TANGENCIES BETWEEN HOLOMORPHIC MAPS AND HOLOMORPHIC LAMINATIONS

A. EREMENKO AND A. GABRIELOV

(Communicated by Mario Bonk)

ABSTRACT. We prove that the set of leaves of a holomorphic lamination of codimension one that are tangent to a germ of a holomorphic map is discrete.

Let F be a *holomorphic lamination* of codimension one in an open set V in a complex Banach space B . In this paper, this means that $V = W \times \mathbf{C}$, where W is a neighborhood of the origin in some Banach space, and the leaves L_λ of the lamination are disjoint graphs of holomorphic functions $\mathbf{w} \mapsto f(\lambda, \mathbf{w})$, $W \rightarrow \mathbf{C}$. For holomorphic functions in a Banach space we refer to [5]. Here λ is a parameter and we assume that the dependence of f on λ is continuous. A natural choice of this parameter is such that $\lambda = f(\lambda, 0)$, in which case the continuity with respect to λ follows from the so-called λ -lemma of Mane-Sullivan-Sad and Lyubich; see, for example, [5]. With this choice of the parameter, our definition of a lamination coincides with that of a holomorphic motion of $\mathbf{C}$ parametrized by W .

Let $\gamma : U \rightarrow V$ be a holomorphic map with $U \subset \mathbf{C}^n$. We say that γ is *tangent* to the lamination at a point $\mathbf{z}_0 \in U$ if the image of the derivative $\gamma'(\mathbf{z}_0)$ is contained in the tangent space $T_L(\gamma(\mathbf{z}_0))$, where L is the leaf passing through $\gamma(\mathbf{z}_0)$. A leaf for which this holds is called a *tangent leaf* to γ .

Theorem. *Let K be a compact subset of U . Then the set of leaves tangent to γ at the points of K is finite.*

For the case of holomorphic curves ($n = 1$) this result is contained in [1, Lemma 9.1] where it is credited to Douady. Artur Avila, in a conversation with the authors, proposed to extend this result to arbitrary holomorphic maps. According to Avila, this generalization has several applications to holomorphic dynamics [1].

Proof. We assume without loss of generality that

$$f(0, \mathbf{w}) \equiv 0$$

and that L_0 is tangent to γ at $\mathbf{z}_0 = 0$.

We have to show that other tangent leaves cannot accumulate to L_0 . Suppose the contrary; that is, suppose that there is a sequence $\lambda_k \rightarrow 0$ such that L_{λ_k} are tangent to γ , and let L_{λ_k} be the graphs of the functions $f_k(\mathbf{w}) = f(\lambda_k, \mathbf{w})$. We may assume that tangency points $\mathbf{z}_k \rightarrow 0$.

Received by the editors October 25, 2008, and, in revised form, October 28, 2009.

2010 *Mathematics Subject Classification.* Primary 32B99; Secondary 37F99.

The first author was supported by NSF grant DMS-0555279.

The second author was supported by NSF grant DMS-0801050.

©2010 American Mathematical Society
 Reverts to public domain 28 years from publication

We make several preliminary reductions. □

1. Let

$$\gamma(\mathbf{z}) = (\phi(\mathbf{z}), \psi(\mathbf{z})) \in W \times \mathbf{C}.$$

Consider the new lamination in $U \times \mathbf{C}$ whose leaves are the graphs of $f^*(\lambda, \mathbf{z}) = f(\lambda, \phi(\mathbf{z}))$ and the new map $\gamma^*(\mathbf{z}) = (\mathbf{z}, \psi(\mathbf{z}))$. Then γ^* is tangent to a leaf L^* if and only if γ is tangent to L . This reduces our problem to the case where W is an open set in $\mathbf{C}^n$ and the map γ is a graph of a function ψ of the same variable as the functions f_k . From now on we assume that $U = W$ and $\gamma(\mathbf{w}) = (\mathbf{w}, \psi(\mathbf{w}))$.

2. Now we reduce the problem to the case where ψ is a monomial. For this we use the desingularization theorem of Hironaka [4, 2, 3, 6].

Let X be a complex analytic manifold, and let ψ be an analytic function on X . Then there exists a complex analytic manifold M and a proper surjective map $\pi : M \rightarrow X$ such that the restriction of π onto the complement of the π -preimage of the set $\{\psi = 0, \psi' = 0\}$ is injective and for each point $\mathbf{z}_0 \in \pi^{-1}(\{\psi = 0\})$ there is a local coordinate system with the origin at $\mathbf{z}_0$ such that $\psi \circ \pi$ is a monomial $z_1^{m_1} \dots z_n^{m_n}$. We choose such M corresponding to $X = W$.

Let $Y = W \times \mathbf{C}$, and let $S \subset Y$ be the set of points $(\mathbf{w}, t)$ with $t \neq 0$ such that the graph of $t = \psi(\mathbf{w})$ is tangent to the lamination. In our proof by contradiction, we assume that the origin belongs to the closure of S . Let $N = M \times \mathbf{C}$, and let $\rho : N \rightarrow Y$ be the map defined by $\rho(\mathbf{z}, t) = (\pi(\mathbf{z}), t)$. Then $\rho^{-1}(F)$ is the lamination whose leaves are the components of the ρ -preimages of the leaves of F , and the set $T = \rho^{-1}(S)$ has a limit point $(\mathbf{z}_0, 0)$ with $\pi(\mathbf{z}_0) = 0$ since π is proper. Also, the set T is exactly the set of those points in N where the graph $t = \psi \circ \pi(\mathbf{z})$ is tangent to the lamination $\rho^{-1}(F)$ since ρ is injective in a neighborhood of each point of T . (Any point $(\mathbf{z}, t)$ where ρ is not injective satisfies $\psi(\pi(\mathbf{z})) = 0$, while at every point of T we have $\psi(\pi(\mathbf{z})) \neq 0$.) This reduces our problem to the case where ψ is a monomial.

3. We may assume now that $W = \{\mathbf{z} : |\mathbf{z}| < 2\}$. Thus, we are in the situation

$$\psi(z) = z_1^{m_1} \dots z_n^{m_n},$$

and $\{f_k\}$ is a family of holomorphic functions on W with disjoint graphs, $f_k(\mathbf{z}) \neq 0$ for $\mathbf{z} \in W$, and $f_k \rightarrow 0$ as $k \rightarrow \infty$ uniformly on W . Moreover, for some sequence $\mathbf{z}_k \rightarrow 0$ we have

$$(1) \quad f_k(\mathbf{z}_k) = \prod_{j=1}^n z_{j,k}^{m_j},$$

$$(2) \quad \text{grad } f_k(\mathbf{z}_k) = (m_1 z_{1,k}^{m_1-1} z_{2,k}^{m_2} \dots, m_2 z_{1,k}^{m_1} z_{2,k}^{m_2-1} \dots),$$

assuming zero values for the components with $m_j = 0$. We may assume that $|f_k| \leq 1$ in W . Setting $f_k = \exp g_k$ we obtain that $\Re g_k \leq 0$. Now we put

$$h_k(\mathbf{z}) = g_k(\mathbf{z} + \mathbf{z}_k) - g_k(\mathbf{z}_k).$$

Then the h_k are defined in the unit ball and satisfy

$$\Re h_k(\mathbf{z}) \leq \sum_{j:m_j>0} m_j \log |z_{j,k}|^{-1}, h_k(0) = 0.$$

From this we conclude that

$$(3) \quad \left| \frac{\partial h_k}{\partial z_j}(0) \right| \leq 2 \sum_{j:m_j>0} m_j \log |z_{j,k}|^{-1}.$$

This follows from the following lemma.

Lemma. *Let h be an analytic function in the unit disc, $h(0) = 0$, and $\Re h \leq A$, where $A > 0$. Then $|h'(0)| \leq 2A$.*

Proof. The function

$$\phi(z) = \frac{z}{2Az - z}$$

maps the half-plane $\{\Re z < A\}$ onto the unit disc, $\phi(0) = 0$, and $\phi'(0) = 1/(2A)$. Now the statement follows from the Schwarz lemma applied to $\phi \circ h$ and the chain rule:

$$|(\phi \circ h)'(0)| = |\phi'(0)h'(0)| = |h'(0)/2A| \leq 1.$$

□

On the other hand, (1) and (2) imply that, for $m_j > 0$,

$$(4) \quad \left| \frac{\partial h_k}{\partial z_j}(0) \right| = \frac{m_j}{|z_{j,k}|}.$$

Assume without loss of generality that $m_1 > 0$ and

$$|z_{1,k}| = \min_{j:m_j>0} |z_{j,k}|.$$

Then the right-hand side of (3) is at most

$$\text{const} \log |z_{1,k}|^{-1},$$

while the right-hand side of (4) for $j = 1$ is

$$\frac{m_1}{|z_{1,k}|}.$$

As $|z_{1,k}| \rightarrow 0$, we obtain a contradiction which proves our theorem.

ACKNOWLEDGMENT

We thank the referee for useful comments.

REFERENCES

- [1] A. Avila, M. Lyubich and W. de Melo, Regular or stochastic dynamics in real analytic families of unimodal maps, *Invent. Math.* 154 (2003), no. 3, 451–550. MR2018784 (2006i:37083)
- [2] E. Bierstone and P.D. Milman, Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant, *Invent. Math.* 128 (1997), 207–302. MR1440306 (98e:14010)
- [3] S. Encinas and O. Villamayor, A new proof of desingularization over fields of characteristic zero, *Revista Matemática Iberoamericana* 19 (2003), 339–353. MR2023188 (2004m:14017)
- [4] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero. I, *Ann. of Math. (2)* 79 (1964), 109–203; II, *ibid.* (2) 79 (1964), 205–326. MR0199184 (33:7333)

- [5] J. Hubbard, Teichmüller theory and applications to geometry, topology, and dynamics, Vol. 1, Matrix Editions, Ithaca, NY, 2006. MR2245223 (2008k:30055)
- [6] J. Włodarczyk, Simple Hironaka resolution in characteristic zero, J. Amer. Math. Soc. 18 (4) (2005), 779-822. MR2163383 (2006f:14014)

DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47907
E-mail address: `eremenko@math.purdue.edu`

DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47907
E-mail address: `agabriel@math.purdue.edu`