INVARIANT SUBSPACES IN BERGMAN SPACE OVER THE BIDISC

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ABSTRACT. In this paper, we investigate the doubly commuting condition restricted to invariant subspaces of the Bergman space over the bidisc. This condition was first introduced by V. Mandrekar in the setting of the Hardy space over the bidisc.

1. INTRODUCTION

Let $\mathbb{D}$ denote the unit disc in the complex plane. We use $H(\mathbb{D})$ to denote the collection of all functions holomorphic in $\mathbb{D}$. Let $H^2(\mathbb{D})$ denote the classical Hardy space. That is,

$$H^2(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) : \sup_{0 \leq r < 1} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 d\theta < \infty \right\}.$$

In [3], A. Beurling showed that if $\mathcal{M}$ is a (closed) subspace of $H^2(\mathbb{D})$ for which multiplication by the coordinate function maps $\mathcal{M}$ into itself, then $\mathcal{M}$ is of the form $\phi H^2(\mathbb{D})$ where ϕ is an inner function.

It is natural to consider the validity of Beurling's theorem in the unit bidisc. First, let us recall the definition of the classical Hardy spaces over the unit bidisc. The bidisc, $\mathbb{D}^2$, is the cartesian product of two copies of the unit disc. We use the notation $H(\mathbb{D}^2)$ to denote the collections of all functions holomorphic in $\mathbb{D}^2$. Holomorphic in $\mathbb{D}^2$ means holomorphic in each variable. The Hardy space over the bidisc is then given by

$$H^2(\mathbb{D}^2) = \left\{ f \in H(\mathbb{D}^2) : \sup_{0 \leq r < 1} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |f(re^{i\theta_1}, re^{i\theta_2})|^2 d\theta_1 d\theta_2 < \infty \right\}.$$

The question of the validity of Beurling's theorem in the bidisc becomes the following question. Is every subspace $\mathcal{M}$ of $H^2(\mathbb{D}^2)$ that is invariant under multiplication by each of the coordinate functions also generated by a single inner function? The answer to this question is no! W. Rudin showed in [6] that there are invariant subspaces $\mathcal{M}$ in $H^2(\mathbb{D}^2)$ that are not even finitely generated. Further, he showed that there are invariant subspaces of $H^2(\mathbb{D}^2)$ that contain no bounded elements,

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again in contrast with the one variable case, where every invariant subspace contains and is generated by a bounded function. One may then ask, what additional conditions are required so that $\mathcal{M}$ is generated by a single inner functions? In [5], V. Mandrekar answered this question by proving the following theorem.

Theorem 1 (Mandrekar). *Suppose $\mathcal{M}$ is a subspace of $H^2(\mathbb{D}^2)$ invariant under T_z and T_w . Then T_z and T_w doubly commute on $\mathcal{M}$ if and only if*

$$\mathcal{M} = \phi H^2(\mathbb{D}^2),$$

where ϕ is an inner function.

In the above theorem we used T_z and T_w to denote multiplication by z and w , respectively. We say T_z and T_w doubly commute on $\mathcal{M}$ when these operators commute with each other and with each other's adjoint.

Mandrekar's proof of the above theorem relied on a multivariate version of the Wold decomposition. This multivariate decomposition, due to Słociński [8], says that if T_1 and T_2 are bounded linear operators on a Hilbert space $\mathcal{H}$ for which T_1 and T_2 are isometry, doubly commute and $\bigcap_{n=0}^{\infty} T_i^n(\mathcal{H}) = \{0\}$, $i = 1, 2$, then

$$\mathcal{H} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \oplus T_1^n T_2^m ((\mathcal{H} \ominus T_1(\mathcal{H})) \cap (\mathcal{H} \ominus T_2(\mathcal{H}))).$$

It is precisely this doubly commuting condition, introduced by Mandrekar, that we wish to investigate in the Bergman space over the bidisc. Before we begin this investigation, we recall some definitions and known results regarding invariant subspaces in the Bergman space.

First, we recall that the Bergman space over the unit disc is defined by

$$A^2(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) : \int_0^1 \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 r d\theta dr < \infty \right\}.$$

In [1], A. Aleman, S. Richter and C. Sundberg showed that Beurling's theorem held in $A^2(\mathbb{D})$ in the following sense.

Theorem 2 (Aleman, Richter and Sundberg). *Let $\mathcal{M}$ be an invariant subspace of $A^2(\mathbb{D})$. Then $\mathcal{M} = [\mathcal{M} \ominus z\mathcal{M}]$.*

In the above theorem, $[\mathcal{M} \ominus z\mathcal{M}]$ denotes the smallest subspace of $A^2(\mathbb{D})$ that contains the set $\mathcal{M} \ominus z\mathcal{M}$ and is invariant under multiplication by z . In contrast with Beurling's theorem, where $\mathcal{M} \ominus z\mathcal{M}$ is spanned by a single inner function, C. Apostol, H. Bercovici, C. Foias and C. Pearcy in [2] showed that if n is any positive integer or ∞ , then there exists an invariant subspace $\mathcal{M}$ of $A^2(\mathbb{D})$ such that $\dim(\mathcal{M} \ominus z\mathcal{M}) = n$.

Aleman, Richter and Sundberg gave a very involved, "hard" analysis proof of Theorem 2. Recently, in [7], S. Shimorin gave a "soft" analysis proof of Theorem 2, by first proving a decomposition theorem similar to the aforementioned Wold decomposition.

Our goal in this paper is to develop a multivariate version of Shimorin's theorem and use it to shed light on the doubly commuting condition in the Bergman space over the bidisc.

2. MAIN RESULTS

Let $\mathcal{K}$ be a subset of a Hilbert space $\mathcal{H}$ and let T_1 and T_2 be two operators on $\mathcal{H}$. We write $[\mathcal{K}]_{T_1}$ to denote the smallest subspace of $\mathcal{H}$ containing $\mathcal{K}$ that is invariant under T_1 . Similarly, we write $[\mathcal{K}]_{T_1, T_2}$ to denote the smallest subspace of $\mathcal{H}$ containing $\mathcal{K}$ that is invariant under both T_1 and T_2 .

Theorem 3. *Let T_1 and T_2 be two commuting linear operators on a Hilbert space $\mathcal{H}$ with the property $\|T_i x + y\|^2 \leq 2(\|x\|^2 + \|T_i y\|^2)$ ($x, y \in \mathcal{H}, i = 1, 2$). Then,*

- (1) T_1 and T_2 doubly commute on $\mathcal{H}$, and
- (2) $\bigcap_{n=1}^{\infty} T_i^n(\mathcal{H}) = \{0\}$ ($i = 1, 2$)

if and only if

- (a) $\mathcal{H} = [(\mathcal{H} \ominus T_1(\mathcal{H})) \cap (\mathcal{H} \ominus T_2(\mathcal{H}))]_{T_1, T_2}$,
- (b) T_2 commutes with $T_1^* T_1$, and
- (c) $[(\mathcal{H} \ominus T_1(\mathcal{H})) \cap (\mathcal{H} \ominus T_2(\mathcal{H}))]_{T_2} \subseteq \mathcal{H} \ominus T_1(\mathcal{H})$.

To prove our theorem, we use the single operator result due to Shimorin (see [4]).

Theorem 4 (Shimorin). *Let T be a linear operator on a Hilbert space $\mathcal{H}$ with the properties:*

- (i) $\|Tx + y\|^2 \leq 2(\|x\|^2 + \|Ty\|^2)$ ($x, y \in \mathcal{H}$),
- (ii) $\bigcap_{n=1}^{\infty} T^n(\mathcal{H}) = \{0\}$.

Then $\mathcal{H} = [(\mathcal{H} \ominus T(\mathcal{H}))]_T$.

Henceforth, we will refer to the inequality in statement (i) of Theorem 4 as the Shimorin inequality.

We now prove Theorem 3.

Proof. First, suppose that (1) and (2) hold. (b) follows directly from (1). To prove (a), let $E_1 = \mathcal{H} \ominus T_1(\mathcal{H})$, $E_2 = \mathcal{H} \ominus T_2(\mathcal{H})$ and $E = E_1 \cap E_2$. We need to show that $\mathcal{H} = [E]_{T_1, T_2}$. Let $x \in E_1$. Then $T_1^* x = 0$, since the $\ker T_1^* = E_1$. Therefore, for all y in $\mathcal{H}$, we have

$$\begin{aligned} \langle T_2 x, T_1 y \rangle_{\mathcal{H}} &= \langle T_1^* T_2 x, y \rangle_{\mathcal{H}} \\ &= \langle T_2 T_1^* x, y \rangle_{\mathcal{H}} \quad (\text{by (1)}) \\ &= 0. \end{aligned}$$

Therefore, $T_2 E_1 \perp T_1 \mathcal{H}$, and hence $T_2 E_1 \subseteq E_1$. Similarly, we have

$$\begin{aligned} \langle T_2^* x, T_1 y \rangle_{\mathcal{H}} &= \langle T_1^* T_2^* x, y \rangle_{\mathcal{H}} \\ &= \langle T_2^* T_1^* x, y \rangle_{\mathcal{H}} \quad (\text{by (1)}) \\ &= 0. \end{aligned}$$

Therefore, $T_2^* E_1 \perp T_1 \mathcal{H}$, and hence $T_2^* E_1 \subseteq E_1$.

Now, let $F = E_1 \ominus T_2(E_1)$. We now show that $F \subseteq E$. Let $x \in F$. Then $x \perp T_2(E_1)$, and hence $T_2^* x \perp E_1$. Also, since $x \in E_1$ and E_1 reduces T_2 , we have $T_2^* x \in E_1$. Therefore, $T_2^* x = 0$. Since $\ker T_2^* = E_2$, we get $x \in E_2$. Hence $x \in E$.

We now observe that $E_1 = [E]_{T_2}$. Indeed,

$$\begin{aligned} E_1 &= [F]_{T_2} \quad (\text{by Theorem 4}) \\ &\subseteq [E]_{T_2} \quad (\text{since } F \subseteq E) \\ &\subseteq E_1 \quad (\text{since } E \subseteq E_1 \text{ and } E_1 \text{ is invariant under } T_2). \end{aligned}$$

We point out that F actually equals E . Also, the above sequence of containments gives us statement (c). In fact, it shows that equality actually holds in statement (c).

We finally observe that

$$\begin{aligned}\mathcal{H} &= [E_1]_{T_1} \quad (\text{by Theorem 4}) \\ &= [[E]_{T_2}]_{T_1} \quad (\text{by above}) \\ &= [E]_{T_1, T_2}.\end{aligned}$$

For the converse, suppose that (a), (b) and (c) hold. Let $E_1 = \mathcal{H} \ominus T_1(\mathcal{H})$ and $E_2 = \mathcal{H} \ominus T_2(\mathcal{H})$. If we let $E'_1 = [E_1 \cap E_2]_{T_2}$, then $\mathcal{H} = [E'_1]_{T_1}$. By supposition, $E'_1 \subseteq E_1$. Let x be an element of $\mathcal{H}$. Then

$$x = \lim_{k \rightarrow \infty} x_k$$

with

$$x_k = \sum_{n=0}^{N_k} T_1^n x_{n,k},$$

where each N_k is a nonnegative integer and $x_{n,k}$ is a member of E'_1 . We now have

$$\begin{aligned}T_1^* T_2 x_k &= \sum_{n=0}^{N_k} T_1^* T_2 T_1^n x_{n,k} \\ &= T_1^* T_2 x_{0,k} + \sum_{n=1}^{N_k} T_1^* T_2 T_1^n x_{n,k} \\ &= \sum_{n=1}^{N_k} T_1^* T_2 T_1^n x_{n,k} \quad (\text{since } T_2 x_{0,k} \in E_1)\end{aligned}$$

and

$$\begin{aligned}T_2 T_1^* x_k &= \sum_{n=0}^{N_k} T_2 T_1^* T_1^n x_{n,k} \\ &= T_2 T_1^* x_{0,k} + \sum_{n=1}^{N_k} T_2 T_1^* T_1^n x_{n,k} \\ &= \sum_{n=1}^{N_k} T_2 T_1^* T_1^n x_{n,k} \quad (\text{since } x_{0,k} \in E_1) \\ &= \sum_{n=1}^{N_k} T_1^* T_1 T_2 T_1^{n-1} x_{n,k} \quad (\text{by supposition (b)}) \\ &= \sum_{n=1}^{N_k} T_1^* T_2 T_1^n x_{n,k} \quad (\text{since } T_1 \text{ and } T_2 \text{ commute}).\end{aligned}$$

So $T_1^* T_2 x_k = T_2 T_1^* x_k$, and therefore T_1 and T_2 doubly commute on $\mathcal{H}$. Indeed,

$$\begin{aligned}\|T_1^* T_2 x - T_2 T_1^* x\| &\leq \|T_1^* T_2 x - T_1^* T_2 x_k\| + \|T_2 T_1^* x_k - T_2 T_1^* x\| \\ &\leq \|T_1^* T_2\| \|x - x_k\| + \|T_2 T_1^*\| \|x - x_k\|,\end{aligned}$$

and letting k go to infinity, we get our desired conclusion.

By Theorem 3.6 from [7], we know that $\mathcal{H} = [E_1]_{T_1} \oplus \bigcap_{n=1}^{\infty} T_1^n(\mathcal{H})$. Since $E'_1 \subseteq E_1$ and $\mathcal{H} = [E'_1]_{T_1}$, we may conclude that $\bigcap_{n=1}^{\infty} T_1^n(\mathcal{H}) = \{0\}$. Similarly, if we define $E'_2 = [E_1 \cap E_2]_{T_1}$, then $\mathcal{H} = [E'_2]_{T_2}$. As shown above, we need only show that $E'_2 \subseteq E_2$ to conclude that $\bigcap_{n=1}^{\infty} T_2^n(\mathcal{H}) = \{0\}$. To see this, let x be an element of E'_2 . Then x can be written as

$$x = \lim_{k \rightarrow \infty} x_k$$

with

$$x_k = \sum_{n=0}^{N_k} T_1^n x_{n,k},$$

where each N_k is a nonnegative integer and $x_{n,k}$ is a member of $E_1 \cap E_2$. Now, since we already showed under our assumptions that T_1 and T_2 doubly commute, we see that $T_2^* T_1^n x_{n,k} = T_1^n T_2^* x_{n,k} = 0$ for every n and k . Therefore, $T_2^* x_k = 0$ for every k . Hence,

$$\begin{aligned} \|T_2^* x\| &= \|T_2^* x - T_2^* x_k\| \\ &\leq \|T_2^*\| \|x - x_k\|, \end{aligned}$$

and letting k go to infinity, we get that $T_2^* x = 0$. Therefore, $E'_2 \subseteq E_2$ as desired. $\square$

From this general operator theorem, we have the following theorem regarding invariant subspace of

$$A^2(\mathbb{D}^2) = \left\{ f \in H(\mathbb{D}^2) : \int_{-1}^1 \int_{-1}^1 \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |f(r_1 e^{i\theta_1}, r_2 e^{i\theta_2})|^2 r_1 r_2 d\theta_1 d\theta_2 dr_1 dr_2 < \infty \right\}.$$

Theorem 5. Suppose $\mathcal{M}$ is a (closed) subspace of $A^2(\mathbb{D}^2)$ invariant under T_z and T_w . Then T_z and T_w doubly commute on $\mathcal{M}$ if and only if $\mathcal{M} = [(\mathcal{M} \ominus z\mathcal{M}) \cap (\mathcal{M} \ominus w\mathcal{M})]_{T_z, T_w}$, $[(\mathcal{M} \ominus z\mathcal{M}) \cap (\mathcal{M} \ominus w\mathcal{M})]_{T_w} \subseteq \mathcal{M} \ominus z\mathcal{M}$, and T_w commutes with $T_z^* T_z$.

For any nonnegative integers n and m , $\mathcal{M} = z^n w^m A^2(\mathbb{D}^2)$ satisfies the conditions of this theorem.

The only condition in Theorem 3 that needs to be verified is that T_z and T_w satisfy the Shimorin inequality. This is done in the following lemma.

Lemma 1. The operator $T_z : A^2(\mathbb{D}^2) \rightarrow A^2(\mathbb{D}^2)$ defined by

$$(T_z f)(z, w) = z f(z, w)$$

satisfies

$$\|T_z f + g\|_{A^2(\mathbb{D}^2)}^2 \leq 2 \left(\|f\|_{A^2(\mathbb{D}^2)}^2 + \|T_z g\|_{A^2(\mathbb{D}^2)}^2 \right), \quad f, g \in A^2(\mathbb{D}^2).$$

The inequality holds when T_z is replaced by T_w .

Before we prove this lemma, we recall a well known fact regarding the norm of a function in $A^2(\mathbb{D}^2)$. If f is in $A^2(\mathbb{D}^2)$ with power series expansion $f(z, w) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n} z^m w^n$, then

$$\|f\|_{A^2(\mathbb{D}^2)}^2 = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{|a_{m,n}|^2}{(m+1)(n+1)}.$$

The proof of Lemma 1 follows.

Proof. Note that for $f(z, w) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n} z^m w^n$,

$$(T_z f)(z, w) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n} z^{m+1} w^n = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} a_{m-1,n} z^m w^n.$$

Thus

$$\|T_z f\|_{A^2(\mathbb{D}^2)}^2 = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{|a_{m-1,n}|^2}{(m+1)(n+1)}.$$

Then what we need to show is that for every $f(z, w) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n} z^m w^n$ and $g(z, w) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{m,n} z^m w^n$, we have

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{|b_{0,n}|^2}{n+1} + \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{|a_{m-1,n} + b_{m,n}|^2}{(m+1)(n+1)} \\ & \leq 2 \left(\sum_{m,n} \frac{|a_{m,n}|^2}{(m+1)(n+1)} + \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{|b_{m-1,n}|^2}{(m+1)(n+1)} \right) \\ & = 2 \left(\sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{|a_{m-1,n}|^2}{m(n+1)} + \sum_{n=0}^{\infty} \frac{|b_{0,n}|^2}{2(n+1)} + \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{|b_{m,n}|^2}{(m+2)(n+1)} \right). \end{aligned}$$

Thus the inequality holds if we can verify that for any integers m and complex numbers a and b ,

$$\frac{|a+b|^2}{m+1} \leq 2 \left(\frac{|a|^2}{m} + \frac{|b|^2}{m+2} \right);$$

this is indeed the case (see p. 277 of [4]). That the inequality holds for T_w is seen by exchanging the roles of m and n . $\square$

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