mathcal{D}_1$ such that $\lim_{t \rightarrow +\infty} [x, y]_1(t) = c \neq 0$. Then there exists $T_0 \geq t_0$ such that $|[x, y]_1(t)| \geq |c|/2 > 0$ for all $t \geq T_0$. We have from (2.1) and (2.2) that

$$\begin{aligned}\sigma(t) |[x, y]_1(t)| &\leq \sum_{j=1}^n \left(|\Delta^{j-1} x(t-j)| \sum_{k=j}^n \sum_{s=0}^{k-j} C_{k-j}^s \sigma(t) |\Delta^s q_k(t)| |\Delta^{2k-j-s} \bar{y}(t+s-k)| \right) \\ &\quad + \sum_{j=1}^n \left(|\Delta^{j-1} \bar{y}(t-j)| \sum_{k=j}^n \sum_{s=0}^{k-j} C_{k-j}^s \sigma(t) |\Delta^s q_k(t)| |\Delta^{2k-j-s} x(t+s-k)| \right),\end{aligned}$$

which, together with the second condition in (3.1) and Cauchy's inequality, implies that

$$\begin{aligned}\sigma(t) |[x, y]_1(t)| &\leq M \sum_{j=1}^n \sum_{k=j}^n \sum_{s=0}^{k-j} C_{k-j}^s (|\Delta^{j-1} x(t-j)|^2 + |\Delta^{j-1} \bar{y}(t-j)|^2 \\ &\quad + |\Delta^{2k-j-s} \bar{y}(t+s-k)|^2 + |\Delta^{2k-j-s} x(t+s-k)|^2).\end{aligned}$$

Since $x, y \in \mathcal{D}_1$, $|\Delta^k x|$ and $|\Delta^k \bar{y}|$ belong to $l^2[0, +\infty)$ for any $k \geq 1$. Hence,

$$\sum_{t=T_0}^{+\infty} \sigma(t) |[x, y]_1(t)| < +\infty.$$

On the other hand, it follows from the first condition in (3.1) that

$$\sum_{t=T_0}^{+\infty} \sigma(t)|[y, z]_1(t)| \geq \frac{|c|}{2} \sum_{t=T_0}^{+\infty} \sigma(t) = +\infty.$$

This is a contradiction. Therefore, $\lim_{t \rightarrow +\infty} [x, y]_1(t) = 0$ holds for all $x, y \in \mathcal{D}_1$. By Lemma 2.4, equation (2.9) is in l.p.c. at $t = +\infty$. $\square$

Reversing the transformation from (1.1) to (2.9), we get the following result:

Theorem 3.4. Assume that there exist a non-negative function $\sigma(t)$ defined on I , a constant $M > 0$ and an integer $t_0 \geq n$ such that $\sigma(t)$ satisfies the first condition in (3.1) and

$$(3.2) \quad \sigma(t)|\Delta^m[p_s(t+k+v)w^{-1/2}(t+v)w^{-1/2}(t-j+v)]| \leq M, \quad t \geq t_0$$

for all $1 \leq s \leq n, 1 \leq j \leq s, 0 \leq k \leq s-j, 0 \leq v \leq j-1, 0 \leq m \leq j-1-v$. Then equation (1.1) is in l.p.c. at $t = +\infty$.

Proof. Since equations (1.1) and (2.9) have the same limit case at $t = +\infty$ under the transformation (2.6), it suffices to show that the second condition in (3.1) holds by Theorem 3.3.

From (2.4), (2.5) and (2.8) we have

$$(3.3) \quad q_n(t) = p_n(t)w^{-1/2}(t)w^{-1/2}(t-n),$$

$$(3.4) \quad q_j(t) = \sum_{s=j}^n \sum_{k=0}^{s-j} C_s^k C_s^{s-j-k} p_s(t+k)w^{-1/2}(t)w^{-1/2}(t-j) \\ - \sum_{s=j+1}^n \sum_{k=0}^{s-j} C_s^k C_s^{s-j-k} q_s(t+k), \quad 1 \leq j \leq n-1.$$

When $s = n, j = n$ and $k = 0$, it follows from (3.2) that

$$\sigma(t)|\Delta^m[p_n(t+v)w^{-1/2}(t+v)w^{-1/2}(t-n+v)]| \leq M$$

for all $0 \leq v \leq n-1, 0 \leq m \leq n-1-v$, which together with (3.3) yields that

$$(3.5) \quad \sigma(t)|\Delta^m q_n(t+v)| \leq M, \quad 0 \leq v \leq n-1, 0 \leq m \leq n-1-v.$$

In particular, setting $v = 0$ in (3.5), one has

$$\sigma(t)|\Delta^m q_n(t)| \leq M, \quad 0 \leq m \leq n-1.$$

When $s = n-1, j = n-1$ and $k = 0$, it follows from (3.2) that

$$(3.6) \quad |\sigma(t)\Delta^m[p_{n-1}(t+v)w^{-1/2}(t+v)w^{-1/2}(t-n+1+v)]| \leq M$$

for all $0 \leq v \leq n-2, 0 \leq m \leq n-2-v$. When $s = n, j = n-1$, (3.2) implies that

$$(3.7) \quad \sigma(t)|\Delta^m[p_n(t+v+k)w^{-1/2}(t+v)w^{-1/2}(t-n+1+v)]| \leq M$$

for all $0 \leq k \leq 1, 0 \leq v \leq n-2, 0 \leq m \leq n-2-v$. If $0 \leq v \leq n-2$ and $0 \leq k \leq 1$, then $0 \leq v+k \leq n-1$. So, from (3.5) we have

$$(3.8) \quad \sigma(t)|\Delta^m q_n(t+k+v)| \leq M$$

for all $0 \leq k \leq 1, 0 \leq v \leq n-2, 0 \leq m \leq n-2-v$. Hence, from (3.6), (3.7) and (3.8) we get

$$(3.9) \quad \sigma(t)|\Delta^m q_{n-1}(t+v)| \leq M, \quad 0 \leq v \leq n-2, 0 \leq m \leq n-2-v.$$

Setting $v = 0$ in (3.9), one has

$$\sigma(t)|\Delta^m q_{n-1}(t)| \leq M, \quad 0 \leq m \leq n-2.$$

When $s = n-2, j = n-2$ and $k = 0$, it follows from (3.2) that

$$(3.10) \quad \sigma(t)|\Delta^m [p_{n-2}(t+v)w^{-1/2}(t+v)w^{-1/2}(t-n+2+v)]| \leq M$$

for all $0 \leq v \leq n-3, 0 \leq m \leq n-3-v$. When $s = n-1, j = n-2$, (3.2) implies that

$$(3.11) \quad \sigma(t)|\Delta^m [p_{n-1}(t+v+k)w^{-1/2}(t+v)w^{-1/2}(t-n+2+v)]| \leq M$$

for all $0 \leq k \leq 1, 0 \leq v \leq n-3, 0 \leq m \leq n-3-v$. When $s = n, j = n-2$, (3.2) yields that

$$(3.12) \quad \sigma(t)|\Delta^m [p_n(t+v+k)w^{-1/2}(t+v)w^{-1/2}(t-n+2+v)]| \leq M$$

for all $0 \leq k \leq 2, 0 \leq v \leq n-3, 0 \leq m \leq n-3-v$. If $0 \leq v \leq n-3$ and $0 \leq k \leq 1$, then $0 \leq v+k \leq n-2$. So from (3.9) we get

$$(3.13) \quad \sigma(t)|\Delta^m q_{n-1}(t+k+v)| \leq M$$

for all $0 \leq k \leq 1, 0 \leq v \leq n-3, 0 \leq m \leq n-3-v$. It is evident that $0 \leq v+k \leq n-1$ if $0 \leq v \leq n-3$ and $0 \leq k \leq 2$. So it follows from (3.5) that

$$(3.14) \quad \sigma(t)|\Delta^m q_n(t+k+v)| \leq M$$

for all $0 \leq k \leq 2, 0 \leq v \leq n-3, 0 \leq m \leq n-3-v$. Hence, from (3.10)-(3.14) we obtain

$$(3.15) \quad |\sigma(t)\Delta^m q_{n-2}(t+v)| \leq M, \quad 0 \leq v \leq n-3, 0 \leq m \leq n-3-v.$$

Setting $v = 0$ in (3.15), one has

$$\sigma(t)|\Delta^m q_{n-2}(t)| \leq M, \quad 0 \leq m \leq n-3.$$

With a similar argument one can conclude that

$$(3.16) \quad \sigma(t)|\Delta^m q_s(t+v)| \leq M$$

for all $1 \leq s \leq n, 0 \leq v \leq s-1, 0 \leq m \leq s-1-v$ by using (3.2). Therefore, the second condition in (3.1) holds by setting $v = 0$ in (3.16). By Theorem 3.3, equation (2.9) is in l.p.c. at $t = +\infty$, which is equivalent to equation (1.1) being in l.p.c. at $t = +\infty$. The proof is complete. $\square$

Note that if there exist a constant $K > 0$ and an integer $t_0 \geq 0$ such that $|f(t)| \leq Kt$ for $t \geq t_0$, then there exists a constant $N > 0$ such that $|\Delta^m f(t+v)| \leq Nt$ for $t \geq t_0 + 1$ and $0 \leq m, v \leq n$. Thus the following result is a direct consequence of Theorem 3.4 by taking $\sigma(t) = t^{-1}$:

Corollary 3.5. *If there exist a constant $M > 0$ and an integer $t_0 \geq n$ such that*

$$|p_s(t+k)| \leq Mt w^{1/2}(t)w^{1/2}(t-j), \quad t \geq t_0,$$

for all $1 \leq s \leq n, 1 \leq j \leq s$ and $0 \leq k \leq s-j$, then equation (1.1) is in l.p.c. at $t = +\infty$.

Now we give two remarks on Theorems 3.3 and 3.4.

Remark 3.6. Note that condition (3.1) of Theorem 3.3 does not impose a restriction on $q_0(t)$, and condition (3.2) of Theorem 3.4 does not place a restriction on $p_0(t)$.

Remark 3.7. In the case of $n = 1$, condition (3.2) is

$$|\sigma(t)p_1(t)w^{-1/2}(t)w^{-1/2}(t-1)| \leq M, \quad t \geq t_0.$$

It can be verified that the conditions of Theorem 3.4 in this case are equivalent to those of [2, Corollary 3.1] and [9, Theorem 10] as well as those of [13, Theorem 3.1] when the coefficients are real.

At the end of this section, two criteria of the strong limit point case for equations (2.9) and (1.1) are established, respectively.

Theorem 3.8. *If $q_j(t)$, $0 \leq j \leq n$, satisfy all the conditions in Theorem 3.3 and Lemma 2.7, then equation (2.9) is in s.l.p.c. at $t = +\infty$.*

Proof. Since $q_j(t)$, $0 \leq j \leq n$, satisfy the conditions in Lemma 2.7, we have that $\lim_{t \rightarrow +\infty} S_1(y)(t)$ exists and is finite or infinite for all $y \in \mathcal{D}_1$. With a similar argument to that used in the proof of Theorem 3.3, it follows that $\lim_{t \rightarrow +\infty} S_1(y)(t) = 0$ for all $y \in \mathcal{D}_1$. By Lemma 2.6, (2.9) is in s.l.p.c. at $t = +\infty$. The proof is complete. $\square$

Theorem 3.9. *If $p_j(t)$, $0 \leq j \leq n$, and $w(t)$ satisfy all the conditions in Theorem 3.4 and Lemma 2.7, then equation (1.1) is in s.l.p.c. at $t = +\infty$.*

Proof. Since the proof is similar to that of Theorems 3.4 and 3.8, its details are omitted. $\square$

Remark 3.10. Sun and Shi [14] established several strong limit point criteria for the Hamiltonian system (1.3) with the weight function $W(t) = \text{diag}\{W_1(t), W_2(t)\}$, where $W_1(t)$ and $W_2(t)$ are $n \times n$ Hermitian matrices, and it is required that $W_1(t) > 0$. Although the $2n$ th-order equation (1.1) can be converted into system (1.3) by (1.2), its weight function $W(t) = \text{diag}\{w(t), 0, \dots, 0\}$ from (1.4) does not satisfy the above condition in the case of $n \geq 2$. So, Theorems 3.8 and 3.9 are not included by their results in the case of $n \geq 2$.

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