ESSENTIAL NORMS OF COMPOSITION OPERATORS BETWEEN BLOCH TYPE SPACES

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ABSTRACT. For $\alpha > 0$, the α -Bloch space is the space of all analytic functions f on the unit disk D satisfying

$$\|f\|_{B^\alpha} = \sup_{z \in D} |f'(z)|(1 - |z|^2)^\alpha < \infty.$$

Let φ be an analytic self-map of D . We show that for $0 < \alpha, \beta < \infty$, the essential norm of the composition operator C_φ mapping from B^α to B^β can be given by the following formula:

$$\|C_\varphi\|_e = \left(\frac{e}{2\alpha}\right)^\alpha \limsup_{n \rightarrow \infty} n^{\alpha-1} \|\varphi^n\|_{B^\beta}.$$

1. INTRODUCTION

Let D be the unit disk in the complex plane $\mathbb{C}$, and let $H(D)$ be the space of analytic functions on D . For an analytic self-map φ of D , the *composition operator* C_φ for $f \in H(D)$ is defined as

$$C_\varphi(f) = f \circ \varphi.$$

The study of composition operators started in the late 60s and became fairly active after that. One of the reasons is that it provides connections between operator theory and complex analysis and helps us to gain a deeper understanding of both areas. For general references on the theory of composition operators, see the two books [2] and [9].

Recently, there has been a lot of work on composition operators between Bloch type spaces. See, for example, [4], [5], [8] and [11]. Recall that, for $0 < \alpha < \infty$, an analytic function f is said to be in the Bloch type space B^α , or α -Bloch space, if

$$\|f\|_{B^\alpha} = \sup_{z \in D} |f'(z)|(1 - |z|^2)^\alpha < \infty.$$

We also say that $f \in B_0^\alpha$, the little α -Bloch space, if

$$\lim_{|z| \rightarrow 1} |f'(z)|(1 - |z|^2)^\alpha = 0.$$

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The space B^α is a Banach space under the norm

$$\|f\|_{B^\alpha} = |f(0)| + \sup_{z \in D} |f'(z)|(1 - |z|^2)^\alpha,$$

and B_0^α is the closure of polynomials in B^α . It is known that when $\alpha = 1$, $B^\alpha = B$, the classical Bloch space; when $0 < \alpha < 1$, $B^\alpha = \text{Lip}_{1-\alpha}$, the analytic Lipschitz space which contains analytic functions f on D satisfying

$$|f(z) - f(w)| \leq C|z - w|^{1-\alpha}$$

for some constant $C > 0$ and all $z, w \in D$; when $\alpha > 1$, $B^\alpha = H_{\alpha-1}^\infty$, the weighted Banach space of analytic functions that contains analytic functions f on D satisfying

$$\sup_{z \in D} |f(z)|(1 - |z|^2)^{\alpha-1} < \infty.$$

We refer to [12] for the above facts.

Recall that the essential norm $\|T\|_e$ of a bounded operator T between Banach spaces X and Y is defined as the distance from T to the space of compact operators from X to Y . The essential norms of composition operators between Bloch type spaces have been determined by several authors. The following result is known:

Theorem A. *Let $0 < \alpha, \beta < \infty$ and φ be an analytic self-map of the unit disk D . Then the essential norm of the composition operator $C_\varphi : B^\alpha \rightarrow B^\beta$ is*

$$\|C_\varphi\|_e = \lim_{s \rightarrow 1} \sup_{|\varphi(z)| > s} |\varphi'(z)| \frac{(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^\alpha}.$$

The result was proved by Montes-Rodríguez in [6] for the case where $\alpha = \beta = 1$, and in [7] for the case where $0 < \alpha = \beta < \infty$. The general case was proved by Contreras and Hernandez-Díaz in [1]. When $0 < \alpha \leq 1$, the result was also proved by MacCluer and the author in [3].

Recently, Wulan, Zheng and Zhu obtained the following result in [10].

Theorem B. *Let φ be an analytic self-map of D . Then C_φ is compact on the Bloch space B if and only if*

$$\lim_{n \rightarrow \infty} \|\varphi^n\|_B = 0.$$

Notice that here φ^n means the n -th power of φ . A natural question that arises from this result is whether we can get an essential norm formula for composition operators on B in terms of φ^n . The purpose of this paper is to solve this problem. Actually, we obtain an essential norm formula for composition operators from B^α to B^β for any $0 < \alpha, \beta < \infty$ in terms of φ^n . Here is our main result.

Theorem 1.1. *Let $0 < \alpha, \beta < \infty$. Let φ be an analytic self-map of the unit disk D . Then the essential norm of the composition operator $C_\varphi : B^\alpha \rightarrow B^\beta$ is*

$$(1.1) \quad \|C_\varphi\|_e = \left(\frac{e}{2\alpha}\right)^\alpha \limsup_{n \rightarrow \infty} n^{\alpha-1} \|\varphi^n\|_{B^\beta}.$$

The paper is organized as follows. In section 2 we first give a new characterization of bounded composition operators from B^α to B^β in terms of φ^n . In section 3 we prove the lower bound in Theorem 1.1, and in section 4 we prove the upper bound.

2. BOUNDEDNESS

In this section we give the following new characterization of bounded composition operators from B^α to B^β .

Theorem 2.1. *Let $0 < \alpha, \beta < \infty$. Let φ be an analytic self-map of the unit disk D . Then $C_\varphi : B^\alpha \rightarrow B^\beta$ is bounded if and only if*

$$(2.1) \quad \sup_{n \in \mathbb{N}} n^{\alpha-1} \|\varphi^n\|_{B^\beta} < \infty.$$

In order to prove this result, we need the following lemma.

Lemma 2.2. *Let $\alpha > 0$, $n \in \mathbb{N}$ and $0 \leq x \leq 1$. Let $H_{n,\alpha}(x) = x^{n-1}(1-x^2)^\alpha$. Then $H_{n,\alpha}$ has the following properties:*

(i)

$$\max_{0 \leq x \leq 1} H_{n,\alpha}(x) = H_{n,\alpha}(r_n) = \begin{cases} 1, & n = 1 \\ \left(\frac{2\alpha}{n-1+2\alpha}\right)^\alpha \left(\frac{n-1}{n-1+2\alpha}\right)^{(n-1)/2}, & n \geq 2 \end{cases}$$

where

$$(2.2) \quad r_n = \begin{cases} 0, & n = 1 \\ \left(\frac{n-1}{n-1+2\alpha}\right)^{1/2}, & n \geq 2. \end{cases}$$

(ii) For $n \geq 1$, $H_{n,\alpha}$ is increasing on $[0, r_n]$ and decreasing on $[r_n, 1]$.

(iii) For $n \geq 1$, $H_{n,\alpha}$ is decreasing on $[r_n, r_{n+1}]$, and so

$$\min_{x \in [r_n, r_{n+1}]} H_{n,\alpha}(x) = H_{n,\alpha}(r_{n+1}) = \left(\frac{2\alpha}{n+2\alpha}\right)^\alpha \left(\frac{n}{n+2\alpha}\right)^{(n-1)/2}.$$

Consequently,

$$\lim_{n \rightarrow \infty} n^\alpha \min_{x \in [r_n, r_{n+1}]} H_{n,\alpha}(x) = \left(\frac{2\alpha}{e}\right)^\alpha.$$

The proof is an easy exercise in calculus. We omit the details here.

Proof of Theorem 2.1. Let $C_\varphi : B^\alpha \rightarrow B^\beta$ be bounded. To show that (2.1) is true, we follow the lines of the proof for the lower bound in Theorem 2.1 of [6]. Let the integer $n \geq 2$. Consider the function z^n . By Lemma 2.2,

$$\|z^n\|_{B^\alpha} = \max_{z \in D} n|z|^{n-1}(1-|z|^2)^\alpha = n \left(\frac{2\alpha}{n-1+2\alpha}\right)^\alpha \left(\frac{n-1}{n-1+2\alpha}\right)^{(n-1)/2},$$

where the maximum is attained at any point on the circle with radius

$$r_n = \left(\frac{n-1}{n-1+2\alpha}\right)^{1/2}.$$

Thus

$$(2.3) \quad \lim_{n \rightarrow \infty} n^{\alpha-1} \|z^n\|_{B^\alpha} = \left(\frac{2\alpha}{e}\right)^\alpha.$$

Hence, there is a constant $C > 0$, independent of n , such that $\|z^n\|_{B^\alpha} \leq Cn^{1-\alpha}$. Let $f_n(z) = z^n/\|z^n\|_{B^\alpha}$. Then $\|f\|_{B^\alpha} = \|f_n\|_{B^\alpha} = 1$. Thus

$$\infty > \|C_\varphi\| \geq \|C_\varphi f_n\|_{B^\beta} = \frac{\|\varphi^n\|_{B^\beta}}{\|z^n\|_{B^\alpha}} \geq \frac{1}{C} n^{\alpha-1} \|\varphi^n\|_{B^\beta}.$$

Thus (2.1) is true.

Conversely, let (2.1) be true. Then obviously $\|\varphi\|_{B^\beta} < \infty$. If $\sup_{z \in D} |\varphi(z)| < 1$, then there is a number r , with $0 < r < 1$, such that $\sup_{z \in D} |\varphi(z)| < r$. In this case we can easily check that $C_\varphi : B^\alpha \rightarrow B^\beta$ is bounded. In the following, we assume that $\sup_{z \in D} |\varphi(z)| = 1$.

For any integer $n \geq 1$, let

$$D_n = \{z \in D : r_n \leq |\varphi(z)| \leq r_{n+1}\},$$

where r_n is given by (2.2). Let m be the smallest positive integer such that $D_m \neq \emptyset$. Since $\sup_{z \in D} |\varphi(z)| = 1$, D_n is not empty for every integer $n \geq m$, and $D = \bigcup_{n=m}^{\infty} D_n$. By Lemma 2.2, for every $n \geq m$,

$$\min_{z \in D_n} n^\alpha |\varphi(z)|^{n-1} (1 - |\varphi(z)|^2)^\alpha \geq n^\alpha H_{n,\alpha}(r_{n+1}) = \left(\frac{2\alpha n}{n+2\alpha}\right)^\alpha \left(\frac{n}{n+2\alpha}\right)^{(n-1)/2},$$

where $H_{n,\alpha}(x) = x^{n-1}(1-x^2)^\alpha$. Thus

$$\lim_{n \rightarrow \infty} \min_{z \in D_n} n^\alpha |\varphi(z)|^{n-1} (1 - |\varphi(z)|^2)^\alpha \geq \left(\frac{2\alpha}{e}\right)^\alpha.$$

Therefore, there exists a constant $\delta > 0$ such that, for any $n \geq m$,

$$\min_{z \in D_n} n^\alpha |\varphi(z)|^{n-1} (1 - |\varphi(z)|^2)^\alpha \geq \delta.$$

Let $f \in B^\alpha$. Then

$$\begin{aligned} \|C_\varphi f\|_{B^\beta} &= \sup_{z \in D} |f'(\varphi(z))| |\varphi'(z)| (1 - |z|^2)^\beta \\ &= \sup_{n \geq m} \sup_{z \in D_n} |f'(\varphi(z))| |\varphi'(z)| (1 - |z|^2)^\beta \\ &= \sup_{n \geq m} \sup_{z \in D_n} \frac{|f'(\varphi(z))| |\varphi'(z)| (1 - |z|^2)^\beta n^\alpha |\varphi(z)|^{n-1} (1 - |\varphi(z)|^2)^\alpha}{n^\alpha |\varphi(z)|^{n-1} (1 - |\varphi(z)|^2)^\alpha} \\ &\leq \frac{1}{\delta} \sup_{n \geq m} \sup_{z \in D_n} n^{\alpha-1} |f'(\varphi(z))| (1 - |\varphi(z)|^2)^\alpha |(\varphi^n)'(z)| (1 - |z|^2)^\beta \\ &\leq \frac{1}{\delta} \|f\|_{B^\alpha} \sup_{n \geq 1} n^{\alpha-1} \|\varphi^n\|_{B^\beta}. \end{aligned}$$

Thus C_φ is bounded from B^α to B^β . The proof is complete. $\square$

It is well-known that, as a simple consequence of the Schwartz-Pick lemma, C_φ is bounded on the Bloch space B for any analytic self-map φ of D . Hence, if we let $\alpha = \beta = 1$ in Theorem 2.1, we get the following corollary.

Corollary 2.3. *For any analytic self-map φ of the unit disk D ,*

$$\sup_{n \in \mathbb{N}} \|\varphi^n\|_B < \infty.$$

3. THE LOWER ESTIMATE

Proof of Theorem 1.1: The lower estimate. Let $n \in \mathbb{N}$. Consider the function z^n . By Lemma 2.2,

$$\|z^n\|_{B^\alpha} = \max_{z \in D} n|z|^{n-1}(1-|z|^2)^\alpha = n \left(\frac{2\alpha}{n-1+2\alpha} \right)^\alpha \left(\frac{n-1}{n-1+2\alpha} \right)^{(n-1)/2},$$

where the maximum is attained at any point on the circle with radius

$$r_n = \left(\frac{n-1}{n-1+2\alpha} \right)^{1/2}.$$

Clearly,

$$\lim_{n \rightarrow \infty} n^{\alpha-1} \|z^n\|_{B^\alpha} = \left(\frac{2\alpha}{e} \right)^\alpha.$$

Let $f_n(z) = z^n / \|z^n\|_{B^\alpha}$. Then $\|f_n\|_{B^\alpha} = 1$ and $f_n \rightarrow 0$ weakly in B^α (see [3]). Thus, if K is any compact operator from B^α to B^β , then

$$\lim_{n \rightarrow \infty} \|Kf_n\|_{B^\beta} = 0.$$

Hence,

$$\|C_\varphi - K\| \geq \limsup_{n \rightarrow \infty} \|(C_\varphi - K)f_n\|_{B^\beta} \geq \limsup_{n \rightarrow \infty} \|C_\varphi f_n\|_{B^\beta}.$$

Thus, for $C_\varphi : B^\alpha \rightarrow B^\beta$,

$$\begin{aligned} \|C_\varphi\|_e &\geq \limsup_{n \rightarrow \infty} \|C_\varphi f_n\|_{B^\beta} = \limsup_{n \rightarrow \infty} \frac{1}{\|z^n\|_{B^\alpha}} \|\varphi^n\|_{B^\beta} \\ &= \limsup_{n \rightarrow \infty} \frac{1}{n^{\alpha-1} \|z^n\|_{B^\alpha}} n^{\alpha-1} \|\varphi^n\|_{B^\beta} \\ &= \left(\frac{e}{2\alpha} \right)^\alpha \limsup_{n \rightarrow \infty} n^{\alpha-1} \|\varphi^n\|_{B^\beta}. \end{aligned}$$

□

4. THE UPPER ESTIMATE

We need several lemmas to obtain the upper bound in Theorem 1.1.

For $r \in (0, 1)$, let $K_r f(z) = f(rz)$. Then K_r is a compact operator on the space B^α or B_0^α for any positive number α , with $\|K_r\| \leq 1$. The following lemma is from [3].

Lemma 4.1. *Let $0 < \alpha < 1$. Then there is a sequence $\{r_k\}$, with $0 < r_k < 1$ tending to 1, such that the compact operator $L_n = \frac{1}{n} \sum_{k=1}^n K_{r_k}$ on B_0^α satisfies:*

- (i) *For any $t \in [0, 1)$, $\lim_{n \rightarrow \infty} \sup_{\|f\|_{B^\alpha} \leq 1} \sup_{|z| \leq t} |(I - L_n)f'(z)| = 0$.*
- (ii) $\lim_{n \rightarrow \infty} \sup_{\|f\|_{B^\alpha} \leq 1} \sup_{z \in D} |(I - L_n)f(z)| = 0$.
- (iii) $\limsup_{n \rightarrow \infty} \|I - L_n\| \leq 1$.

Furthermore, these statements hold as well for the sequence of biadjoints L_n^{**} on B^α .

The analogue of Lemma 4.1 for the case $\alpha = 1$ is next, which is also given in [3].

Lemma 4.2. *There is a sequence $\{r_k\}$, with $0 < r_k < 1$ tending to 1, such that the compact operator $L_n = \frac{1}{n} \sum_{k=1}^n K_{r_k}$ acting on B_0 satisfies:*

- (i) *For any $t \in [0, 1)$, $\lim_{n \rightarrow \infty} \sup_{\|f\|_{B \leq 1}} \sup_{|z| \leq t} |((I - L_n)f)'(z)| = 0$.*
- (iia) $\limsup_{n \rightarrow \infty} \sup_{\|f\|_{B \leq 1}} \sup_{|z| > s} |(I - L_n)f(z)| \left(\log \frac{1}{1 - |z|^2} \right)^{-1} \leq 1$, *for s sufficiently close to 1, and*
- (iib) $\lim_{n \rightarrow \infty} \sup_{\|f\|_{B \leq 1}} \sup_{|z| \leq s} |(I - L_n)f(z)| = 0$, *for the above s .*
- (iii) $\limsup_{n \rightarrow \infty} \|I - L_n\| \leq 1$.

Furthermore, the same is true for the sequence of biadjoints L_n^{**} on B .

For the case where $\alpha > 1$, we have the following lemma.

Lemma 4.3. *Let $\alpha > 0$. Then there is a sequence $\{r_k\}$, with $0 < r_k < 1$ tending to 1, such that the compact operator $L_n = \frac{1}{n} \sum_{k=1}^n K_{r_k}$ on B_0^α satisfies:*

- (i) *For any $s \in [0, 1)$, $\lim_{n \rightarrow \infty} \sup_{\|f\|_{B^\alpha \leq 1}} \sup_{|z| \leq s} |((I - L_n)f)'(z)| = 0$.*
- (ii) *For any $t \in [0, 1)$, $\lim_{n \rightarrow \infty} \sup_{\|f\|_{B^\alpha \leq 1}} \sup_{|z| \leq t} |(I - L_n)f(z)| = 0$.*
- (iii) $\limsup_{n \rightarrow \infty} \|I - L_n\| \leq 1$.

Furthermore, these statements hold as well for the sequence of biadjoints L_n^{**} on B^α .

Proof. The proofs of (i) and (iii) and the statement about the biadjoints L_n^{**} are the same as the proof of Proposition 2.1 in [7], so we omit them. To prove (ii), notice that $(I - L_n)f(0) = 0$ for all $n \in \mathbb{N}$, so by Exercise 11 on p. 75 of [12], we have

$$(4.1) \quad (I - L_n)f(z) = \int_D \frac{((I - L_n)f)'(w)(1 - |w|^2)^{1+\alpha}}{\bar{w}(1 - \bar{z}w)^{2+\alpha}} dA(w).$$

Given $t \in [0, 1)$ and any $\varepsilon > 0$, we choose $s \in [0, 1)$ sufficiently close to 1 such that

$$\frac{1 - s}{s(1 - t)^{2+\alpha}} < \varepsilon.$$

Hence

$$\begin{aligned} I_1 &= \sup_{\|f\|_{B^\alpha \leq 1}} \sup_{|z| \leq t} \int_{D \setminus D_s} \frac{|((I - L_n)f)'(w)|(1 - |w|^2)^{1+\alpha}}{|w||1 - \bar{z}w|^{2+\alpha}} dA(w) \\ &\leq \sup_{\|f\|_{B^\alpha \leq 1}} \|(I - L_n)f\|_{B^\alpha} \sup_{|z| \leq t} \int_{D \setminus D_s} \frac{1 - |w|^2}{|w||1 - \bar{z}w|^{2+\alpha}} dA(w) \\ &\leq \frac{1}{s(1 - t)^{2+\alpha}} \sup_{|z| \leq t} \int_{D \setminus D_s} (1 - |w|^2) dA(w) \\ &\leq \frac{1}{s(1 - t)^{2+\alpha}} (1 - s) < \varepsilon \end{aligned}$$

as s is sufficiently close to 1. Here for the second inequality we have used (iii). Fix this s . From (i), there exists $N > 0$ such that for any integer $n > N$,

$$\sup_{\|f\|_{B^\alpha \leq 1}} \sup_{|z| \leq s} |((I - L_n)f)'(z)| < \varepsilon.$$

For such n we have

$$\begin{aligned}
I_2 &= \sup_{\|f\|_{B^\alpha} \leq 1} \sup_{|z| \leq t} \int_{D_s} \frac{|((I - L_n)f)'(w)|(1 - |w|^2)^{1+\alpha}}{|w||1 - \bar{z}w|^{2+\alpha}} dA(w) \\
&\leq \varepsilon \sup_{|z| \leq t} \int_{D_s} \frac{(1 - |w|^2)^{1+\alpha}}{|w||1 - \bar{z}w|^{2+\alpha}} dA(w) \\
&\leq \frac{\varepsilon}{(1 - t)^{2+\alpha}} \sup_{|z| \leq t} \int_{D_s} \frac{(1 - |w|^2)^{1+\alpha}}{|w|} dA(w) \\
&< C\varepsilon
\end{aligned}$$

for some positive constant C . From (4.1), upon combining the above inequalities, we get that for $n > N$,

$$\begin{aligned}
&\sup_{\|f\|_{B^\alpha} \leq 1} \sup_{|z| \leq t} |(I - L_n)f(z)| \\
&\leq \sup_{\|f\|_{B^\alpha} \leq 1} \sup_{|z| \leq t} \int_D \frac{|((I - L_n)f)'(w)|(1 - |w|^2)^{1+\alpha}}{|w||1 - \bar{z}w|^{2+\alpha}} dA(w) \\
&\leq I_1 + I_2 < (1 + C)\varepsilon.
\end{aligned}$$

Thus

$$\lim_{n \rightarrow \infty} \sup_{\|f\|_{B^\alpha} \leq 1} \sup_{|z| \leq t} |(I - L_n)f(z)| = 0.$$

The proof is complete. $\square$

Proof of Theorem 1.1: The upper estimate. If $\limsup_{n \rightarrow \infty} n^{\alpha-1} \|\varphi^n\|_{B^\beta} = \infty$, then the result is trivially true. Now suppose

$$\limsup_{n \rightarrow \infty} n^{\alpha-1} \|\varphi^n\|_{B^\beta} < \infty.$$

Then, by Theorem 2.1, C_φ is a bounded operator from B^α to B^β and, clearly, $\|\varphi\|_{B^\beta} < \infty$.

Notice that if $\sup_{z \in D} |\varphi(z)| < 1$, then we can easily see that $C_\varphi : B^\alpha \rightarrow B^\beta$ is compact (see, for example, p. 129 of [2]) and that both sides of (1.1) are 0, so (1.1) is trivially true. Hence, in the following we assume that $\sup_{z \in D} |\varphi(z)| = 1$.

Let $\{L_n\}$ be the sequence of operators given in Lemmas 4.1, 4.2 and 4.3. Since each L_n is compact as an operator from B^α to B^α , $C_\varphi L_n : B^\alpha \rightarrow B^\beta$ is also compact and we have

$$\begin{aligned}
\|C_\varphi\|_e &\leq \limsup_{n \rightarrow \infty} \|C_\varphi - C_\varphi L_n\| = \limsup_{n \rightarrow \infty} \|C_\varphi(I - L_n)\| \\
&= \limsup_{n \rightarrow \infty} \sup_{\|f\|_{B^\alpha} \leq 1} \|C_\varphi(I - L_n)f\|_{B^\beta}.
\end{aligned}$$

We bound this last expression from above by

$$\begin{aligned}
&\limsup_{n \rightarrow \infty} \sup_{\|f\|_{B^\alpha} \leq 1} |(I - L_n)f(\varphi(0))| \\
&\quad + \limsup_{n \rightarrow \infty} \sup_{\|f\|_{B^\alpha} \leq 1} \sup_{z \in D} |((I - L_n)f)'(\varphi(z))| |\varphi'(z)| (1 - |z|^2)^\beta.
\end{aligned}$$

Lemma 4.1 (ii), Lemma 4.2 (iib) and Lemma 4.3 (ii) guarantee that

$$\limsup_{n \rightarrow \infty} \sup_{\|f\|_{B^\alpha} \leq 1} |(I - L_n)f(\varphi(0))| = 0.$$

Now we need only consider the term

$$J = \sup_{\|f\|_{B^\alpha} \leq 1} \sup_{z \in D} |((I - L_n)f)'(\varphi(z))| |\varphi'(z)| (1 - |z|^2)^\beta.$$

For any integer $n \geq 1$, let r_n be given by (2.2), and let

$$D_n = \{z \in D : r_n \leq |\varphi(z)| \leq r_{n+1}\}.$$

Let m be the smallest positive integer such that $D_m \neq \emptyset$. Since $\sup_{z \in D} |\varphi(z)| = 1$, D_n is not empty for every integer $n \geq m$, and $D = \bigcup_{n=m}^{\infty} D_n$. Now we divide J into two parts:

$$\begin{aligned} J &= \sup_{\|f\|_{B^\alpha} \leq 1} \sup_{m \leq k \leq N-1} \sup_{z \in D_k} |((I - L_n)f)'(\varphi(z))| |\varphi'(z)| (1 - |z|^2)^\beta \\ &\quad + \sup_{\|f\|_{B^\alpha} \leq 1} \sup_{k \geq N} \sup_{z \in D_k} |((I - L_n)f)'(\varphi(z))| |\varphi'(z)| (1 - |z|^2)^\beta \\ &= J_1 + J_2. \end{aligned}$$

Here N is a positive integer determined as follows. Write the function under three supremum signs in J_2 as

$$\frac{|((I - L_n)f)'(\varphi(z))| |\varphi'(z)| (1 - |z|^2)^\beta k^\alpha |\varphi(z)|^{k-1} (1 - |\varphi(z)|^2)^\alpha}{k^\alpha |\varphi(z)|^{k-1} (1 - |\varphi(z)|^2)^\alpha}.$$

By Lemma 2.2, for $z \in D_k$,

$$k^\alpha |\varphi(z)|^{k-1} (1 - |\varphi(z)|^2)^\alpha \geq \left(\frac{2\alpha k}{k + 2\alpha} \right)^\alpha \left(\frac{k}{k + 2\alpha} \right)^{(k-1)/2}.$$

It is easy to see that

$$\lim_{k \rightarrow \infty} \left[\left(\frac{2\alpha k}{k + 2\alpha} \right)^\alpha \left(\frac{k}{k + 2\alpha} \right)^{(k-1)/2} \right]^{-1} = \left(\frac{e}{2\alpha} \right)^\alpha.$$

Hence, for any $\varepsilon > 0$, we can choose $N > m + 1$ large enough such that for any $k \geq N$,

$$\left[\left(\frac{2\alpha k}{k + 2\alpha} \right)^\alpha \left(\frac{k}{k + 2\alpha} \right)^{(k-1)/2} \right]^{-1} < \left(\frac{e}{2\alpha} \right)^\alpha + \varepsilon.$$

For such N we have

$$\begin{aligned} J_2 &\leq \left[\left(\frac{e}{2\alpha} \right)^\alpha + \varepsilon \right] \sup_{\|f\|_{B^\alpha} \leq 1} \|(I - L_n)f\|_{B^\alpha} \sup_{k \geq N} \sup_{z \in D_k} k^{\alpha-1} |(\varphi^k)'(z)| (1 - |z|^2)^\beta \\ &\leq \left[\left(\frac{e}{2\alpha} \right)^\alpha + \varepsilon \right] \|I - L_n\| \sup_{k \geq N} k^{\alpha-1} \|\varphi^k\|_{B^\beta}. \end{aligned}$$

Thus by (iii) of Lemmas 4.1, 4.2 and 4.3,

$$\limsup_{n \rightarrow \infty} J_2 \leq \left[\left(\frac{e}{2\alpha} \right)^\alpha + \varepsilon \right] \sup_{k \geq N} k^{\alpha-1} \|\varphi^k\|_{B^\beta}.$$

For J_1 , by (i) of Lemmas 4.1, 4.2 and 4.3, we have

$$\limsup_{n \rightarrow \infty} J_1 = \limsup_{n \rightarrow \infty} \sup_{\|f\|_{B^\alpha} \leq 1} \sup_{r_m \leq |\varphi(z)| \leq r_{N-1}} |((I - L_n)f)'(\varphi(z))| = 0.$$

Hence, for any N sufficiently large we have

$$\|C_\varphi\|_e \leq \limsup_{n \rightarrow \infty} J_2 \leq \left[\left(\frac{e}{2\alpha} \right)^\alpha + \varepsilon \right] \sup_{k \geq N} k^{\alpha-1} \|\varphi^k\|_{B^\beta}.$$

Hence

$$\|C_\varphi\|_e \leq \left[\left(\frac{e}{2\alpha} \right)^\alpha + \varepsilon \right] \limsup_{k \rightarrow \infty} k^{\alpha-1} \|\varphi^k\|_{B^\beta}.$$

Since ε is an arbitrary positive number, we have

$$\|C_\varphi\|_e \leq \left(\frac{e}{2\alpha} \right)^\alpha \limsup_{k \rightarrow \infty} k^{\alpha-1} \|\varphi^k\|_{B^\beta}.$$

The proof is complete. $\square$

By (2.3) we know that

$$\lim_{n \rightarrow \infty} n^{\alpha-1} \|z^n\|_{B^\alpha} = \left(\frac{2\alpha}{e} \right)^\alpha.$$

Combining this equation and Theorem 1.1, we immediately obtain the following corollary.

Corollary 4.4. *Let $0 < \alpha, \beta < \infty$. Let φ be an analytic self-map of the unit disk D . Then the essential norm of the composition operator $C_\varphi : B^\alpha \rightarrow B^\beta$ is*

$$\|C_\varphi\|_e = \limsup_{n \rightarrow \infty} \frac{\|\varphi^n\|_{B^\beta}}{\|z^n\|_{B^\alpha}} = \limsup_{n \rightarrow \infty} \frac{\|C_\varphi(z^n)\|_{B^\beta}}{\|z^n\|_{B^\alpha}}.$$

The following result is also an immediate consequence of Theorem 1.1 and Corollary 4.4.

Corollary 4.5. *Let $0 < \alpha, \beta < \infty$. Let φ be an analytic self-map of the unit disk D . Then the following conditions are equivalent:*

- (i) C_φ is compact from B^α to B^β .
- (ii) $\limsup_{n \rightarrow \infty} n^{\alpha-1} \|\varphi^n\|_{B^\beta} = 0$.
- (iii) $\limsup_{n \rightarrow \infty} \frac{\|\varphi^n\|_{B^\beta}}{\|z^n\|_{B^\alpha}} = 0$.

Corollary 4.4 raises a natural question: For what other spaces X and Y of analytic functions on D is the essential norm of the composition operator C_φ from X to Y given by

$$\|C_\varphi\|_e = \limsup_{n \rightarrow \infty} \frac{\|C_\varphi(z^n)\|_Y}{\|z^n\|_X}?$$

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