HEWITT-MARCZEWSKI-PONDICZERY TYPE THEOREM FOR ABELIAN GROUPS AND MARKOV'S POTENTIAL DENSITY

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ABSTRACT. For an uncountable cardinal τ and a subset S of an abelian group G , the following conditions are equivalent:

- (i) $|\{ns : s \in S\}| \geq \tau$ for all integers $n \geq 1$;
- (ii) there exists a group homomorphism $\pi : G \rightarrow \mathbb{T}^{2^\tau}$ such that $\pi(S)$ is dense in $\mathbb{T}^{2^\tau}$.

Moreover, if $|G| \leq 2^{2^\tau}$, then the following item can be added to this list:

- (iii) there exists an isomorphism $\pi : G \rightarrow G'$ between G and a subgroup G' of $\mathbb{T}^{2^\tau}$ such that $\pi(S)$ is dense in $\mathbb{T}^{2^\tau}$.

We prove that the following conditions are equivalent for an uncountable subset S of an abelian group G that is either (almost) torsion-free or divisible:

- (a) S is $\mathcal{T}$ -dense in G for some Hausdorff group topology $\mathcal{T}$ on G ;
- (b) S is $\mathcal{T}$ -dense in some precompact Hausdorff group topology $\mathcal{T}$ on G ;
- (c) $|\{ns : s \in S\}| \geq \min\{\tau : |G| \leq 2^{2^\tau}\}$ for every integer $n \geq 1$.

This partially resolves a question of Markov going back to 1946.

We use $\mathbb{N}$ and $\mathbb{P}$ to denote the set of positive natural numbers and prime numbers, respectively. The symbol $\mathfrak{c}$ denotes the cardinality of the continuum, and ω denotes the first infinite cardinal. For an infinite cardinal κ , let

$$\log \kappa = \min\{\tau \geq \omega : \kappa \leq 2^\tau\}.$$

The group of integer numbers is denoted by $\mathbb{Z}$.

For a subset X of an abelian group G , we use $\langle X \rangle$ to denote the subgroup of G generated by X , and we let $nX = \{nx : x \in X\}$ for every $n \in \mathbb{Z}$. As usual,

$$t(G) = \{g \in G : ng = 0 \text{ for some } n \in \mathbb{N}\}$$

denotes the *torsion part* of G . An abelian group G is said to be:

- (i) *torsion-free* if $t(G) = \{0\}$;
- (ii) *torsion* if $t(G) = G$;
- (iii) *bounded* if $nG = \{0\}$ for some $n \in \mathbb{N}$;
- (iv) *divisible* if $nG = G$ for every $n \in \mathbb{N}$.

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Recall that a group homomorphism $\varphi : G \rightarrow H$ from an abelian group G to an abelian group H is a *monomorphism* if $\ker \varphi = \{0\}$.

A topological group is *precompact* if it is isomorphic to a subgroup of some compact Hausdorff group. As usual, $w(X)$ denotes the *weight* of a topological space X .

1. INTRODUCTION

The classical result of Hewitt-Marczewski-Pondiczery states: If τ is an infinite cardinal, I is a set such that $|I| \leq 2^\tau$, and for every $i \in I$ a space X_i has a dense subset of size $\leq \tau$, then the product $X = \prod \{X_i : i \in I\}$ also has a dense subset of size $\leq \tau$ (see, for example, [8, Theorem 2.3.15]). In this paper we investigate the following “algebraic version” of this theorem. Let κ be an infinite cardinal and $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the circle group. Given a fixed subset S of an abelian group G , we attempt to find a group homomorphism $\pi : G \rightarrow \mathbb{T}^\kappa$ such that $\pi(S)$ becomes dense in $\mathbb{T}^\kappa$. Of particular interest is the special case when π can be chosen to be a monomorphism, that is, when the group G and the subgroup $\pi(G)$ of $\mathbb{T}^\kappa$ become isomorphic. Our choice of the target group is justified by the fact that every abelian group G is isomorphic to a subgroup of $\mathbb{T}^\kappa$ for a suitable cardinal κ . To ensure a closer resemblance to the Hewitt-Marczewski-Pondiczery theorem, we pay special attention to the case $\kappa = 2^\tau$ for some infinite cardinal τ by addressing the following question: Given a subset S of an abelian group G such that $|S| \geq \tau$ and $|G| \leq 2^{2^\tau}$, does there exist a monomorphism $\pi : G \rightarrow \mathbb{T}^{2^\tau}$ such that $\pi(S)$ becomes dense in $\mathbb{T}^{2^\tau}$? We completely resolve this problem in the case when S is uncountable. Moreover, we provide a complete answer to this problem even for a countable set S such that no multiple nS of S (for $n \in \mathbb{N}$) is contained in a finitely generated subgroup of G . The remaining case is being resolved by the authors in [6].

The origin of this setting can be traced back to the 1916 paper of Weyl [11]. We recall the classical Weyl’s uniform distribution theorem: Given a faithfully indexed set $S = \{a_n : n \in \mathbb{N}\} \subseteq \mathbb{Z}$, the set of all $\alpha \in \mathbb{T}$ such that the set $S\alpha = \{a_n\alpha : n \in \mathbb{N}\} \subseteq \mathbb{T}$ is uniformly distributed has full measure 1. Since uniform distribution implies density in $\mathbb{T}$, it follows that $S\alpha$ is dense in $\mathbb{T}$ for almost all $\alpha \in \mathbb{T}$. Every $\alpha \in \mathbb{T}$ uniquely determines a homomorphism $h_\alpha : \mathbb{Z} \rightarrow \mathbb{T}$ such that $h_\alpha(1) = \alpha$. Furthermore, $\alpha \in \mathbb{T}$ generates a dense subgroup $\langle \alpha \rangle$ of $\mathbb{T}$ iff α is non-torsion iff the homomorphism h_α is a monomorphism. Hence, one can state (a consequence of) Weyl’s theorem by simply saying that for every infinite subset S of $\mathbb{Z}$, there exists a monomorphism $\pi : \mathbb{Z} \rightarrow \mathbb{T}$ such that $\pi(S)$ is dense in $\mathbb{T}$. Tkachenko and Yaschenko [10] consider homomorphisms $\pi : G \rightarrow \mathbb{T}^\omega$ such that $\pi(S)$ is dense in $\mathbb{T}^\omega$, for a certain class of groups G . They use such homomorphisms as a technical tool in addressing the problem suggested first in 1946 by Markov [9].

According to Markov [9], a subset S of a group G is called *potentially dense* (in G) provided that G admits some Hausdorff group topology $\mathcal{T}$ such that S is dense in $(G, \mathcal{T})$. The last section of [9] is exclusively dedicated to the following problem: Which subsets of a group G are potentially dense in G ? Markov [9] proved that every infinite subset of $\mathbb{Z}$ is potentially dense in $\mathbb{Z}$. This was strengthened in [7, Lemma 5.2] by showing that every infinite subset of $\mathbb{Z}$ is dense in some precompact metric group topology on $\mathbb{Z}$. (Apparently, the authors of [9] and [7] were unaware that both these results easily follow from Weyl’s uniform distribution theorem.) Further progress was obtained by Tkachenko and Yaschenko [10], who proved the

following theorem: If an abelian group G of size at most $\mathfrak{c}$ is either almost torsion-free or has exponent p for some prime p , then every infinite subset of G is potentially dense in G . (According to [10], an abelian group G is *almost torsion-free* if $r_p(G)$ is finite for every prime p .)

In [6], the authors resolved Markov's problem for *countable* sets: A countable subset S of an abelian group G is potentially dense in G if and only if $|G| \leq 2^{\mathfrak{c}}$ and S is Zariski dense in G . Recall that a subset S of an abelian group G is said to be *Zariski dense* in G provided that if $k \in \mathbb{N}$, $g_1, g_2, \dots, g_k \in G$, $n_1, n_2, \dots, n_k \in \mathbb{N}$ and each $s \in S$ satisfies the equation $n_i s = g_i$ for some $i = 1, 2, \dots, k$ (depending on s), then every $g \in G$ also satisfies some equation $n_j g = g_j$, for a suitable $j = 1, 2, \dots, k$ ([5]; see also [4, Section 5]).

In this paper we investigate the remaining case of *uncountable* sets. In particular, we obtain a new sufficient condition that guarantees that a subset S of an abelian group G is potentially dense in G . Moreover, when this condition is satisfied, we prove that the topology $\mathcal{T}$ on G such that S is $\mathcal{T}$ -dense in G can be chosen to be precompact. When S is uncountable and G belongs to a wide class of abelian groups (for example, almost torsion-free or divisible groups), our sufficient condition turns out to be also necessary for the potential density of S in G . And last but not least, our sufficient condition is rather powerful in the countable case as well, because the only case that is *not* covered by it is when nS , for a suitable $n \in \mathbb{N}$, is contained in a finitely generated subgroup of G . Therefore, it is only this special case that still requires the substantially more sophisticated techniques from [6] to prove potential density (in some precompact group topology).

2. SENDING A GIVEN SUBSET OF AN ABELIAN GROUP DENSELY IN $\mathbb{T}^{\kappa}$

Definition 2.1. Let τ be an infinite cardinal. We say that a subset S of an abelian group G is τ -wide if $nS \setminus \langle S' \rangle \neq \emptyset$ for every $n \in \mathbb{N}$ and each $S' \subseteq S$ with $|S'| < \tau$.

Our next proposition collects four simple facts that clarify the above definition and facilitate future references.

Proposition 2.2. *Let S be a subset of an abelian group G .*

- (i) *If S is uncountable, then S is τ -wide if and only if $\min\{|nS| : n \in \mathbb{N}\} \geq \tau$.*
- (ii) *If S contains an infinite independent subset S' , then S is $|S'|$ -wide.*
- (iii) *If G is torsion, then S is ω -wide if and only if nS is infinite for every $n \in \mathbb{N}$ (that is, S is almost 0-torsion in the sense of [3]). Therefore, G contains an ω -wide set S if and only if G is unbounded (and in such a case the ω -wide sets coincide with the Zariski dense sets).*
- (iv) *If S is not ω -wide, then nS is contained in a finitely generated subgroup of G , for a suitable $n \in \mathbb{N}$.*

Theorem 2.3. *Let τ be an infinite cardinal and S be a τ -wide subset of an abelian group G . Then there exist a subgroup H of G and a monomorphism $\varphi : H \rightarrow \mathbb{T}^{2^\tau}$ such that $|H| = \tau$ and $\varphi(H \cap S)$ is dense in $\mathbb{T}^{2^\tau}$.*

The proof of this theorem is postponed until Section 5.

Corollary 2.4. *Let τ be an infinite cardinal and S be a τ -wide subset of an abelian group G . Then:*

- (i) *there exists a group homomorphism $\pi : G \rightarrow \mathbb{T}^{2^\tau}$ such that $\pi(S)$ is dense in $\mathbb{T}^{2^\tau}$;*

- (ii) if one additionally assumes that $|G| \leq 2^{2^\tau}$, then π from the item (i) can be chosen to be a monomorphism.

Proof. Let H and φ be as in the conclusion of Theorem 2.3. The proof now branches into two cases, depending on which item of our corollary holds.

(i) Since $\mathbb{T}^{2^\tau}$ is a divisible group, there exists a group homomorphism $\pi : G \rightarrow \mathbb{T}^{2^\tau}$ extending φ .

(ii) Since $|H| = \tau < 2^{2^\tau}$ and $r_p(G) \leq |G| \leq 2^{2^\tau} = |\mathbb{T}^{2^\tau}| = r_p(\mathbb{T}^{2^\tau})$ for every $p \in \mathbb{P} \cup \{0\}$, we can extend φ to a monomorphism $\pi : G \rightarrow \mathbb{T}^{2^\tau}$; see [3, Lemma 3.17].

Returning back to the common part of the proof, note that $\varphi(H \cap S) \subseteq \pi(S)$ and $\varphi(H \cap S)$ is dense in $\mathbb{T}^{2^\tau}$, so $\pi(S)$ must be dense in $\mathbb{T}^{2^\tau}$ as well. $\square$

Theorem 2.5. *Let κ be a cardinal such that $\kappa > \mathfrak{c}$, and let $\tau = \log \kappa$. For a subset S of an abelian group G , the following conditions are equivalent:*

- (i) *there exists a group homomorphism $\varpi : G \rightarrow \mathbb{T}^\kappa$ such that $\varpi(S)$ is dense in $\mathbb{T}^\kappa$;*
- (ii) $\tau \leq \min\{|nS| : n \in \mathbb{N}\}$;
- (iii) S is τ -wide.

Proof. (i) $\rightarrow$ (ii) Let $n \in \mathbb{N}$. Since $\varpi(S)$ is dense in $\mathbb{T}^\kappa$, $w(n\mathbb{T}^\kappa) \leq 2^{|nS|}$ by Lemma 3.1(ii) below. Since $\mathbb{T}^\kappa$ is divisible, $n\mathbb{T}^\kappa = \mathbb{T}^\kappa$, and so $\kappa = w(\mathbb{T}^\kappa) \leq 2^{|nS|}$, which yields $\tau = \log \kappa \leq |nS|$. This proves (ii).

(ii) $\rightarrow$ (iii) Since $\kappa > \mathfrak{c}$, τ is uncountable. Applying (ii) with $n = 1$, we conclude that S is uncountable. Then S is τ -wide by (ii) and Proposition 2.2(i).

(iii) $\rightarrow$ (i) Let π be as in the conclusion of Corollary 2.4(i). From $\tau = \log \kappa$ it follows that $\kappa \leq 2^\tau$. Let $\psi : \mathbb{T}^{2^\tau} \rightarrow \mathbb{T}^\kappa$ be the projection on the first κ coordinates. Since ψ is a homomorphism, so is $\varpi = \psi \circ \pi : G \rightarrow \mathbb{T}^\kappa$. Since ψ is continuous and $\pi(S)$ is dense in $\mathbb{T}^{2^\tau}$, the set $\psi(\pi(S)) = \varpi(S)$ is dense in $\psi(\mathbb{T}^{2^\tau}) = \mathbb{T}^\kappa$. $\square$

Theorem 2.6. *Let κ be a cardinal such that $\kappa > \mathfrak{c}$, and let $\tau = \log \kappa$. For a subset S of an abelian group G , the following conditions are equivalent:*

- (i) *there exists a monomorphism $\varpi : G \rightarrow \mathbb{T}^\kappa$ such that $\varpi(S)$ is dense in $\mathbb{T}^\kappa$;*
- (ii) $|G| \leq 2^\kappa$ and $\tau \leq \min\{|nS| : n \in \mathbb{N}\}$;
- (iii) $|G| \leq 2^\kappa$ and S is τ -wide.

Proof. (i) $\rightarrow$ (ii) Clearly, $|G| = |\varpi(G)| \leq |\mathbb{T}^\kappa| = 2^\kappa$. The other inequality in item (ii) follows from the implication (i) $\rightarrow$ (ii) of Theorem 2.5.

(ii) $\rightarrow$ (iii) follows from the implication (ii) $\rightarrow$ (iii) of Theorem 2.5.

(iii) $\rightarrow$ (i) Let H and φ be as in the conclusion of Theorem 2.3. From $\tau = \log \kappa$ it follows that $\tau \leq \kappa \leq 2^\tau$.

For every $h \in H \setminus \{0\}$, there exists $\xi_h \in 2^\tau$ such that $\varphi(h)(\xi_h) \neq 0$. Define $\Xi = \{\xi_h \in 2^\tau : h \in H \setminus \{0\}\} \subseteq 2^\tau$. Let $\mu : \mathbb{T}^{2^\tau} \rightarrow \mathbb{T}^\Xi$ be the projection. Then $\mu \upharpoonright_{\varphi(H)} : \varphi(H) \rightarrow \mathbb{T}^\Xi$ is a monomorphism. Since $|\Xi| \leq |H \setminus \{0\}| \leq |H| = \tau \leq \kappa$, it follows that there exists a continuous surjective homomorphism $\psi : \mathbb{T}^{2^\tau} \rightarrow \mathbb{T}^\kappa$ such that $\chi = \psi \circ \varphi : H \rightarrow \mathbb{T}^\kappa$ is a monomorphism. Since ψ is continuous and $\varphi(H \cap S)$ is dense in $\mathbb{T}^{2^\tau}$, the set $\psi(\varphi(H \cap S)) = \chi(H \cap S)$ is dense in $\psi(\mathbb{T}^{2^\tau}) = \mathbb{T}^\kappa$.

Since $|H| = \tau \leq \kappa < 2^\kappa$ and $r_p(G) \leq |G| \leq 2^\kappa = |\mathbb{T}^\kappa| = r_p(\mathbb{T}^\kappa)$ for every $p \in \mathbb{P} \cup \{0\}$, we can extend χ to a monomorphism $\varpi : G \rightarrow \mathbb{T}^\kappa$; see [3, Lemma 3.17]. Since $\chi(H \cap S) \subseteq \varpi(S)$ and $\chi(H \cap S)$ is dense in $\mathbb{T}^\kappa$, the set $\varpi(S)$ must be dense in $\mathbb{T}^\kappa$ as well. $\square$

The counterpart of Theorems 2.5 and 2.6 for $\kappa \leq \mathfrak{c}$ is proved in [6].

3. MARKOV'S POTENTIAL DENSITY

We start this section with a simple necessary condition for potential density.

Lemma 3.1. *Let S be a dense subset of a Hausdorff group G . Then:*

- (i) *for every $n \in \mathbb{N}$, the set nS is dense in the subgroup nG of G ;*
- (ii) *$w(nG) \leq 2^{|nS|}$ and $|nG| \leq 2^{2^{|nS|}}$ for each $n \in \mathbb{N}$.*

Proof. (i) The map $g \mapsto ng$ that sends G to the subgroup nG of G is continuous. Since S is dense in nG , nS must be dense in nG .

(ii) Let $n \in \mathbb{N}$. From (i) we conclude that nS is a dense subset of the Tychonoff space nG . Therefore, $w(nG) \leq 2^{|nS|}$ by [8, Theorem 1.5.7] and $|nG| \leq 2^{2^{|nS|}}$ by [8, Theorem 1.5.3]. $\square$

Corollary 3.2. *If S is a potentially dense subset of a group G , then*

$$(3.1) \quad \log \log |nG| \leq |nS| \quad \text{for all } n \in \mathbb{N}.$$

Our next theorem provides a general sufficient condition for potential density that also allows it to be realized by some precompact group topology.

Theorem 3.3. *Let τ be an infinite cardinal and S be a τ -wide subset of an abelian group G such that $|G| \leq 2^{2^\tau}$. Then there exists a precompact Hausdorff group topology $\mathcal{T}$ on G such that S is dense in $(G, \mathcal{T})$.*

Proof. Let $\pi : G \rightarrow \mathbb{T}^{2^\tau}$ be a monomorphism from Corollary 2.4(ii), and let $\mathcal{T}'$ be the topology $\pi(G)$ inherits from $\mathbb{T}^{2^\tau}$. Then $(\pi(G), \mathcal{T}')$ is a precompact group. Since $\pi(S)$ is dense in $\mathbb{T}^{2^\tau}$, we conclude that $\pi(S)$ is $\mathcal{T}'$ -dense in $\pi(G)$. Since π is an isomorphism between G and $\pi(G)$, $\mathcal{T} = \{\pi^{-1}(U) : U \in \mathcal{T}'\}$ is the required topology. $\square$

The characterization of the countable potentially dense subsets of an abelian group G with $|G| \leq 2^{\mathfrak{c}}$ can be found in [6]. Our next corollary shows that Theorem 3.3 is sufficiently useful in obtaining some particular cases of that characterization.

Corollary 3.4. *Let G be an unbounded torsion abelian group with $|G| \leq 2^{\mathfrak{c}}$. For a countable subset S of G , the following conditions are equivalent:*

- (i) *S is potentially dense in G ;*
- (ii) *there exists a precompact Hausdorff group topology $\mathcal{T}$ on G such that S is dense in $(G, \mathcal{T})$;*
- (iii) *S is ω -wide.*

Proof. The implication (iii) $\rightarrow$ (ii) follows from Theorem 3.3, while the implication (ii) $\rightarrow$ (i) is trivial. According to Proposition 2.2(iii), to prove the implication (i) $\rightarrow$ (iii), it suffices to check that nS is infinite for every $n \in \mathbb{N}$. Assume that nS is finite for some $n \in \mathbb{N}$. By Lemma 3.1(i), this yields that nG is finite as well. Consequently, G is bounded, a contradiction. $\square$

In connection with the last corollary, it is worth mentioning that the unboundeness of a torsion abelian group G is a necessary condition for the existence of an ω -wide subset of G ; see Proposition 2.2(iii).

Corollary 3.5. *Let G be an abelian group such that $|nG| = |G|$ for every $n \in \mathbb{N}$. For an uncountable subset S of G , the following conditions are equivalent:*

- (i) S is potentially dense in G ;
- (ii) S is $\mathcal{T}$ -dense in some precompact Hausdorff group topology $\mathcal{T}$ on G ;
- (iii) $\log \log |nG| \leq |nS|$ for every $n \in \mathbb{N}$;
- (iv) $\log \log |G| \leq \min\{|nS| : n \in \mathbb{N}\}$.

Proof. The implication (ii) $\rightarrow$ (i) is obvious, the implication (i) $\rightarrow$ (iii) is proved in Corollary 3.2, and the implication (iii) $\rightarrow$ (iv) holds due to our assumption on G . It remains only to check the implication (iv) $\rightarrow$ (ii). From (iv) and Proposition 2.2(i) we conclude that S is τ -wide for $\tau = \log \log |G|$. Since $|G| \leq 2^{2^\tau}$, Theorem 3.3 applies. $\square$

Corollary 3.6. *Let G be an abelian group satisfying one of the following conditions:*

- (a) G is divisible;
- (b) $|t(G)| < |G|$;
- (c) $|t(G)| \leq \omega$;
- (d) G is almost torsion-free.

For every uncountable subset S of G , conditions (i)–(iv) of Corollary 3.5 are equivalent.

Proof. Let S be an uncountable subset of G . In particular, $|G| \geq |S| > \omega$. It suffices to show that G satisfies the assumption of Corollary 3.5. Fix $n \in \mathbb{N}$. If (a) holds, then $|nG| = |G|$ trivially holds, as the homomorphism $\eta_n : G \rightarrow G$ defined by $\eta_n(g) = ng$ for $g \in G$ is surjective. If (b) holds, then from $\ker \eta_n \subseteq t(G)$ and $|t(G)| < |G|$ it follows that $|nG| = |G|$. If (c) holds, then $t(G) \leq \omega < |G|$, and so (c) is a particular case of (b). Finally, note that every almost torsion-free group satisfies (c). $\square$

Remark 3.7. For an uncountable subset S of an abelian group G satisfying items (c) or (d) of Corollary 3.6, the following simplified condition can be added to the list of equivalent items (i)–(iv) of Corollary 3.5:

- (v) $\log \log |G| \leq |S|$.

Indeed, since $t(G)$ is at most countable and S is uncountable, we must have $|nS| = |S|$ for every $n \in \mathbb{N}$. This establishes the implication (v) $\rightarrow$ (iv). The reverse implication (iv) $\rightarrow$ (v) is trivial.

Let S be an uncountable subset of an abelian group G . Corollaries 3.5 and 3.6 show that (3.1) is not only a necessary but also a sufficient condition for the potential density of S in G when G belongs to a wide class of groups including divisible and (almost) torsion-free groups. According to Remark 3.7, the weaker (and much simpler) condition $\log \log |G| \leq |S|$, obtained by taking $n = 1$ in (3.1), is also sufficient for the potential density of S in G when the torsion part $t(G)$ of G is at most countable. Our next example demonstrates that this simpler condition (v) is no longer sufficient for the potential density of S in G when one weakens the assumption $|t(G)| \leq \omega$ to $|t(G)| < |G|$. (Note that $|G| \geq |S| > \omega$.)

Example 3.8. Let $\mathbb{Z}(2) = \mathbb{Z}/2\mathbb{Z}$ be the abelian group with two elements. Define $G = \mathbb{Z}(2)^\omega \times \mathbb{Z}^{2^c}$ and $S = \mathbb{Z}(2)^\omega \times C$, where C is any infinite cyclic subgroup of $\mathbb{Z}^{2^c}$. Then $\log \log |G| = \mathfrak{c} = |S|$ and $|t(G)| = |\mathbb{Z}(2)^\omega \times \{0\}| = \mathfrak{c} < |G|$. Since $\log \log |2G| > \omega = |2S|$, the set S cannot be potentially dense in G by Corollary 3.2.

Observe that a group G with the properties from Example 3.8 must necessarily have an uncountable torsion part $t(G)$; see Remark 3.7. Example 3.8 provides a negative answer to [4, Question 45].

4. TECHNICAL LEMMAS

We call a connected open subset V of $\mathbb{T}$ an *open arc*, and we use $l(V)$ to denote the length of V .

Lemma 4.1. *Suppose that V is an open arc in $\mathbb{T}$, $z, z' \in \mathbb{T}$, $m, n \in \mathbb{N}$, $1 \leq n < m$ and $2/m < l(V)$. Then there exists $y \in V$ such that $my = z$ and $ny \neq z'$.*

Proof. Since $2/m < l(V)$, the arc V contains two solutions y_0 and y_1 of the equation $my = z$ with

$$(4.1) \quad l(C(y_0, y_1)) = 1/m,$$

where $C(y_0, y_1)$ is the shortest arc in $\mathbb{T}$ connecting y_0 and y_1 .

Suppose that $ny_0 = ny_1 = z'$. Then $n(y_0 - y_1) = 0$, and so $l(C(y_0, y_1)) \geq 1/n$. Together with (4.1), this gives $m \leq n$, a contradiction. Therefore, $ny_j \neq z'$ for some $j = 0, 1$, and so we can take that y_j as our y . $\square$

If G and H are groups, then $G \cong H$ means that G and H are isomorphic.

Lemma 4.2. *Let τ and κ be infinite cardinals such that $\tau \leq \kappa$. Let K be a subgroup of $\mathbb{T}^\kappa$ such that $|K| \leq \tau$. For every $\gamma < \kappa$, let V_γ be an open arc in $\mathbb{T}$ such that the set $\{l(V_\gamma) : \gamma < \kappa\}$ has a positive lower bound. Then there exists $f \in \prod\{V_\gamma : \gamma < \kappa\}$ having the following properties:*

- (i) $\langle f \rangle \cong \mathbb{Z}$;
- (ii) $\langle f \rangle \cap K = \{0\}$.

Proof. Fix $k \in \mathbb{N}$ such that $2/k < l(V_\gamma)$ for every $\gamma < \kappa$. Since $|K| \leq \tau$, we can choose an enumeration $K \times \mathbb{N} = \{(h_\alpha, n_\alpha) : \alpha < \tau\}$ of the set $K \times \mathbb{N}$. By transfinite recursion on $\alpha < \tau$ we will select $\gamma_\alpha < \kappa$ and $y_{\gamma_\alpha} \in \mathbb{T}$ satisfying the following properties:

- (i _{α}) $\gamma_\alpha \notin \{\gamma_\beta : \beta < \alpha\}$,
- (ii _{α}) $y_{\gamma_\alpha} \in V_{\gamma_\alpha}$,
- (iii _{α}) $n_\alpha y_{\gamma_\alpha} \neq h_\alpha(\gamma_\alpha)$.

Basis of recursion. Select $\gamma_0 < \kappa$ arbitrarily. Apply Lemma 4.1 to $V = V_{\gamma_0}$, $z = 0$, $z' = h_0(\gamma_0)$, $n = n_0$ and $m = n_0(k+1)$ to choose $y_{\gamma_0} \in \mathbb{T}$ satisfying (ii₀) and (iii₀). Condition (i₀) is vacuous.

Recursive step. Let $\alpha < \tau$, and suppose that $\gamma_\beta < \kappa$ and $y_{\gamma_\beta} \in \mathbb{T}$ satisfying (i _{β})–(iii _{β}) have already been selected for all $\beta < \alpha$. We will now choose $\gamma_\alpha < \kappa$ and $y_{\gamma_\alpha} \in \mathbb{T}$ satisfying (i _{α})–(iii _{α}). Since $|\alpha| < \tau \leq \kappa$, we can choose $\gamma_\alpha < \kappa$ satisfying (i _{α}). Now we apply Lemma 4.1 to $V = V_{\gamma_\alpha}$, $z = 0$, $z' = h_\alpha(\gamma_\alpha)$, $n = n_\alpha$ and $m = n_\alpha(k+1)$ to choose $y_{\gamma_\alpha} \in \mathbb{T}$ satisfying (ii _{α}) and (iii _{α}).

The recursion being complete, choose $y_\gamma \in V_\gamma$ arbitrarily for every $\gamma \in \kappa \setminus \{\gamma_\alpha : \alpha < \tau\}$. Define $f \in \mathbb{T}^\kappa$ by $f(\gamma) = y_\gamma$ for each $\gamma < \kappa$. Then $f \in \prod\{V_\gamma : \gamma < \kappa\}$. We claim that

$$(4.2) \quad nf \notin K \quad \text{for every } n \in \mathbb{N}.$$

Indeed, let $n \in \mathbb{N}$ and $h \in K$ be arbitrary. Then $(h, n) \in K \times \mathbb{N}$, and so $(h, n) = (h_\alpha, n_\alpha)$ for some $\alpha < \kappa$. Now $nf(\gamma_\alpha) = n_\alpha y_{\gamma_\alpha} \neq h_\alpha(\gamma_\alpha) = h(\gamma_\alpha)$ by (iii) _{α} . Thus, $nf \neq h$. Since $h \in K$ was arbitrary, this proves (4.2). From (4.2) we immediately get both (i) and (ii). $\square$

Lemma 4.3. *Let τ and κ be infinite cardinals such that $\tau \leq \kappa$. Let K be a subgroup of $\mathbb{T}^\kappa$ such that $|K| \leq \tau$. Assume that $f' \in K$, $m \in \mathbb{N}$ and $m \geq 2$. For every $\gamma < \kappa$, let V_γ be an open arc in $\mathbb{T}$ such that $2/m < l(V_\gamma)$. Then there exists $f \in \prod\{V_\gamma : \gamma < \kappa\}$ satisfying the following properties:*

- (i) $mf = f'$;
- (ii) $nf \notin K$ for all $n \in \mathbb{N}$ with $1 \leq n < m$.

Proof. Since $|K| \leq \tau$, we can choose an enumeration $K \times \{1, 2, \dots, m-1\} = \{(h_\alpha, n_\alpha) : \alpha < \tau\}$ of the set $K \times \{1, 2, \dots, m-1\}$. By transfinite recursion on $\alpha < \tau$ we will select $\gamma_\alpha < \kappa$ and $y_{\gamma_\alpha} \in \mathbb{T}$ with the following properties:

- (i) _{α} $\gamma_\alpha \notin \{\gamma_\beta : \beta < \alpha\}$,
- (ii) _{α} $y_{\gamma_\alpha} \in V_{\gamma_\alpha}$,
- (iii) _{α} $my_{\gamma_\alpha} = f'(\gamma_\alpha)$,
- (iv) _{α} $n_\alpha y_{\gamma_\alpha} \neq h_\alpha(\gamma_\alpha)$.

Basis of recursion. Select $\gamma_0 < \kappa$ arbitrarily, and apply Lemma 4.1 to $V = V_{\gamma_0}$, $z = f'(\gamma_0)$, $z' = h_0(\gamma_0)$, $n = n_0$ and m to choose $y_{\gamma_0} \in \mathbb{T}$ satisfying (ii)₀, (iii)₀ and (iv)₀. Condition (i)₀ is vacuous.

Recursive step. Let $\alpha < \tau$, and suppose that $\gamma_\beta < \kappa$ and $y_{\gamma_\beta} \in \mathbb{T}$ satisfying (i) _{β} –(iv) _{β} have already been selected for all $\beta < \alpha$. We will now choose $\gamma_\alpha < \kappa$ and $y_{\gamma_\alpha} \in \mathbb{T}$ satisfying (i) _{α} –(iv) _{α} . Since $|\alpha| < \tau \leq \kappa$, we can choose $\gamma_\alpha < \kappa$ satisfying (i) _{α} . Now we apply Lemma 4.1 to $V = V_{\gamma_\alpha}$, $z = f'(\gamma_\alpha)$, $z' = h_\alpha(\gamma_\alpha)$, $n = n_\alpha$ and m to choose $y_{\gamma_\alpha} \in \mathbb{T}$ satisfying (ii) _{α} , (iii) _{α} and (iv) _{α} .

The recursion being complete, for every $\gamma \in \kappa \setminus \{\gamma_\alpha : \alpha < \tau\}$, apply Lemma 4.1 to $V = V_\gamma$, $z = f'(\gamma)$, $z' = 0$, $n = 1$ and m to choose $y_\gamma \in V_\gamma$ such that $my_\gamma = f'(\gamma)$.

Define $f \in \mathbb{T}^\kappa$ by $f(\gamma) = y_\gamma$ for every $\gamma < \kappa$. Then $f \in \prod\{V_\gamma : \gamma < \kappa\}$ and (i) is satisfied. To prove (ii), choose $n \in \mathbb{N}$ such that $1 \leq n < m$. Let $h \in K$ be arbitrary. Then $(h, n) \in K \times \{1, 2, \dots, m-1\}$, and so $(h, n) = (h_\alpha, n_\alpha)$ for some $\alpha < \kappa$. Now $nf(\gamma_\alpha) = n_\alpha y_{\gamma_\alpha} \neq h_\alpha(\gamma_\alpha) = h(\gamma_\alpha)$ by (iv) _{α} . Therefore, $nf \neq h$. Since $h \in K$ was arbitrary, this proves $nf \notin K$. $\square$

Lemma 4.4. *Let G and G^* be abelian groups, and let K and K^* be subgroups of G and G^* , respectively. Suppose also that $x \in G$, $x^* \in G^*$, $m \in \mathbb{N}$, $m \geq 2$, and $\psi : K \rightarrow K^*$ is an isomorphism satisfying the following properties:*

- (a) $mx \in K$ and $mx^* \in K^*$;
- (b) $nx \notin K$ and $nx^* \notin K^*$ whenever $n \in \mathbb{N}$ and $1 \leq n < m$;
- (c) $\psi(mx) = mx^*$.

Then there exists a unique isomorphism $\varphi : K + \langle x \rangle \rightarrow K^ + \langle x^* \rangle$ extending ψ such that $\varphi(x) = x^*$.*

Proof. Define φ by

$$(4.3) \quad \varphi(h + kx) = \psi(h) + kx^* \text{ for } h \in K \text{ and } k \in \mathbb{Z}.$$

To check that this definition is correct, suppose that

$$(4.4) \quad h + kx = h' + k'x,$$

with $h, h' \in K$ and $k, k' \in \mathbb{Z}$. Hence, $(k' - k)x = h - h' \in K$, so by (a) and (b), one has $k' - k = lm$ for some $l \in \mathbb{Z}$, which yields $h - h' = lmx$. This gives

$$\psi(h) - \psi(h') = \psi(h - h') = \psi(lmx) = l\psi(mx) = lmx^* = (k' - k)x^* = k'x^* - kx^*$$

by (c), and so $\psi(h) + kx^* = \psi(h') + k'x^*$. Comparing this with (4.4), we conclude that (4.3) correctly defines a homomorphism φ . From (4.3) we get $\varphi(x) = x^*$ and $\varphi \upharpoonright_K = \psi$. Moreover, φ is surjective and unique with these properties.

To prove that φ is a monomorphism, assume that $\varphi(h + kx) = 0$ for some $h \in K$ and $k \in \mathbb{Z}$. Then $\psi(h) + kx^* = 0$, so $kx^* \in K^*$. Consequently, m divides k by (a) and (b). Then $kx \in K$, and so $h + kx \in K$ as well. Therefore, $\varphi(h + kx) = \psi(h + kx) = 0$ yields $h + kx = 0$, as ψ is a monomorphism. $\square$

5. PROOF OF THEOREM 2.3

Fix a countable base $\mathcal{V}$ for the topology of $\mathbb{T}$ consisting of non-empty open arcs of $\mathbb{T}$ such that $\mathbb{T} \in \mathcal{V}$. Consider the Tychonoff product topology on 2^τ , and let $\mathcal{B}$ be the canonical base for 2^τ consisting of non-empty clopen subsets of 2^τ such that $|\mathcal{B}| = \tau$. Let

$$\mathbb{U} = \{\mathcal{U} \in [\mathcal{B}]^{<\omega} : \mathcal{U} \text{ is a cover of } 2^\tau \text{ by pairwise disjoint sets}\}.$$

For $\xi \in 2^\tau$ and $\mathcal{U} \in \mathbb{U}$, let $U_{\xi, \mathcal{U}} \in \mathcal{U}$ denote the unique $U \in \mathcal{U}$ such that $\xi \in U$. Define

$$\mathbb{E} = \{(\mathcal{U}, v) : \mathcal{U} \in \mathbb{U} \text{ and } v : \mathcal{U} \rightarrow \mathcal{V} \text{ is a function}\}.$$

For $(\mathcal{U}, v) \in \mathbb{E}$, let

$$F(\mathcal{U}, v) = \left\{ f \in \mathbb{T}^{2^\tau} : f(\xi) \in v(U_{\xi, \mathcal{U}}) \text{ for all } \xi \in 2^\tau \right\} = \prod \{v(U_{\xi, \mathcal{U}}) : \xi \in 2^\tau\}.$$

Clearly, $|\mathbb{E}| = \tau$, so we can fix an enumeration $\mathbb{E} = \{(\mathcal{U}_\alpha, v_\alpha) : \alpha < \tau\}$ of $\mathbb{E}$ such that $\mathcal{U}_0 = \{2^\tau\}$ and $v_0(2^\tau) = \mathbb{T}$. For each $\alpha < \tau$, choose $n_\alpha \in \mathbb{N}$ such that

$$(5.1) \quad 2/n_\alpha < \min\{l(v(U)) : U \in \mathcal{U}\}.$$

By transfinite recursion on $\alpha < \tau$ we will choose an element $x_\alpha \in S$ and define a map $\varphi_\alpha : H_\alpha = \langle \{x_\beta : \beta \leq \alpha\} \rangle \rightarrow \mathbb{T}^{2^\tau}$ satisfying the following conditions:

- (i $_\alpha$) $\varphi_\alpha(x_\alpha) \in F(\mathcal{U}_\alpha, v_\alpha)$,
- (ii $_\alpha$) φ_α is a monomorphism,
- (iii $_\alpha$) $\varphi_\alpha \upharpoonright_{H_\beta} = \varphi_\beta$ for all $\beta < \alpha$.

Basis of recursion. Pick $x_0 \in S$ arbitrarily, and let $\varphi_0 : \langle x_0 \rangle \rightarrow \mathbb{T}^{2^\tau} = F(\mathcal{U}_0, v_0)$ be an arbitrary monomorphism. Then conditions (i $_0$) and (ii $_0$) are satisfied, while the condition (iii $_0$) is vacuous.

Recursive step. Let $\alpha < \tau$, and suppose that $x_\beta \in S$ and a map $\varphi_\beta : H_\beta \rightarrow \mathbb{T}^{2^\tau}$ satisfying (i $_\beta$), (ii $_\beta$) and (iii $_\beta$) have already been constructed for every $\beta < \alpha$. We are going to define $x_\alpha \in S$ and a map $\varphi_\alpha : H_\alpha \rightarrow \mathbb{T}^{2^\tau}$ satisfying (i $_\alpha$), (ii $_\alpha$) and (iii $_\alpha$).

Define

$$H'_\alpha = \langle \{x_\beta : \beta < \alpha\} \rangle = \bigcup_{\beta < \alpha} H_\beta.$$

Since (ii $_\beta$) and (iii $_\beta$) hold for every $\beta < \alpha$,

$$\varphi'_\alpha = \bigcup_{\beta < \alpha} \varphi_\beta : H'_\alpha \rightarrow \mathbb{T}^{2^\tau}$$

is a monomorphism. Since $\{x_\beta : \beta < \alpha\} \subseteq S$, $|\alpha| < \tau$ and S is τ -wide, one has $n_\alpha!S \setminus \langle \{x_\beta : \beta < \alpha\} \rangle = n_\alpha!S \setminus H'_\alpha \neq \emptyset$, and so there exists $x_\alpha \in S$ such that $n_\alpha!x_\alpha \notin H'_\alpha$. In particular,

$$(5.2) \quad nx_\alpha \notin H'_\alpha \text{ for all } n \leq n_\alpha.$$

Let $\kappa = 2^\tau$ and $K = \varphi'_\alpha(H'_\alpha)$. For $\xi \in 2^\tau$, define $V_\xi = v_\alpha(U_{\xi, \mathcal{U}_\alpha})$. Then (5.1) yields

$$(5.3) \quad 2/n_\alpha < l(v_\alpha(U_{\xi, \mathcal{U}_\alpha})) = l(V_\xi) \quad \text{for every } \xi \in 2^\tau.$$

We need to consider two cases.

Case 1. $\{n \in \mathbb{N} : nx_\alpha \in H'_\alpha\} = \emptyset$. In this case, $\langle x_\alpha \rangle \cong \mathbb{Z}$ and the sum $\langle x_\alpha \rangle + H'_\alpha = \langle x_\alpha \rangle \oplus H'_\alpha$ is direct. Since $\{l(V_\gamma) : \gamma < \kappa\}$ has a positive lower bound by (5.3), we can apply Lemma 4.2 to choose

$$(5.4) \quad f \in \prod \{V_\xi : \xi \in 2^\tau\} = F(\mathcal{U}_\alpha, v_\alpha)$$

with $\langle f \rangle \cong \mathbb{Z}$ and $\langle f \rangle \cap K = \{0\}$. Then the sum $K + \langle f \rangle = K \oplus \langle f \rangle$ is direct as well. Since $\langle x_\alpha \rangle \cong \langle f \rangle \cong \mathbb{Z}$, there exists a unique monomorphism $\varphi_\alpha : H_\alpha = H'_\alpha \oplus \langle x_\alpha \rangle \rightarrow K \oplus \langle f \rangle \subseteq \mathbb{T}^{2^\tau}$ extending φ'_α such that $\varphi_\alpha(x_\alpha) = f$.

Case 2. $\{n \in \mathbb{N} : nx_\alpha \in H'_\alpha\} \neq \emptyset$. Let $m = \min\{n \in \mathbb{N} : nx_\alpha \in H'_\alpha\}$ and $f' = \varphi'_\alpha(mx_\alpha) \in K$. Then $m > n_\alpha$ by (5.2), so from (5.3) we obtain $2/m < 2/n_\alpha < l(V_\xi)$ for every $\xi \in 2^\tau$. Since $n_\alpha \geq 1$, we have $m \geq 2$. Applying Lemma 4.3, we get f satisfying (5.4) such that $mf = f'$ and $nf \notin K$ for all $n \in \mathbb{N}$ with $1 \leq n < m$. Observe that $\mathbb{T}^{2^\tau}$ (taken as G^*), H'_α (taken as K), K (taken as K^*), x_α (taken as x), f (taken as x^*), φ'_α (taken as ψ) and m satisfy the assumptions of Lemma 4.4. Denoting φ from the conclusion of this lemma by φ_α , we obtain a monomorphism $\varphi_\alpha : H_\alpha \rightarrow \mathbb{T}^{2^\tau}$ extending φ'_α such that $\varphi_\alpha(x_\alpha) = f$.

The monomorphism φ_α obviously satisfies (i $_\alpha$), (ii $_\alpha$) and (iii $_\alpha$) in both cases.

The recursive construction being complete, let

$$H = \bigcup_{\alpha < \tau} H_\alpha \quad \text{and} \quad \varphi = \bigcup_{\alpha < \tau} \varphi_\alpha.$$

Since (ii $_\alpha$) and (iii $_\alpha$) are satisfied for every $\alpha < \tau$, $\varphi : H \rightarrow \mathbb{T}^{2^\tau}$ is a monomorphism.

It remains only to check that $\varphi(H \cap S)$ is dense in $\mathbb{T}^{2^\tau}$. Let W be a non-empty open subset of $\mathbb{T}^{2^\tau}$. Choose pairwise distinct $\xi_1, \dots, \xi_i \in 2^\tau$ and $V_1, \dots, V_i \in \mathcal{V}$ such that

$$\left\{ f \in \mathbb{T}^{2^\tau} : f(\xi_j) \in V_j \text{ for all } j \leq i \right\} \subseteq W.$$

Select $\mathcal{U} = \{U_1, \dots, U_i\} \in \mathbb{U}$ which separates the ξ_j 's; that is, $j, k \leq i$ and $\xi_j \in U_k$ imply $j = k$. Define $v : \mathcal{U} \rightarrow \mathcal{V}$ by $v(U_j) = V_j$ for $j \leq i$. Then $(\mathcal{U}, v) \in \mathbb{E}$, and so $(\mathcal{U}, v) = (\mathcal{U}_\alpha, v_\alpha)$ for some $\alpha < \tau$. Note that

$$(5.5) \quad F(\mathcal{U}_\alpha, v_\alpha) = F(\mathcal{U}, v) \subseteq \left\{ f \in \mathbb{T}^{2^\tau} : f(\xi_j) \in V_j \text{ for all } j \leq i \right\} \subseteq W.$$

Since $x_\alpha \in S \cap H_\alpha \subseteq H \cap S$ and $\varphi(x_\alpha) = \varphi_\alpha(x_\alpha) \in F(\mathcal{U}_\alpha, v_\alpha)$ by (i $_\alpha$), it follows from (5.5) that $\varphi(x_\alpha) \in \varphi(H \cap S) \cap W \neq \emptyset$.

6. QUESTIONS

All instances of potential density in Section 3 were witnessed by a precompact Hausdorff group topology. This makes it natural to ask the following question.

Question 6.1. Let S be a potentially dense subset of an abelian group G . Does there exist a Hausdorff precompact group topology $\mathcal{T}$ on G such that S is $\mathcal{T}$ -dense in G ?

The answer to this question is positive when S is countable [6].

It was shown in [5] that every potentially dense subset S of an abelian group G must be Zariski dense in G . (We refer the reader to the Introduction for the definition of Zariski density.) Corollary 3.2 gives another necessary condition for the potential density of S in G . One may wonder if these two conditions, combined together, are also sufficient.

Question 6.2. Let S be an infinite subset of an Abelian group G such that:

- (a) S is Zariski dense in G , and
- (b) $\log \log |nG| \leq |nS|$ (equivalently, $|nG| \leq 2^{2^{|nS|}}$) for all $n \in \mathbb{N}$.

Is S potentially dense in G ? Does there exist a Hausdorff precompact group topology $\mathcal{T}$ on G such that S is $\mathcal{T}$ -dense in G ?

We note that this question is an appropriate modification of [4, Question 45], which is necessary in view of Example 3.8.

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