THE REAL PLANK PROBLEM AND SOME APPLICATIONS

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ABSTRACT. K. Ball has proved the “complex plank problem”: if $(x_k)_{k=1}^n$ is a sequence of norm 1 vectors in a *complex* Hilbert space $(H, \langle \cdot, \cdot \rangle)$, then there exists a unit vector x for which

$$|\langle x, x_k \rangle| \geq 1/\sqrt{n}, \quad k = 1, \dots, n.$$

In general, this result is not true on *real* Hilbert spaces. However, in special cases we prove that the same result holds true. In general, for some unit vector x we have derived the estimate

$$|\langle x, x_k \rangle| \geq \max \left\{ \sqrt{\lambda_1/n}, 1/\sqrt{\lambda_n n} \right\},$$

where λ_1 is the smallest and λ_n is the largest eigenvalue of the Hermitian matrix $A = [\langle x_j, x_k \rangle]$, $j, k = 1, \dots, n$. We have also improved known estimates for the norms of homogeneous polynomials which are products of linear forms on real Hilbert spaces.

1. INTRODUCTION AND BACKGROUND

Let $(x_k)_{k=1}^n$ be a sequence of norm 1 vectors in a *complex* Hilbert space $(H, \langle \cdot, \cdot \rangle)$ and let $(t_k)_{k=1}^n$ be a sequence of non-negative numbers satisfying

$$\sum_{k=1}^n t_k^2 = 1.$$

Then there is a unit vector x for which

$$|\langle x, x_k \rangle| \geq t_k,$$

for every k . For the proof of the previous result one can take $t_k = 1/\sqrt{n}$; small modifications are needed to handle the general case. This is K. Ball’s “complex plank problem” [4, Theorem 2]. In fact, this result can be extended to *complex* $L_p(\mu)$ spaces. It was proved in [28] that if f_k , $1 \leq k \leq n$, are unit vectors in the dual of $L_p(\mu)$, then there is a point $x \in B_{L_p(\mu)}$ such that for every k

$$(1.1) \quad |f_k(x)| \geq \begin{cases} n^{-1/p} & \text{if } 1 \leq p \leq 2, \\ n^{-1/p'} & \text{if } 2 \leq p \leq \infty, \end{cases}$$

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where $1/p + 1/p' = 1$. Observe that for $p = 1$ or $p = \infty$ we have $|f_k(x)| \geq 1/n$. But, this last estimate is true for any unit vectors $f_k \in X^*$, $1 \leq k \leq n$, where X^* is the dual of a real or complex Banach space X [3].

In this work we investigate the real analogue of the “complex plank problem”. We also consider the problem of improving known lower bounds for the norms of homogeneous polynomials which are products of linear forms on *real* Hilbert spaces.

For convenience we recall the basic definitions needed to discuss polynomials on Banach spaces. If X is a Banach space over $\mathbb{K}$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we denote by B_X , $\overline{B}_X$ and S_X the open unit ball, the closed unit ball and the unit sphere of X respectively. We let $\mathcal{L}^s({}^nX)$ denote the Banach space of all continuous symmetric n -linear forms $L : X^n \rightarrow \mathbb{K}$ with the norm

$$\|L\| = \sup \{|L(x_1, \dots, x_n)| : \|x_1\| \leq 1, \dots, \|x_n\| \leq 1\}.$$

A function $P : X \rightarrow \mathbb{K}$ is a continuous n -homogeneous polynomial if there is a continuous symmetric n -linear form $L : X^n \rightarrow \mathbb{K}$ for which $P(x) = L(x, \dots, x)$ for all $x \in X$. In this case it is convenient to write $P = \widehat{L}$. Let $\mathcal{P}({}^nX)$ denote the Banach space of all continuous n -homogeneous polynomials $P : X \rightarrow \mathbb{K}$ with the norm

$$\|P\| = \sup \{|P(x)| : \|x\| \leq 1\}.$$

Each $\widehat{L} \in \mathcal{P}({}^nX)$ is associated with a unique $L \in \mathcal{L}^s({}^nX)$ with the property that $\widehat{L}(x) = L(x, \dots, x)$. We recall a well-known polarization formula (see [10, Proposition 1.5]),

$$(1.2) \quad L(x_1, \dots, x_n) = \frac{1}{n!} \int_0^1 r_1(t) \cdots r_n(t) \widehat{L} \left(\sum_{j=1}^n r_j(t) x_j \right) dt,$$

where $r_k(t) = \text{sgn} \sin 2^n \pi t$ is the k th Rademacher function defined on $[0, 1]$. The Rademacher functions $\{r_k\}$ form an orthonormal set in $L_2([0, 1], dt)$, where dt denotes Lebesgue measure on $[0, 1]$.

Let $\mathbb{T}^n$ be the n -fold product of the circle group and let λ be Haar measure on $\mathbb{T}^n$. Thus $d\lambda(\theta) = (1/2\pi)^n d\theta_1 \cdots d\theta_n$, and for every $\widehat{L} \in \mathcal{P}({}^nX)$, where X is a *complex* Banach space, we also have from [10, Proposition 1.5] the following polarization formula:

$$(1.3) \quad L(x_1, \dots, x_n) = \frac{1}{n!} \int_{\mathbb{T}^n} e^{-i\theta_1} \cdots e^{-i\theta_n} \widehat{L} \left(\sum_{j=1}^n e^{i\theta_j} x_j \right) d\lambda(\theta).$$

It follows from the polarization formula that

$$\|\widehat{L}\| \leq \|L\| \leq \frac{n^n}{n!} \|\widehat{L}\|,$$

for every $L \in \mathcal{L}^s({}^nX)$. However, the right hand inequality can be tightened for many Banach spaces. A famous result, investigated by S. Banach [5] (see also [14]), asserts that if H is a Hilbert space, then

$$(1.4) \quad \|L\| = \|\widehat{L}\|$$

for every $L \in \mathcal{L}^s({}^nH)$. For general background on polynomials, we refer to [10].

2. THE REAL PLANK PROBLEM

Let $x_1, \dots, x_{2n}$ be unit vectors in the Euclidean space $\mathbb{R}^2$ distributed uniformly around the circle. As was observed by Pádraig Kirwan [29, p. 706], for the arbitrary unit vector x in the plane there is an x_k , $1 \leq k \leq 2n$, for which

$$|\langle x, x_k \rangle| \leq \sin(\pi/2n) \leq \pi/2n.$$

Now, this last inequality shows that in general the “complex plank problem” [4, Theorem 2] cannot be true on any *real* Hilbert space for n large enough.

2.1. How far do we get with calculus? In the early 1930’s, Tarski asked the following question: If a convex body of minimum width 1 is covered by a union of planks (a *plank* is the region between two parallel hyperplanes in $\mathbb{R}^n$), must the widths of the planks add up to at least 1?

The problem was solved in the affirmative by T. Bang [6] in 1951. Bang used the following simple lemma which was the starting point of several surprising results. For fixed x_j , $j = 1, \dots, n$, in a finite dimensional Hilbert space $(H, \langle \cdot, \cdot \rangle)$, the same lemma was also used by M. Marcus and H. Minc in [20] trying to find a significant lower bound for the expression $\prod_{j=1}^n |\langle x_j, u \rangle|^2$ as a function of the unit vector u .

Lemma 2.1 ([6, 20]). *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and let $x_j \in H$, $1 \leq j \leq n$, be unit vectors. If*

$$\|\varepsilon_1^* x_1 + \dots + \varepsilon_n^* x_n\| = \max_{\varepsilon_j = \pm 1} \|\varepsilon_1 x_1 + \dots + \varepsilon_n x_n\|,$$

then

$$\Re \langle \varepsilon_k^* x_k, \sum_{j=1}^n \varepsilon_j^* x_j \rangle \geq 1, \quad k = 1, \dots, n.$$

Moreover,

$$\|\varepsilon_1^* x_1 + \dots + \varepsilon_n^* x_n\|^2 = n + \sum_{\substack{j,k=1 \\ j \neq k}} \Re \langle \varepsilon_j^* x_j, \varepsilon_k^* x_k \rangle \geq n.$$

Proposition 2.2. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and let $x_i \in H$, $1 \leq i \leq n$, be linearly independent unit vectors. Take $y_i = \varepsilon_i^* x_i$, $i = 1, \dots, n$, as in the previous lemma and consider the function*

$$f(t) = \|t_1 y_1 + \dots + t_n y_n\|^2,$$

where $t = (t_1, \dots, t_n)$, $t_1, \dots, t_n \geq 0$, with $t_1 + \dots + t_n = 1$. If the determinants of all diagonal submatrices of order ≥ 3 of the matrix

$$(2.1) \quad \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & \langle y_1, y_1 \rangle & \langle y_1, y_2 \rangle & \dots & \langle y_1, y_n \rangle \\ 1 & \langle y_2, y_1 \rangle & \langle y_2, y_2 \rangle & \dots & \langle y_2, y_n \rangle \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \langle y_n, y_1 \rangle & \langle y_n, y_2 \rangle & \dots & \langle y_n, y_n \rangle \end{pmatrix}$$

are negative, then f achieves its minimum at some interior point $t^* = (t_1^*, \dots, t_n^*)$, $t_1^*, \dots, t_n^* > 0$, with $t_1^* + \dots + t_n^* = 1$. Moreover,

$$|\langle x_i, y^* / \|y^*\| \rangle| \geq 1/\sqrt{n}, \quad i = 1, \dots, n,$$

where $y^* = t_1^* y_1 + \dots + t_n^* y_n$.

Proof. Without loss of generality we can take H to be a *real* Hilbert space. Consider the quadratic form

$$(2.2) \quad f(t) = \|t_1 y_1 + \cdots + t_n y_n\|^2 = \sum_{i,j=1}^n \langle y_i, y_j \rangle t_i t_j.$$

To find the extreme points of f subject to the side condition $g(t) = 0$, where $g(t) = t_1 + \cdots + t_n - 1$, we use the method of Lagrange multipliers. We form the auxiliary function $h = f - \lambda g$ and the bordered Hessian matrix (2.1). Since the determinants of all diagonal submatrices of order ≥ 3 of the bordered Hessian matrix are negative, f achieves its minimum at some interior point $t^* = (t_1^*, \dots, t_n^*)$, $t_1^*, \dots, t_n^* > 0$, with $t_1^* + \cdots + t_n^* = 1$. Since f is a homogeneous polynomial of degree 2, we can apply Euler's identity to obtain

$$\langle y_i, t_1^* y_1 + \cdots + t_n^* y_n \rangle = \lambda/2 = f(t^*), \quad i = 1, \dots, n,$$

and therefore

$$nf(t^*) = \sum_{i=1}^n t_i^* \langle y_i, y_1 + \cdots + y_n \rangle \geq 1.$$

Hence,

$$|\langle x_i, y^* / \|y^*\| \rangle| = \|t_1^* y_1 + \cdots + t_n^* y_n\| \geq 1/\sqrt{n}, \quad i = 1, \dots, n,$$

where $y^* = t_1^* y_1 + \cdots + t_n^* y_n$. □

Corollary 2.3. *Under the hypothesis of the previous proposition,*

$$\sup_{\|x\|=1} |\langle x, x_1 \rangle \cdots \langle x, x_n \rangle| \geq n^{-n/2},$$

with equality if and only if $x_1, \dots, x_n$ are orthonormal unit vectors.

Observe that Proposition 2.2 extends Theorem 2 in [16].

2.2. A more general result.

Theorem 2.4. *Let $(x_k)_{k=1}^n$ be a sequence of norm 1 vectors in a Hilbert space $(H, \langle \cdot, \cdot \rangle)$. Then there is a unit vector x for which*

$$|\langle x, x_k \rangle| \geq \max \left\{ \sqrt{\lambda_1/n}, 1/\sqrt{\lambda_n n} \right\},$$

for every k , where λ_1 is the smallest and λ_n is the largest eigenvalue of the Gram matrix $A = [\langle x_j, x_k \rangle]$, $j, k = 1, \dots, n$, based on the unit vectors $x_1, \dots, x_n$.

Proof. Let $A = [\langle x_j, x_k \rangle]$, $j, k = 1, \dots, n$, be the Gram matrix which is an $n \times n$ positive semi-definite Hermitian matrix. Then, for every vector $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ (or $z = (z_1, \dots, z_n) \in \mathbb{R}^n$)

$$(2.3) \quad \lambda_1 \|z\|_2^2 \leq z^* A z = \sum_{j,k=1}^n \langle x_j, x_k \rangle z_j \bar{z}_k = \left\| \sum_{j=1}^n x_j z_j \right\|^2 \leq \lambda_n \|z\|_2^2,$$

where λ_1 is the smallest and λ_n is the largest eigenvalue of the Gram matrix A (see for instance [15]).

In particular, if $t = (t_1, \dots, t_n)$, $t_1, \dots, t_n \geq 0$, with $t_1 + \cdots + t_n = 1$, then

$$\left\| \sum_{j=1}^n x_j t_j \right\| \geq \sqrt{\lambda_1/n}.$$

Therefore, the convex hull of $\{x_1, \dots, x_n\}$ is disjoint from the open ball about the origin of radius $\sqrt{\lambda_1/n}$ in H . Since the convex hull of a finite set is closed, from the geometric form of the Hahn-Banach theorem there exists a norm-one element $x \in H$, such that

$$|\langle x, y \rangle| \geq \sqrt{\lambda_1/n}, \quad \text{for every } y \in \text{co}\{x_1, \dots, x_n\},$$

where λ_1 is the smallest eigenvalue of the Gram matrix A .

On the other hand, it follows from (2.3) that

$$\|\varepsilon_1 x_1 + \dots + \varepsilon_n x_n\| \leq \sqrt{\lambda_n n}, \quad \varepsilon_j = \pm 1,$$

where λ_n is the largest eigenvalue of the Gram matrix A . If

$$\|\varepsilon_1^* x_1 + \dots + \varepsilon_n^* x_n\| = \max_{\varepsilon_j = \pm 1} \|\varepsilon_1 x_1 + \dots + \varepsilon_n x_n\|,$$

then

$$|\langle y^* / \|y^*\|, x_k \rangle| \geq 1/\sqrt{\lambda_n n}, \quad k = 1, \dots, n,$$

where $y^* = \varepsilon_1^* y_1 + \dots + \varepsilon_n^* y_n$. □

Corollary 2.5. *Let $(x_k)_{k=1}^n$ be a sequence of norm 1 vectors in a Hilbert space $(H, \langle \cdot, \cdot \rangle)$. Then*

$$\sup_{\|x\|=1} |\langle x, x_1 \rangle \cdots \langle x, x_n \rangle| \geq \max \left\{ (\lambda_1/n)^{n/2}, (1/\lambda_n n)^{n/2} \right\},$$

where λ_1 is the smallest and λ_n is the largest eigenvalue of the Gram matrix $A = [\langle x_j, x_k \rangle]$, $j, k = 1, \dots, n$.

If $x_1, x_2, \dots, x_n$ are linearly independent vectors, $A = [\langle x_j, x_k \rangle]$ is an $n \times n$ positive definite Hermitian matrix, and in this case $\lambda_n \geq \lambda_1 > 0$.

Remark 2.6. If $x_1, x_2, \dots, x_n$ are unit vectors in the Euclidean space $\mathbb{R}^n$, it was proved in [22, 28] that

$$\sup_{\|x\|=1} |\langle x, x_1 \rangle \cdots \langle x, x_n \rangle| \geq (\lambda_1/n)^{n/2}.$$

The previous estimate can be improved. It was shown in [23] that

$$\sup_{\|x\|=1} |\langle x, x_1 \rangle \cdots \langle x, x_n \rangle| \geq \sqrt{\lambda_1 \lambda_2 \cdots \lambda_n} \cdot n^{-n/2},$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ denote the eigenvalues of the Gram matrix $A = [\langle x_j, x_k \rangle]$, $j, k = 1, \dots, n$. Notice that $\lambda_1 + \lambda_2 + \dots + \lambda_n = n$.

3. PRODUCT OF LINEAR FORMS

The simplest, and in many cases the most useful, homogeneous polynomials are obtained by multiplying linear functionals together. The discussion for general Banach spaces is facilitated by some notation.

Definition 3.1. The n th linear polarization constant of a Banach space X over $\mathbb{K}$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, is defined by

$$\begin{aligned} \mathbb{K}(n; X) &:= \inf \{ M > 0 : \|f_1\| \cdots \|f_n\| \leq M \|f_1 \cdots f_n\|, \quad \forall f_1, \dots, f_n \in X^* \} \\ &= 1 / \inf_{f_1, \dots, f_n \in S_{X^*}} \sup_{\|x\|=1} |f_1(x) \cdots f_n(x)|. \end{aligned}$$

3.1. Some basic results. The following result is a consequence of the plank problem [3].

Theorem 3.2 (Benítez-Sarantopoulos-Tonge, [7]).

- (1) If X is any Banach space, then $\mathbb{K}(n; X) \leq n^n$. In particular, if $\dim(L_1(\mu)) \geq n$, then $\mathbb{K}(n; L_1(\mu)) = n^n$.
- (2) If X is a complex Banach space with $\mathbb{K}(n; X) = n^n$, then X contains $(1 + \epsilon)$ -isomorphic copies of ℓ_1^n for all $\epsilon > 0$. Moreover, if X has dimension n , then $\mathbb{K}(n; X) = n^n$ if and only if X is isometrically isomorphic to ℓ_1^n . In this case, the only $f_k \in S_{X^*}$ ($1 \leq k \leq n$) which satisfy

$$\inf_{f_1, \dots, f_n \in S_{X^*}} \sup_{\|x\|=1} |f_1(x) \cdots f_n(x)| = n^n$$

are the coordinate functionals.

For complex $L_p(\mu)$ spaces, inequalities (1.1) imply the following result (see [28]).

Proposition 3.3 (Sz. Gy. Révész and Y. Sarantopoulos, [28]). For a complex $L_p(\mu)$ space we have

$$(3.1) \quad \mathbb{C}(n; L_p(\mu)) \leq \begin{cases} n^{n/p} & \text{if } 1 \leq p \leq 2, \\ n^{n/p'} & \text{if } 2 \leq p \leq \infty, \end{cases}$$

where $1/p + 1/p' = 1$. If $\dim(L_p(\mu)) \geq n$, then for $1 \leq p \leq 2$ we have $\mathbb{C}(n; L_p(\mu)) = n^{n/p}$.

If $(H, \langle \cdot, \cdot \rangle)$ is a real or complex Hilbert space and $x_k \in H$, $1 \leq k \leq n$, are unit vectors, we consider the problem of obtaining the best possible lower bound for the n th homogeneous polynomial $\hat{F}(x) := \langle x, x_1 \rangle \cdots \langle x, x_n \rangle$, $x \in H$. It was an open problem in [7] to determine whether

$$(3.2) \quad \sup_{\|x\|=1} |\langle x, x_1 \rangle \cdots \langle x, x_n \rangle| \geq n^{-n/2},$$

with equality if and only if $x_1, \dots, x_n$ are orthonormal. In the complex case the problem was solved by J. Arias-de-Reyna [1]. In fact, this is a corollary of K. Ball's "complex plank problem" [4, Theorem 2]).

Theorem 3.4 (J. Arias-de-Reyna, [1]). If H is a complex Hilbert space, then $\mathbb{C}(n; H) \leq n^{n/2}$. In other words, if $x_k \in H$, $1 \leq k \leq n$, are unit vectors, then

$$\sup_{\|x\|=1} |\langle x, x_1 \rangle \cdots \langle x, x_n \rangle| \geq n^{-n/2}.$$

If $\dim(H) \geq n$, then $\mathbb{C}(n; H) = n^{n/2}$.

Lemma 3.5 ([7]). If M is a closed subspace of the Banach space X and $S : X \rightarrow M$ is a continuous projection, then

$$\mathbb{K}(n; M) \leq \|S\|^n \mathbb{K}(n; X).$$

If M is a 1-complemented closed subspace of X , i.e. if there exists a projection $S : X \rightarrow M$, with $\|S\| = 1$, from the previous lemma it follows that $\mathbb{K}(n; M) \leq \mathbb{K}(n; X)$. This last observation proves the following result (cf. [16, Theorem 1]).

Proposition 3.6. Let H_1 and H_2 be two Hilbert spaces of dimension n_1 and n_2 respectively. If $n_1 \leq n_2$, then for $n \in \mathbb{N}$

$$\mathbb{K}(n; H_1) \leq \mathbb{K}(n; H_2).$$

For *real* Hilbert spaces and for small values of n , $n \leq 5$, inequality (3.2) holds; see for instance [25]. However, in general this problem is still open for real Hilbert spaces. For a discussion and various estimates, we refer to [11, 12, 16, 22, 25, 28]. To the best of our knowledge, the first occurrence of an inequality similar to (3.2), but with the lower bound n^{-n} , appeared in [20] in the case of a d -dimensional unitary space H , $n \leq d$. It was used to produce a lower bound for the permanent of a positive semi-definite Hermitian matrix in terms of its main diagonal entries. Also (see Proposition 2.4.4 in [19]), the optimal lower bound for the maximum of the product of n norms over the unit sphere S^{m-1} of $\mathbb{R}^m$ is exactly

$$c_n = \min_{x_1, \dots, x_n \in S^{m-1}} \max_{x \in S^{m-1}} |\langle x, x_1 \rangle \cdots \langle x, x_n \rangle|.$$

Remark 3.7. Let $(x_k)_{k=1}^n$ be a sequence of norm 1 vectors in a *real* Hilbert space $(H, \langle \cdot, \cdot \rangle)$. It is not known if Theorem 3.4 holds in the case of a *real* Hilbert space. However, we may consider the complex Hilbert space $\tilde{H} = H \oplus iH$, with norm $\|x + iy\| := \sqrt{\|x\|^2 + \|y\|^2}$, which is the *natural complexification* of H . If $\tilde{P} \in \mathcal{P}({}^n\tilde{H})$ is the unique complex extension of the n -homogeneous polynomial $P(x) := \langle x, x_1 \rangle \cdots \langle x, x_n \rangle$ defined on H , it is known [24] that $\|\tilde{P}\| \leq 2^{(n-2)/2} \|P\|$. Therefore,

$$\sup_{\|x\|=1} |\langle x, x_1 \rangle \cdots \langle x, x_n \rangle| \geq 2^{-(n-2)/2} \cdot n^{-n/2}.$$

In other words, $\mathbb{R}(n; H) \leq 2^{(n-2)/2} n^{n/2}$.

3.2. The permanent function. If $A = (a_{jk})$ is an $n \times n$ positive semi-definite Hermitian matrix, the *permanent* of A is defined by

$$\text{per}(A) := \sum_{\sigma \in S_n} \prod_{j=1}^n a_{j, \sigma(j)},$$

where S_n is the symmetric group on $I_n = \{1, 2, \dots, n\}$. In particular, if $A = [\langle x_j, x_k \rangle]$, $j, k = 1, \dots, n$, is the Gram matrix based on the unit vectors $x_1, \dots, x_n$ of a Hilbert space $(H, \langle \cdot, \cdot \rangle)$, then

$$\text{per}(A) := \sum_{\sigma \in S_n} \prod_{j=1}^n \langle x_j, x_{\sigma(j)} \rangle.$$

It is known that

$$(3.3) \quad 1 \leq \text{per}(A) \leq n!.$$

The first of the two inequalities in (3.3) is the Hadamard theorem for permanents, which is due to M. Marcus [21]; see also [17, 18]. It reduces to equality if and only if the vectors $x_1, \dots, x_n$ are orthonormal. The second inequality is easy to prove by using the triangle and Cauchy-Schwartz inequalities and reduces to equality if and only if $x_1 = \cdots = x_n$.

Consider now $\hat{L}(x) := \langle x, x_1 \rangle \cdots \langle x, x_n \rangle$, which is an n homogeneous polynomial on H . Then $L(x_1, \dots, x_n) = (1/n!) \text{per}(A)$, where L is the symmetric n -linear

form associated to $\widehat{L}$, and (1.4) implies

$$\begin{aligned} \sup_{\|x\|=1} |\langle x, x_1 \rangle \cdots \langle x, x_n \rangle| &= \|\widehat{L}\| \\ &\geq |L(x_1, \dots, x_n)| \\ &= (1/n!) \operatorname{per}(A) \geq 1/n!. \end{aligned}$$

However, $n^{-n/2} > 1/n!$ for $n > 2$, so this estimate is worse than (3.2). But this improves the original estimate from [20].

In general, if $\dim H < n$, the lower bound for the norm of $\widehat{L}$ can be improved.

Example 3.8 (see [2, 30]). Consider the Euclidean space $\mathbb{R}^2$. It was proved in [26] that $\operatorname{per}(A) \geq n!/2^{n-1}$ and that this lower bound is best possible. This is an improvement upon the lower bound of $\operatorname{per}(A)$ in (3.3), and therefore in $\mathbb{R}^2$ we have

$$(3.4) \quad \sup_{\|x\|=1} |\langle x, x_1 \rangle \cdots \langle x, x_n \rangle| \geq (1/n!) \operatorname{per}(A) \geq 2^{1-n}.$$

To see that 2^{1-n} is the best possible lower bound, it is enough to consider the unit vectors $x_k = (\cos k\pi/n, \sin k\pi/n) \in \mathbb{R}^2$, $k = 1, \dots, n$. Then, if $z_k = \exp(ik\pi/n)$, $k = 1, \dots, n$, for any unit vector $x = (\cos \theta, \sin \theta) \in \mathbb{R}^2$ we have

$$\max_{\|x\|=1} |\langle x, x_1 \rangle \cdots \langle x, x_n \rangle| = \frac{1}{2^n} \max_{|z|=1} \prod_{k=1}^n |z^2 + z_k^2| = \frac{1}{2^n} \max_{|z|=1} |z^{2n} + 1| = 2^{1-n}.$$

3.3. Lower bounds for the norm. Let $A = [\langle x_j, x_k \rangle]$, $j, k = 1, \dots, n$, be the Gram matrix based on the unit vectors $x_1, \dots, x_n$ of a Hilbert space $(H, \langle \cdot, \cdot \rangle)$ and let

$$\widehat{L}(x) := \langle x, x_1 \rangle \cdots \langle x, x_n \rangle$$

be the n homogeneous polynomial which is a product of linear forms on H . As we have seen before, $L(x_1, \dots, x_n) = (1/n!) \operatorname{per}(A)$, where L is the symmetric n -linear form associated to $\widehat{L}$. By using the polarization formula (1.2) we express $\operatorname{per}(A)$ in the following convenient form:

$$(3.5) \quad \operatorname{per}(A) = \int_0^1 \prod_{k=1}^n \left\langle \sum_{j=1}^n r_j(t) x_j, r_k(t) x_k \right\rangle dt.$$

If H is a complex Hilbert space, then it follows from (1.3) that

$$(3.6) \quad \operatorname{per}(A) = \int_{\mathbb{T}^n} \prod_{k=1}^n \left\langle \sum_{j=1}^n e^{i\theta_j} x_j, e^{i\theta_k} x_k \right\rangle d\lambda(\theta).$$

By using (3.5) and (3.6) we get lower estimates for the norm of $\widehat{L}$.

Proposition 3.9. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and let $x_j \in H$, $1 \leq j \leq n$, be unit vectors. If $\widehat{L}(x) := \langle x, x_1 \rangle \cdots \langle x, x_n \rangle$, then*

$$(3.7) \quad \|\widehat{L}\| \geq \operatorname{per}(A) / \int_0^1 \left\| \sum_{j=1}^n r_j(t) x_j \right\|^n dt,$$

where $A = [\langle x_j, x_k \rangle]$, $j, k = 1, \dots, n$, is the Gram matrix based on the unit vectors $x_1, \dots, x_n$. If H is a complex Hilbert space, then

$$(3.8) \quad \|\widehat{L}\| \geq \text{per}(A) / \int_{\mathbb{T}^n} \left\| \sum_{j=1}^n e^{i\theta_j} x_j \right\|^n d\lambda(\theta).$$

In order to find a lower bound for the norm of $\widehat{L}$, depending on the unit vectors $x_1, \dots, x_n$, we need a lower bound estimate for $\text{per}(A)$ as well as an upper bound estimate for $\int_0^1 \left\| \sum_{j=1}^n r_j(t) x_j \right\|^n dt$ and $\int_{\mathbb{T}^n} \left\| \sum_{j=1}^n e^{i\theta_j} x_j \right\|^n d\lambda(\theta)$, $n \geq 2$.

Lemma 3.10 (R. Grone and S. Pierce, [13]). *If $A = (a_{jk})$ is an $n \times n$ positive semi-definite Hermitian matrix with $|a_{11}| = \dots = |a_{nn}| = 1$, then*

$$(3.9) \quad \text{per}(A) \geq \frac{1}{n} \|A\|^2,$$

where $\|A\|^2 = \text{trace}(A^*A) = \sum_{j,k} |a_{j,k}|^2$ is the Frobenius norm or the Schatten 2-norm or the Hilbert-Schmidt norm of A . If $A = I_n$, we have equality in (3.9).

Observe that for $n = 2$, $\text{per}(A) = \frac{1}{2} \|A\|^2$.

Theorem 3.11. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and let $x_j \in H$, $1 \leq j \leq n$, be unit vectors. If $\widehat{L}(x) := \langle x, x_1 \rangle \cdots \langle x, x_n \rangle$, for $n = 2, 3, 4$ we have*

$$(3.10) \quad \|\widehat{L}\| \geq \frac{1}{n^{n/2}} \cdot \frac{1 + (1/n) \sum_{j \neq k} |\langle x_j, x_k \rangle|^2}{\left(1 + (c/n^2) \sum_{j \neq k} |\langle x_j, x_k \rangle|^2\right)^{(n-2)/2}},$$

with $c = 2$ if H is a real Hilbert space and $c = 1$ if H is a complex Hilbert space. In particular, $\|\widehat{L}\| \geq 1/n^{n/2}$ for $n = 2, 3, 4$, and we have equality if and only if $x_1, \dots, x_n$ are orthonormal vectors.

Proof. The case $n = 2$ is easy to prove, and so is the case $n = 4$. We prove only the case $n = 3$. If H is a real Hilbert space, by using Hölder's inequality we have

$$\begin{aligned} \int_0^1 \left\| \sum_{j=1}^3 r_j(t) x_j \right\|^3 dt &= \int_0^1 \left\| \sum_{j=1}^3 r_j(t) x_j \right\|^2 \cdot \left\| \sum_{j=1}^3 r_j(t) x_j \right\| dt \\ &\leq \left(\int_0^1 \left\| \sum_{j=1}^3 r_j(t) x_j \right\|^4 dt \right)^{1/2} \left(\int_0^1 \left\| \sum_{j=1}^3 r_j(t) x_j \right\|^2 dt \right)^{1/2} \\ &= \left(3^2 + 2 \sum_{j \neq k} |\langle x_j, x_k \rangle|^2 \right)^{1/2} 3^{1/2} \\ &= 3^{3/2} \left(1 + \frac{2}{3^2} \sum_{j \neq k} |\langle x_j, x_k \rangle|^2 \right)^{1/2}. \end{aligned}$$

Hence,

$$\int_0^1 \left\| \sum_{j=1}^3 r_j(t) x_j \right\|^3 dt \leq 3^{3/2} \left(1 + \frac{2}{3^2} \sum_{j \neq k} |\langle x_j, x_k \rangle|^2 \right)^{1/2}.$$

If H is a *complex* Hilbert space, working in a similar fashion we have

$$\int_{\mathbb{T}^n} \left\| \sum_{j=1}^3 e^{i\theta_j} x_j \right\|^3 d\lambda(\theta) \leq 3^{3/2} \left(1 + \frac{1}{3^2} \sum_{j \neq k} |\langle x_j, x_k \rangle|^2 \right)^{1/2}.$$

Now, the proof of (3.10) for $n = 3$ follows from (3.7) and (3.8) by using the last two inequalities and inequality (3.9) with $a_{jk} = \langle x_j, x_k \rangle$, $1 \leq j, k \leq n$. Finally, it is easy to see that $\|\widehat{L}\| = 1/n^{n/2}$, for $n = 2, 3, 4$, if and only if $x_1, \dots, x_n$ are orthonormal vectors. $\square$

If $x_1, \dots, x_n \in H$, working as in the previous proof we can get upper estimates for $\int_0^1 \left\| \sum_{j=1}^n r_j(t) x_j \right\|^n dt$, $n \geq 5$. But it gets increasingly complicated as n increases. Alternatively, we may think of using the well-known Khintchine-Kahane inequality which states that for $0 < p < q < \infty$ there exists a constant $K(p, q) \geq 1$ such that

$$\left(\int_0^1 \left\| \sum_{j=1}^n r_j(t) x_j \right\|^q dt \right)^{1/q} \leq K(p, q) \left(\int_0^1 \left\| \sum_{j=1}^n r_j(t) x_j \right\|^p dt \right)^{1/p},$$

for $x_1, \dots, x_n \in X$ and X any Banach space X . If $1 < p < q < \infty$, we may take $K(p, q) = (q-1)^{1/2} (p-1)^{-1/2}$ (see [9, 27]). In particular, if $x_1, \dots, x_n$ are unit vectors in a Hilbert space H , then for $q = n \in \mathbb{N}$, $n > 4$, and $p = 4$, the previous inequality implies

$$\begin{aligned} \int_0^1 \left\| \sum_{j=1}^n r_j(t) x_j \right\|^n dt &\leq \left(\frac{n-1}{3} \right)^{n/2} \left(\int_0^1 \left\| \sum_{j=1}^n r_j(t) x_j \right\|^4 dt \right)^{n/4} \\ &\leq \left(\frac{n-1}{3} \right)^{n/2} \left(n^2 + 2 \sum_{j \neq k} |\langle x_j, x_k \rangle|^2 \right)^{n/4} \\ (3.11) \quad &= \left(\frac{n^2 - n}{3} \right)^{n/2} \left(1 + \frac{2}{n^2} \sum_{j \neq k} |\langle x_j, x_k \rangle|^2 \right)^{n/4}. \end{aligned}$$

By using (3.11) and inequality (3.9) with $a_{ij} = \langle x_i, x_j \rangle$, $1 \leq i, j \leq n$, it follows from (3.7) that for $n \geq 4$

$$(3.12) \quad \|\widehat{L}\| \geq \frac{3^{n/2}}{(n^2 - n)^{n/2}} \cdot \frac{1 + \frac{1}{n} \sum_{j \neq k} |\langle x_j, x_k \rangle|^2}{\left(1 + \frac{2}{n^2} \sum_{j \neq k} |\langle x_j, x_k \rangle|^2 \right)^{n/4}}.$$

But this last estimate is far from optimal. So, in order to generalize Theorem 3.11, for $n > 4$, we need the following inequality which is a standard type of interpolation lemma.

Lemma 3.12. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and let $x_j \in H$, $1 \leq j \leq n$, $n \geq 4$, be unit vectors. If H is a real Hilbert space, then*

$$(3.13) \quad \int_0^1 \left| n + 2 \sum_{1 \leq j < k \leq n} r_j(t) r_k(t) \langle x_j, x_k \rangle \right|^{n/2} dt \leq \left(n^{n/(n-2)} + 2^{2/(n-2)} \sum_{j \neq k} |\langle x_j, x_k \rangle|^{n/(n-2)} \right)^{(n-2)/2}.$$

On the other hand, if H is a complex Hilbert space,

$$(3.14) \quad \int_{\mathbb{T}^n} \left| n + \sum_{j \neq k} e^{i(\theta_j - \theta_k)} \langle x_j, x_k \rangle \right|^{n/2} d\lambda(\theta) \leq \left(n^{n/(n-2)} + \sum_{j \neq k} |\langle x_j, x_k \rangle|^{n/(n-2)} \right)^{(n-2)/2}.$$

Proof. In the real case consider the linear operator $S : \ell_2^{n^2} \rightarrow L_2([0, 1], dt)$, defined by

$$(*) \quad S : y = \sum_{j,k=1}^n y_{jk} e_{n(j-1)+k} \mapsto y_{11} + \sum_{1 \leq j < k \leq n} r_j(t) r_k(t) y_{jk},$$

where $(e_m)_{m=1}^{n^2}$ is the standard unit vector basis for $\ell_2^{n^2}$. We have

$$\begin{aligned} \|Sx\| &= \left(\int_0^1 \left| y_{11} + \sum_{1 \leq j < k \leq n} r_j(t) r_k(t) y_{jk} \right|^2 dt \right)^{1/2} \\ &= \left(|y_{11}|^2 + \sum_{1 \leq j < k \leq n} |y_{jk}|^2 \right)^{1/2} \\ &\leq \left(\sum_{j,k=1}^n |y_{jk}|^2 \right)^{1/2}. \end{aligned}$$

Next, if the linear operator $S : \ell_1^{n^2} \rightarrow L_\infty([0, 1], dt)$ is defined by $(*)$, we have

$$\begin{aligned} \|Sx\| &= \sup_{0 \leq t \leq 1} \left| y_{11} + \sum_{1 \leq j < k \leq n} r_j(t) r_k(t) y_{jk} \right| \\ &\leq |y_{11}| + \sum_{1 \leq j < k \leq n} |y_{jk}| \\ &\leq \sum_{j,k=1}^n |y_{jk}|. \end{aligned}$$

Hence, by the Riesz-Thorin interpolation theorem ([8, Theorem 1.1.1]),

$$\left(\int_0^1 \left| y_{11} + \sum_{1 \leq j < k \leq n} r_j(t) r_k(t) y_{jk} \right|^{n/2} dt \right)^{2/n} \leq \left(\sum_{j,k=1}^n |y_{jk}|^{n/(n-2)} \right)^{(n-2)/n}.$$

Now, for $y_{11} = n$, $y_{jk} = 2\langle x_j, x_k \rangle$, $1 \leq j < k \leq n$, and $y_{jk} = 0$ in all other cases, this last inequality implies (3.13) for $n \geq 4$.

In the complex case we consider the linear operator $T : \ell_p^{n^2} \rightarrow L_{p'}(\mathbb{T}^n)$, with $p = 1, 2$ and $p' = \infty, 2$ respectively, defined by

$$(**) \quad T : y = \sum_{j,k=1}^n y_{jk} e_{n(j-1)+k} \mapsto y_{11} + \sum_{\substack{j,k=1 \\ j \neq k}}^n e^{i(\theta_j - \theta_k)} y_{jk}.$$

Working as before, it follows from the Riesz-Thorin interpolation theorem that

$$\left(\int_{\mathbb{T}^n} \left| y_{11} + \sum_{\substack{j,k=1 \\ j \neq k}}^n e^{i(\theta_j - \theta_k)} y_{jk} \right|^{n/2} d\lambda(\theta) \right)^{2/n} \leq \left(\sum_{j,k=1}^n |y_{jk}|^{n/(n-2)} \right)^{(n-2)/n}.$$

This last inequality with $y_{11} = n$, $y_{jk} = \langle x_j, x_k \rangle$, $1 \leq j \neq k \leq n$, and $y_{22} = \dots = y_{nn} = 0$ implies (3.14) for $n \geq 4$. $\square$

Theorem 3.13. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and let $x_j \in H$, $1 \leq j \leq n$, be unit vectors. If $\widehat{L}(x) := \langle x, x_1 \rangle \cdots \langle x, x_n \rangle$, for $n \geq 4$ we have*

$$(3.15) \quad \|\widehat{L}\| \geq \frac{1}{n^{n/2}} \cdot \frac{1 + (1/n) \sum_{j \neq k} |\langle x_j, x_k \rangle|^2}{\left(1 + (c_n/n)^{n/(n-2)} \sum_{j \neq k} |\langle x_j, x_k \rangle|^{n/(n-2)} \right)^{(n-2)/2}},$$

with $c_n = 2^{2/n}$ if H is a real Hilbert space and $c_n = 1$ if H is a complex Hilbert space.

Proof. Let H be a real Hilbert space. Since

$$\int_0^1 \left\| \sum_{i=1}^n r_i(t) x_i \right\|^n dt = \int_0^1 \left| n + 2 \sum_{1 \leq j < k \leq n} r_j(t) r_k(t) \langle x_j, x_k \rangle \right|^{n/2} dt,$$

the proof of (3.15), with $c_n = 2^{2/n}$, follows from (3.7) by using (3.13) and inequality (3.9) with $a_{ij} = \langle x_i, x_j \rangle$, $1 \leq i, j \leq n$.

If H is a complex Hilbert space, then

$$\begin{aligned} \int_{\mathbb{T}^n} \left\| \sum_{j=1}^n e^{i\theta_j} x_j \right\|^n d\lambda(\theta) &= \int_{\mathbb{T}^n} \left(n + \sum_{j \neq k} e^{i(\theta_j - \theta_k)} \langle x_j, x_k \rangle \right)^{n/2} d\lambda(\theta) \\ &\leq \int_{\mathbb{T}^n} \left| n + \sum_{j \neq k} e^{i(\theta_j - \theta_k)} \langle x_j, x_k \rangle \right|^{n/2} d\lambda(\theta). \end{aligned}$$

Now, the proof of (3.15), with $c_n = 1$, follows from (3.8) by using (3.14) and inequality (3.9) with $a_{ij} = \langle x_i, x_j \rangle$, $1 \leq i, j \leq n$. $\square$

Corollary 3.14. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and let $x_j \in H$, $1 \leq j \leq n$, be unit vectors. Then, for n large enough*

$$\sup_{\|x\|=1} |\langle x, x_1 \rangle \cdots \langle x, x_n \rangle| > \frac{1}{n2^{n/4}} \cdot \frac{1}{n^{n/2}}.$$

In particular, the asymptotic growth of the linear polarization constant $\mathbb{R}(n; H)$ of a real Hilbert space H is less than $n2^{n/4} \cdot n^{n/2}$.

Proof. By using Hölder's inequality it follows from (3.15) that

$$\begin{aligned} \|\widehat{L}\| &\geq \frac{1}{n^{n/2}} \cdot \frac{1 + (1/n) \sum_{j \neq k} |\langle x_j, x_k \rangle|^2}{\left(1 + (c_n/n)^2 \sum_{j \neq k} |\langle x_j, x_k \rangle|^2\right)^{n/4} (n^2 - n + 1)^{(n-4)/2(n-2)}} \\ &> \frac{1}{n^{n/2}} \cdot \frac{1}{\left(1 + (c_n/n)^2 \sum_{j \neq k} |\langle x_j, x_k \rangle|^2\right)^{n/4-1} (n^2 - n + 1)^{(n-4)/2(n-2)}}. \end{aligned}$$

But

$$\sum_{\substack{j,k=1 \\ j \neq k}}^n |\langle x_j, x_k \rangle|^2 \leq n^2 - n,$$

and therefore

$$\begin{aligned} \|\widehat{L}\| &> \frac{1}{n^{n/2}} \cdot \frac{1}{\left(1 + (c_n/n)^2 (n^2 - n)\right)^{n/4-1} (n^2 - n + 1)^{(n-4)/2(n-2)}} \\ &= \frac{1}{n^{n/2}} \cdot \frac{(1 + c_n^2 (1 - 1/n)) n^{2/(n-2)}}{(1 + c_n^2 (1 - 1/n))^{n/4} n (1 - 1/n + 1/n^2)^{(n-4)/2(n-2)}}. \end{aligned}$$

Since

$$(1 + c_n^2 (1 - 1/n))^{n/4} \sim c2^{n/4},$$

with $c = \sqrt{2}e^{-1/8}$ if $c_n = 2^{2/n}$ and $c = e^{-1/8}$ if $c_n = 1$, for n large enough we finally have

$$\|\widehat{L}\| > \frac{1}{n^{n/2}} \cdot \frac{1}{n2^{n/4}}.$$

□

Remark 3.15. If $(H, \langle \cdot, \cdot \rangle)$ is a Hilbert space and if $x_j \in H$, $1 \leq j \leq n$, are unit vectors, it is obvious that

$$(3.16) \quad 0 \leq \sum_{\substack{j,k=1 \\ j \neq k}}^n |\langle x_j, x_k \rangle|^2 \leq n^2 - n.$$

The first of the two inequalities in (3.16) reduces to equality if and only if the vectors $x_1, \dots, x_n$ are orthonormal. Since $|\langle x_j, x_k \rangle| \leq 1$, the second inequality reduces to equality if and only if $x_1 = \dots = x_n$. Notice that if $x_1, \dots, x_n$ are orthonormal vectors, then $\sup_{\|x\|=1} |\langle x, x_1 \rangle \cdots \langle x, x_n \rangle| = n^{n/2}$, while in the case $x_1 = \dots = x_n$ it is obvious that $\sup_{\|x\|=1} |\langle x, x_1 \rangle \cdots \langle x, x_n \rangle| = 1$. In the special case for the unit vectors $x_1, \dots, x_n$ we have an estimate of the form

$$\sum_{\substack{j,k=1 \\ j \neq k}}^n |\langle x_j, x_k \rangle|^2 \leq cn,$$

for some constant $c > 0$. Then from the proof of Corollary 3.14 it follows that the asymptotic growth of $\mathbb{K}(n; H)$ is $nn^{n/2}$.

Remark 3.16. Observe that our asymptotic estimate “ $n2^{n/4} \cdot n^{n/2}$ ” for the linear polarization constant $\mathbb{R}(n; H)$ of a *real* Hilbert space H improves the ones given in [11, 28].

REFERENCES

- [1] J. Arias-de-Reyna, Gaussian variables, polynomials and permanents, *Linear Algebra Appl.* 285 (1998) 107–114. MR1653495 (2000a:15008)
- [2] V. Anagnostopoulos and Sz. Révész, Polarization constants for products of linear functionals over $\mathbb{R}^2$ and $\mathbb{C}^2$ and Chebyshev constants of the unit sphere, *Publ. Math. Debrecen* 68(1-2) (2006) 63–75. MR2213542 (2006m:46054)
- [3] K. M. Ball, The plank problem for symmetric bodies, *Invent. Math.* 104 (1991) 535–543. MR1106748 (92c:52003)
- [4] ———, The complex plank problem, *Bull. London Math. Soc.* 33 (2001) 433–442. MR1832555 (2002b:46023)
- [5] S. Banach, Über homogene Polynome in (L^2) , *Studia Math.* 7 (1938) 36–44.
- [6] T. Bang, A solution of the “plank problem”, *Proc. Amer. Math. Soc.* 2 (1951) 990–993. MR0046672 (13:769a)
- [7] C. Benítez, Y. Sarantopoulos and A.M. Tonge, Lower bounds for norms of products of polynomials, *Math. Proc. Cambridge Philos. Soc.* 124 (1998) 395–408. MR1636556 (99h:46077)
- [8] J. Bergh and J. Löfström, Interpolation spaces. An introduction, *Grundlehren der Mathematischen Wissenschaften*, No. 223. Springer-Verlag, Berlin-New York, 1976. MR0482275 (58:2349)
- [9] C. Borell, On the integrability of Banach space valued Walsh polynomials, *Séminaire de Probabilité, XIII* (Univ. Strasbourg, Strasbourg, 1977/78), pp. 1–3, *Lecture Notes in Math.*, 721, Springer, Berlin, 1979. MR544777 (81g:42026)
- [10] S. Dineen, *Complex Analysis on Infinite Dimensional Spaces*, Springer, London, 1999. MR1705327 (2001a:46043)
- [11] P. E. Frenkel, Hafnians and products of real linear functionals, *Math. Res. Lett.* 15(2) (2008) 351–358. MR2385646 (2008m:15016)
- [12] J. C. García-Vázquez and R. Villa, Lower bounds for multilinear forms defined on Hilbert spaces, *Mathematika* 46 (1999) 315–322. MR1832622 (2002c:46045)
- [13] R. Grone and S. Pierce, Permanent inequalities for correlation matrices, *SIAM J. Matrix Anal. Appl.* 9(2) (1988) 194–201. MR938498 (89i:15009)
- [14] L. A. Harris, Bounds on the derivatives of holomorphic functions of vectors, in: *Colloque d’Analyse* (Rio de Janeiro, 1972), L. Nachbin (ed.), *Actualités Sci. Indust.* 1367. Hermann, Paris, 1975, 145–163. MR0477773 (57:17283)
- [15] R. A. Horn and C. R. Johnson, *Matrix Analysis* (corrected reprint of the 1985 original), Cambridge University Press, Cambridge, 1990. MR1084815 (91i:15001)
- [16] Y.J. Leung, W.V. Li and Rakesh, The d th linear polarization constant of $\mathbb{R}^d$, *J. Funct. Anal.* 255(10) (2008) 2861–2871. MR2464193
- [17] E. H. Lieb, Proofs of some conjectures on permanents, *J. Math. Mech.* 16 (1966) 127–134. MR0202745 (34:2605)
- [18] ———, *Inequalities: Selecta of Elliot H. Lieb*, Springer-Verlag, Berlin, 2002. MR1922236 (2003f:01063)
- [19] A. E. Litvak, V. D. Milman and G. Schechtman, Averages of norms and quasi-norms, *Math. Ann.* 312 (1998) 95–124. MR1645952 (2000c:46013)
- [20] M. Marcus and H. Minc, The Pythagorean theorem in certain symmetry classes of tensors, *Trans. Amer. Math. Soc.* 104 (1962) 510–515. MR0139626 (25:3058)
- [21] M. Marcus, The permanent analogue of the Hadamard determinant theorem, *Bull. Amer. Math. Soc.* 69 (1963) 494–496. MR0153688 (27:3649)
- [22] ———, A lower bound for the product of linear forms, *Linear and Multilinear Algebra* 43 (1997) 115–120. MR1613093 (98k:15019)
- [23] M. A. Matolcsi, Geometric estimate on the norm of product of functionals, *Linear Algebra Appl.* 405 (2005) 304–310. MR2148177 (2006d:46005b)

- [24] G. Muñoz, Y. Sarantopoulos and A.M. Tonge, Complexifications of real Banach spaces, polynomials and multilinear maps, *Studia Math.* 134 (1999) 1–33. MR1688213 (2000g:46009)
- [25] A. Pappas and Sz. Révész, Linear polarization constants of Hilbert spaces, *J. Math. Anal. Appl.* 300 (2004), no. 1, 129–146. MR2100242 (2005h:46072)
- [26] T. H. Pate, The best lower bound for the permanent of a correlation matrix of rank two, *Linear and Multilinear Algebra* 51 (2003) 263–278. MR1995658 (2004h:15010)
- [27] G. Pisier, Les inégalités de Khintchine-Kahane, d’après C. Borell (French). *Séminaire sur la Géométrie des Espaces de Banach* (1977–1978), Exp. No. 7, 14 pp., École Polytech., Palaiseau, 1978. MR520209 (81c:60005)
- [28] Sz. Révész and Y. Sarantopoulos, Plank problems, polarization and Chebyshev constants, *J. Korean Math. Soc.* 41(1) (2004) 157–174. MR2048707 (2004m:46033)
- [29] R. A. Ryan and B. Turett, Geometry of spaces of polynomials, *J. Math. Anal. Appl.* 221 (1998) 698–711. MR1621703 (99g:46015)
- [30] Y. Sarantopoulos and A. M. Tonge, Homogeneous polynomials and Hardy-Hilbert’s inequality (preprint).

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