TOWARDS A QUANTUM GALOIS THEORY FOR QUANTUM DOUBLE ALGEBRAS OF FINITE GROUPS

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ABSTRACT. Suppose that G is a finite group and $D(G)$ the quantum double algebra of G . Let $\mathcal{A}$ be the field algebra of G -spin models. There is a natural action of $D(G)$ on $\mathcal{A}$ such that $\mathcal{A}$ becomes a $D(G)$ -module algebra. For a subgroup H of G , there is a Hopf subalgebra $D(G; H)$ of $D(G)$. Based on the concrete construction of a $D(G; H)$ fixed point subalgebra, the paper proves that $D(G; H)$ is Galois closed and thus gives a quantum Galois theory in the field algebra of G -spin models.

1. INTRODUCTION

The classical Galois theory of fields has been generalized in two directions. On the one side, fields are replaced by commutative or even noncommutative rings. On the other side, group actions are replaced by Hopf algebra actions.

As one generalization involving both directions, the Hopf Galois theory has developed rapidly. The definition of Hopf Galois theory has its roots in the Chase-Harrison-Rosenberg approach to Galois theory for groups acting on commutative rings [3]. In 1969 Chase and Sweedler in [4] extended these ideas to coactions of Hopf algebras acting on a commutative ring. In 1981 Kreimer and Takeuchi gave a general definition of finite noncommutative Hopf Galois extensions [13]. Since then the Hopf Galois theory has been studied by many mathematicians; for example, see [2], [18], [7], [15], [19], [20].

The Galois correspondence for group actions has been studied in detail in various forms. C. Dong and G. Mason initiated a systematic search for a vertex operator algebra with a finite automorphism group, which is referred to as the operator content of orbifold models by physicists or as quantum Galois theory for finite groups [6], [9]. There are other Galois correspondences which are relevant to quantum Galois theory for finite groups, especially in the context of subfactors [11], [12].

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The purpose of this paper is to give a quantum Galois theory for quantum double algebras of finite groups acting on the field algebra of G -spin models. Namely, suppose that G is a finite group and $D(G)$ the quantum double algebra of G . Let $\mathcal{A}$ be the field algebra of G -spin models. There is a natural action of $D(G)$ on $\mathcal{A}$ such that $\mathcal{A}$ becomes a $D(G)$ -module algebra. Under such an action, the Hopf subalgebra $D(G; H)$ of $D(G)$, which is determined by a subgroup H of G , is Galois closed.

2. PRELIMINARIES

Suppose that G is a finite group with a unit e . The G -value spin configuration on the 2-dimensional square lattices means the map $\sigma : \mathbb{Z}^2 \rightarrow G$ with energy functional:

$$S(\sigma) = \sum_{(x,y)} f(\sigma_x^{-1} \sigma_y),$$

where the summation runs over the nearest neighborhood pairs of points in $\mathbb{Z}^2$ and $f : G \rightarrow \mathbb{R}$ is a function of positive type. This kind of classical statistical system or the corresponding quantum field theories will be called G -spin models [17]. The main motivation for studying such models is that they can provide the simplest example of quantum symmetry. In general, if G is Abelian, the models have a symmetry structure of $G \times \tilde{G}$, where $\tilde{G}$ is the Pontryagin dual of G . Otherwise, the Pontryagin dual loses its meaning and one often considers the quantum double $D(G)$ of G [14].

Definition 2.1. Let $C(G)$ be the algebra of complex functions on G and let $\mathbb{C}G$ be the group algebra of G over the complex field $\mathbb{C}$. The quantum double $D(G)$ is the crossed product of $C(G)$ and $\mathbb{C}G$ with respect to the adjoint action of the latter on the former.

$D(G)$ is a simple example of the quantum double construction of Drinfeld, whose structure maps are as follows:

$$\begin{aligned} (g_1, h_1)(g_2, h_2) &= \delta_{g_1 h_1, h_1 g_2} (g_1, h_1 h_2), \\ \Delta(g, h) &= \sum_{f \in G} (f, h) \otimes (f^{-1} g, h), \\ \varepsilon((g, h)) &= \delta_{g, e}, \\ S(g, h) &= (h^{-1} g^{-1} h, h^{-1}), \\ (g, h)^* &= (h^{-1} g h, h^{-1}), \end{aligned}$$

where

$$\delta_{x,y} = \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{if } x \neq y \end{cases}$$

and $\{(g, h) : g, h \in G\}$ is the linear basis of $D(G)$ ([5], [8]). Here we use (g, h) instead of $g^* h$ for the basis element in $D(G)$. There is a special element $z = \frac{1}{|G|} \sum_{g \in G} (e, g)$ in $D(G)$, call it a cointegral, such that for $a \in D(G)$, $az = za = \varepsilon(a)z$.

Also as in the traditional quantum field theory, one can define a field algebra $\mathcal{A}$ associated with the models.

Definition 2.2. The local field algebra $\mathcal{A}_l$ of G -spin models is a unital algebra generated by $\{\delta_h(x), \rho_g(l) \mid g, h \in G; x \in \mathbb{Z}, l \in \mathbb{Z} + \frac{1}{2}\}$, subject to

$$\begin{aligned} \sum_{g \in G} \delta_g(x) &= \rho_e(l) = 1; \\ \delta_g(x) \delta_h(x) &= \delta_{g,h}(x) \delta_h(x); \\ \delta_g(x) \delta_h(x') &= \delta_h(x') \delta_g(x), \quad x \neq x'; \\ \rho_g(l) \rho_h(l) &= \rho_{gh}(l); \\ \rho_g(l) \rho_h(l') &= \begin{cases} \rho_h(l') \rho_{h^{-1}gh}(l) & l > l', \\ \rho_{ghg^{-1}}(l) \rho_g(l) & l < l'; \end{cases} \\ \rho_g(l) \delta_h(x) &= \begin{cases} \delta_{gh}(x) \rho_g(l) & l < x, \\ \delta_h(x) \rho_g(l) & l > x. \end{cases} \end{aligned}$$

It becomes a $*$ -algebra by defining $(\delta_g(x))^* = \delta_g(x)$ and $(\rho_g(l))^* = \rho_{g^{-1}}(l)$. Using the inductive limit, $\mathcal{A}_l$ can be extended to a C^* -algebra $\mathcal{A}$, called the field algebra of G -spin models (for details, see [17]).

There is a natural action γ of $D(G)$ on $\mathcal{A}$. For $x \in \mathbb{Z}, l \in \mathbb{Z} + \frac{1}{2}$ and $f, g, h \in G$, set

$$\begin{aligned} \gamma((g, h) \times \delta_f(x)) &= \delta_{g,e} \delta_{hf}(x), \\ \gamma((g, h) \times \rho_f(l)) &= \delta_{g,hfh^{-1}} \rho_g(l). \end{aligned}$$

The γ can be extended continuously to an action of $D(G)$ on $\mathcal{A}$, denote it by γ too, so that $\mathcal{A}$ is a $D(G)$ -module algebra with respect to the γ ([17]). Namely it is a bilinear map with properties that: $\forall a, b \in D(G), S, T \in \mathcal{A}$,

$$\begin{aligned} ab(S) &= a(b(S)); \\ a(ST) &= \sum_{(a)} a_{(1)}(F) a_{(2)}(T); \\ a(S)^* &= (S(a^*)(S))^*. \end{aligned}$$

Here and from now on, by $a(S)$ we always denote the element $\gamma(a \times S)$ in $\mathcal{A}$.

All algebras in this paper will be unital algebras over the complex field $\mathbb{C}$. For general results on Hopf algebras, one can refer to the book of Abe [1]. We shall use m, Δ, ε and S for the multiplication, the comultiplication, the counit and the antipode respectively. Also we shall adopt the summation convention, which is standard in Hopf algebra theory.

3. FIELD ALGEBRA OF G -SPIN MODELS AND OBSERVABLE ALGEBRA

Suppose that H is a subgroup of G . In the following, by $D(G; H)$ we denote the crossed product of $C(G)$ and the group algebra $\mathbb{C}H$ with respect to the adjoint action of the latter on the former. Then $D(G; H)$ is a sub-Hopf algebra of $D(G)$ with a unique cointegral element $z_H = \frac{1}{|H|} \sum_{h \in H} (e, h)$. The main difference between

the quantum double algebra $D(H)$ of the finite group H and $D(G; H)$ is that the former is a quasi-triangular Hopf algebra, while the latter is not [16]. Set

$$\mathcal{A}_H = \{F \in \mathcal{A} \mid a(F) = \varepsilon(a)F, \forall a \in D(G; H)\},$$

where ε is the counit on $D(G; H)$. It is easy to see that

$$\mathcal{A}_H = \{F \in \mathcal{A} \mid z_H(F) = F\}.$$

Since the map z_H is a conditional expectation from $\mathcal{A}$ onto $\mathcal{A}_H$, namely a positive map with the bimodule property

$$z_H(F_1 F F_2) = F_1 z_H(F) F_2, \quad \forall F \in \mathcal{A}, \forall F_1, F_2 \in \mathcal{A}_H,$$

$\mathcal{A}_H$ is a nonzero C^* -subalgebra of $\mathcal{A}$, and we call it an observable algebra of G -spin models associated with the subgroup H [10]. In order to build the quantum Galois theory for $D(G)$ acting on the field algebra of G -spin models, we first give the concrete construction of $\mathcal{A}_H$.

Lemma 3.1. *Suppose that H_1 and H_2 are subgroups of G . If $H_1 \subseteq H_2$, then $\mathcal{A}_{H_1} \supseteq \mathcal{A}_{H_2}$.*

Proof. Since z_{H_i} ($i = 1, 2$) are conditional expectations, namely projections of norm one, from $\mathcal{A}$ onto $\mathcal{A}_{H_i}$, by direct calculation, $z_{H_1} z_{H_2} = z_{H_2} z_{H_1} = z_{H_2}$, the result is clear. $\square$

Now for $\alpha, \beta, g \in G$, $n \in \mathbb{Z}$, and $l \in \mathbb{Z} + \frac{1}{2}$, set

$$w_{\alpha, \beta}^H(n) = \sum_{h \in H} \delta_{h\alpha}(n) \delta_{h\beta}(n+1)$$

and

$$v_g(l) = \sum_{h \in G} \rho_{hg^{-1}h^{-1}}(l) \delta_h\left(l + \frac{1}{2}\right) \rho_{hg^{-1}h}(l+).$$

Lemma 3.2. 1) $w_{\alpha, \beta}^H(n)$ is a selfadjoint idempotent element in $\mathcal{A}_H$ and for $\alpha \in G$, $z_H(\delta_\alpha(n)) = \frac{1}{|H|} \sum_{g \in G} w_{\alpha, g}^H(n)$.

2) $v_g(l)$ is a unitary element in $\mathcal{A}_G$ and thus $z_H(v_g(l)) = v_g(l)$.

Proof. 1) It is easy to see that $w_{\alpha,\beta}^H(n)$ is a self-adjoint idempotent element in $\mathcal{A}_H$. To prove that $w_{\alpha,\beta}^H \in \mathcal{A}_H$, it suffices to show that $z_H(w_{\alpha,\beta}^H(n)) = w_{\alpha,\beta}^H(n)$. Indeed,

$$\begin{aligned}
 z_H(w_{\alpha,\beta}^H(n)) &= z_H\left(\sum_{h \in H} \delta_{h\alpha}(n) \delta_{h\beta}(n+1)\right) \\
 &= \frac{1}{|H|} \sum_{g, h \in H} (e, g) (\delta_{h\alpha}(n) \delta_{h\beta}(n+1)) \\
 &= \frac{1}{|H|} \sum_{g, h \in H} \sum_{x \in G} (x, g) (\delta_{h\alpha}(n)) (x^{-1}, g) (\delta_{h\beta}(n+1)) \\
 &= \frac{1}{|H|} \sum_{g, h \in H} \sum_{x \in G} \delta_{x,e} \delta_{gh\alpha}(n) \delta_{x,e} \delta_{gh\beta}(n+1) \\
 &= \frac{1}{|H|} \sum_{g, h \in H} \delta_{gh\alpha}(n) \delta_{gh\beta}(n+1) \\
 &= \sum_{h \in H} \delta_{h\alpha}(n) \delta_{h\beta}(n+1) \\
 &= w_{\alpha,\beta}^H(n).
 \end{aligned}$$

Now for $\alpha \in G$,

$$\begin{aligned}
 z_H(\delta_\alpha(n)) &= \frac{1}{|H|} \sum_{h \in H} \delta_{h\alpha}(n) \\
 &= \frac{1}{|H|} \sum_{h \in H} \delta_{h\alpha}(n) \left(\sum_{g \in G} \delta_{hg}(n+1)\right) \\
 &= \frac{1}{|H|} \sum_{g \in G} \left(\sum_{h \in H} \delta_{h\alpha}(n) \delta_{hg}(n+1)\right) \\
 &= \frac{1}{|H|} \sum_{g \in G} w_{\alpha,g}^H(n).
 \end{aligned}$$

Similarly we can prove the second part of the lemma and we omit it here. $\square$

Remark. If $H = G$, then $w_{\alpha,\beta}^G(n) = w_{e,\alpha^{-1}\beta}^G(n)$, which is exactly the element $w_{e,\alpha^{-1}\beta}(n)$ given in [17].

Lemma 3.3. *Suppose that $g_i \in G$ and $k_i \in \mathbb{N}$ ($1 \leq i \leq n$, $n \in \mathbb{N}$). If the image of $\delta_{g_1}(k_1) \delta_{g_2}(k_2) \cdots \delta_{g_n}(k_n)$ under that action of z_H is not zero, then it is the linear combination of the products of elements with the form of $w_{x,y}^H(n)$, where $x, y \in G$ and $n \in \mathbb{Z}$.*

Proof. Without loss of generality, one can suppose that $k_i = i$ and that $n \geq 2$. Then

$$\begin{aligned}
 & z_H (\delta_{g_1} (1) \delta_{g_2} (2) \cdots \delta_{g_n} (n)) \\
 &= \frac{1}{|H|} \sum_{h \in H} (e, h) (\delta_{g_1} (1) \delta_{g_2} (2) \cdots \delta_{g_n} (n)) \\
 &= \frac{1}{|H|} \sum_{h \in H} \sum_{((e, h))} (e, h)_{(1)} (\delta_{g_1} (1)) \cdots (e, h)_{(n)} (\delta_{g_n} (n)) \\
 &= \frac{1}{|H|} \sum_{h \in H} \sum_{x_1, \dots, x_n \in G} (x_1, h) (\delta_{g_1} (1)) \cdots \\
 &\quad \times (x_{n-1}^{-1} x_n, h) (\delta_{g_{n-1}} (n-1)) (x_n, h) (\delta_{g_n} (n)) \\
 &= \frac{1}{|H|} \sum_{h \in H} \delta_{hg_1} (1) \delta_{hg_2} (2) \cdots \delta_{hg_n} (n) \\
 &= \frac{1}{|H|} \sum_{h_1, \dots, h_n \in H} \delta_{h_1 g_1} (1) (\delta_{h_1 g_2} (2) \delta_{h_2 g_2} (2)) \cdots (\delta_{h_{n-1} g_n} (n) \delta_{h_n g_n} (n)) \\
 &= \frac{1}{|H|} \sum_{h_1, \dots, h_n \in H} (\delta_{h_1 g_1} (1) \delta_{h_1 g_2} (2)) \cdots \\
 &\quad \times (\delta_{h_{n-1} g_{n-1}} (n-1) \delta_{h_{n-1} g_n} (n)) \delta_{h_n g_n} (n) \\
 &= w_{g_1, g_2}^H (1) \cdots w_{g_{n-1}, g_n}^H (n-1) \left(\frac{1}{|H|} \sum_{h \in H} \delta_{h_n g_n} (n) \right) \\
 &= w_{g_1, g_2}^H (1) \cdots w_{g_{n-1}, g_n}^H (n-1) z_H (\delta_{g_n} (n)) \\
 &= \frac{1}{|H|} \sum_{g \in G} w_{g_1, g_2}^H (1) \cdots w_{g_{n-1}, g_n}^H (n-1) w_{g_n, g}^H (n).
 \end{aligned}$$

This completes the proof. $\square$

Lemma 3.4. Suppose that $x \in \mathbb{Z}$ and $l \in \mathbb{Z} + \frac{1}{2}$. Then $\forall \alpha, g \in G$,

$$\delta_\alpha (x) v_g (l) = \begin{cases} v_g (l) \delta_\alpha (x), & \text{if } x \leq l + \frac{1}{2}, \\ v_g (l) \delta_{\alpha g} (x), & \text{if } x = l - \frac{1}{2}, \\ v_g (l) \delta_\alpha (x), & \text{if } x \geq l + \frac{3}{2}. \end{cases}$$

Proof. If $x \leq l + \frac{1}{2}$, then

$$\begin{aligned}
 \delta_\alpha (x) v_g (l) &= \sum_{h \in G} \delta_\alpha (x) \rho_{hg^{-1}h^{-1}} (l) \delta_h \left(l + \frac{1}{2} \right) \rho_{hg^{-1}h} (l+1) \\
 &= \sum_{h \in G} \rho_{hg^{-1}h^{-1}} (l) \delta_h \left(l + \frac{1}{2} \right) \rho_{hg^{-1}h} (l+1) \delta_\alpha (x) \\
 &= v_g (l) \delta_\alpha (x).
 \end{aligned}$$

Similarly if $x \geq l + \frac{3}{2}$, then $\delta_\alpha(x) v_g(l) = v_g(l) \delta_\alpha(x)$. Now we consider the case of $x = l + \frac{1}{2}$:

$$\begin{aligned}
\delta_\alpha\left(l + \frac{1}{2}\right) v_g(l) &= \sum_{h \in G} \delta_\alpha\left(l + \frac{1}{2}\right) \rho_{hg^{-1}h^{-1}}(l) \delta_h\left(l + \frac{1}{2}\right) \rho_{hg^{-1}h}(l+1) \\
&= \sum_{h \in G} \rho_{hg^{-1}h^{-1}}(l) \delta_{hgh^{-1}\alpha}\left(l + \frac{1}{2}\right) \delta_h\left(l + \frac{1}{2}\right) \rho_{hg^{-1}h}(l+1) \\
&= \sum_{h \in G} \delta_{hgh^{-1}, \alpha} \cdot \rho_{hg^{-1}h^{-1}}(l) \delta_h\left(l + \frac{1}{2}\right) \rho_{hg^{-1}h}(l+1) \\
&= \rho_{\alpha g^{-1}\alpha^{-1}}(l) \delta_{\alpha g}\left(l + \frac{1}{2}\right) \rho_{\alpha g\alpha^{-1}}(l+1). \quad (\text{set } h = \alpha g)
\end{aligned}$$

Through a similar calculation one has

$$v_g(l) \delta_\alpha\left(l + \frac{1}{2}\right) = \rho_{\alpha g^{-1}\alpha^{-1}}(l) \delta_{\alpha g}\left(l + \frac{1}{2}\right) \rho_{\alpha g\alpha^{-1}}(l+1).$$

Thus $\delta_\alpha\left(l + \frac{1}{2}\right) v_g(l) = v_g(l) \delta_\alpha\left(l + \frac{1}{2}\right)$. This completes the proof. $\square$

Since the observable algebra $\mathcal{A}_{\{e\}}$ is a C^* -subalgebra of $\mathcal{A}$ generated by the set [17]

$$\left\{v_g(l), \delta_g(x) \mid g \in G, l \in \mathbb{Z} + \frac{1}{2}, x \in \mathbb{Z}\right\},$$

and with the help of Lemma 2.2–Lemma 2.4, we have the following theorem.

Theorem 3.5. $\mathcal{A}_H$ is a unital C^* -subalgebra of $\mathcal{A}$ generated by the set

$$\left\{v_g(l), w_{\alpha, \beta}^H(x) \mid g, \alpha, \beta \in G, l \in \mathbb{Z} + \frac{1}{2}, x \in \mathbb{Z}\right\},$$

subject to the relations induced by Definition 2.2.

4. TOWARDS GALOIS THEORY FOR QUANTUM DOUBLE ALGEBRA

The following is the main result of the paper which states that the Hopf subalgebra $D(G; H)$ in $D(G)$ is Galois closed. Namely, suppose that $\overline{D(G; H)}$ is the Hopf subalgebra of $D(G)$ generated by the set

$$\{a \in D(G) \mid a(F) = \varepsilon(a) F, \forall F \in \mathcal{A}_H\}.$$

It is clear that $D(G; H) \subseteq \overline{D(G; H)}$. Moreover we have the following result.

Theorem 4.1. $\overline{D(G; H)} = D(G; H)$.

Proof. It is clear that $D(G; H) \subseteq \overline{D(G; H)}$. On the other hand, suppose that $d = \sum_{i=1}^k c_i(s_i, t_i) \in \overline{D(G; H)}$, where $0 \neq c_i \in \mathbb{C}$, $s_i, t_i \in G$ and $(s_i, t_i) \neq (s_j, t_j)$ if $i \neq j$. In order to prove that $d \in D(G; H)$, it suffices to prove that $t_i \in H$ for $1 \leq i \leq k$.

Since $\overline{D(G; H)}$ is a Hopf algebra, one has

$$\Delta(d) = \sum_{i=1}^k \sum_{f \in G} c_i(f, t_i) \otimes (f^{-1} s_i, t_i) \in \overline{D(G; H)} \otimes \overline{D(G; H)}.$$

Now if $t_i \neq t_j$ for $i \neq j$, since $\{(f, t_i) | f \in G, 1 \leq i \leq k\}$ and $\{(f^{-1} s_i, t_i)\}$ are both linearly independent sets in $\overline{D(G; H)}$, one has $(f, t_i) \in \overline{D(G; H)}$ ($f \in G, 1 \leq i \leq k$). In particular we have $(e, t_i) \in \overline{D(G; H)}$, where e is the unit in G and $1 \leq i \leq k$. If $s_i \neq s_j$ for $i \neq j$, we can divide the set $\{1, 2, \dots, k\}$ into different sets of $S_1, S_2, \dots, S_r$ such that $t_i = t_j$ if and only if i and j belong to the same S_j where $1 \leq j \leq r$. Then

$$\Delta(d) = \sum_{j=1}^r \sum_{f \in G} (f, t_j) \otimes \left(\sum_{i \in S_j} c_i(f^{-1} s_i, t_i) \right) \in \overline{D(G; H)} \otimes \overline{D(G; H)},$$

and we also have $(e, t_i) \in \overline{D(G; H)}$ ($1 \leq i \leq k$).

Using the definition of $\overline{D(G; H)}$, $\forall F \in \mathcal{A}_H$,

$$(e, t_i)(F) = (e, t_i)F = F.$$

In particular,

$$(e, t_i) \cdot (w_{e,e}^H(n)) = w_{e,e}^H(n),$$

where $n \in \mathbb{Z}$ and $w_{e,e}^H(n) = \frac{1}{|H|} \sum_{h \in H} \delta_h(n) \delta_h(n+1) \in \mathcal{A}_H$. Notice that for $1 \leq i \leq k$,

$$\begin{aligned} (e, t_i) \cdot (w_{e,e}^H(n)) &= \sum_{h \in H} (e, t_i) (\delta_h(n) \delta_h(n+1)) \\ &= \sum_{h \in H} \sum_{g \in G} (g, t_i) (\delta_h(n)) \cdot (g^{-1}, t_i) (\delta_h(n+1)) \\ &= \sum_{h \in H} \sum_{g \in G} \delta_{g,e} \delta_{t_i h}(n) \delta_{t_i h}(n+1) \\ &= \sum_{h \in H} \delta_{t_i h}(n) \delta_{t_i h}(n+1) \\ &= \left(\sum_{h \in H} \delta_{t_i h}(n) \delta_{t_i h}(n+1) \right), \end{aligned}$$

while

$$\varepsilon((e, t_i)) \cdot (w_{e,e}^H(n)) = \sum_{h \in H} \delta_h(n) \delta_h(n+1).$$

This implies that $t_i \in H$ ($1 \leq i \leq k$) and thus $d \in D(G; H)$. The proof is completed. $\square$

Remark. From the proof of Theorem 3.1 one can see that if $D(G; H)'$ is the sub-coalgebra of $D(G)$ generated by the set

$$\{a \in D(G) | a(F) = \varepsilon(a)F, \forall F \in \mathcal{A}_H\},$$

then $D(G; H)' = D(G; H)$.

Under the embedding map $g \rightarrow (E, g)$, the group algebra $\mathbb{C}G$ is a subalgebra of $D(G)$. The action γ of $D(G)$ on $\mathcal{A}$ can induce an action $\tilde{\gamma}$ of G on $\mathcal{A}_{\{e\}}$, so that G is a finite faithful automorphism group $\mathcal{A}_{\{e\}}$. Similar to the proof of Theorem 4.1, one has

Corollary 4.2. *If H is a subgroup of G , then H is Galois closed. That is to say, if $H' = \{g \in G : \tilde{\gamma}_g(A) = A, \forall A \in \mathcal{A}_H\}$, then $H' = H$.*

Remark. Let $\mathcal{F}$ be a C^* -algebra intermediate between $\mathcal{A}$ and $\mathcal{A}_G$. Does there exist a subgroup H in G so that $\mathcal{A}_H = \mathcal{F}$? The question is under consideration now.

REFERENCES

- [1] E. Abe, Hopf algebras, Cambridge University Press, 1977. MR594432 (83a:16010)
- [2] J. Bergen, A correspondence theorems for modules over Hopf algebras, Proc. AMS, 121(2), 343-345, 1994. MR1211578 (94h:16071)
- [3] S. U. Chase, D. K. Harrison, A. Rosenberg, Galois theory and cohomology of commutative rings, AMS Memoirs, No. 52, 1965. MR0195922 (33:4118)
- [4] S. U. Chase, M. E. Sweedler, Hopf algebras and Galois theory, Lecture Notes in Math., 97, Springer, Berlin, 1969. MR0260724 (41:5348)
- [5] K. A. Dancer, P. S. Isaac, J. Links, Representations of quantum doubles of finite group algebras and spectral parameter dependent solutions of the Yang-Baxter equations, J. Math. Phys., 47, 103511, 2006. MR2268877 (2007h:17016)
- [6] C. Dong, G. Mason, On quantum Galois theory, Duke Math. J., 86, 305-321, 1997. MR1430435 (97k:17042)
- [7] V. O. Ferreira, L. S. I. Murakami, A. Paques, A Hopf-Galois correspondence for free algebras, J. Alg., 276(1), 407-416, 2004. MR2054404 (2005a:16054)
- [8] M. D. Gould, Quantum double finite group algebras and their representations, Bull. Austral. Math. Soc., 48, 275-301, 1993. MR1238802 (94h:16072)
- [9] A. Hanaki, M. Miyamoto, D. Tambara, Quantum Galois theory for finite groups, Duke Math. J., 97, 541-544, 1999. MR1682988 (2000g:17043a)
- [10] L. N. Jiang, Field algebra of G -spin models and its duality, Acta Math. Sinic. (in Chinese), 45(1), 37-42, 2002. MR1923662 (2003g:46067)
- [11] V. F. R. Jones, Subfactors and knots, CBMS, No. 80, AMS, Providence, RI, 1991. MR1134131 (93b:57008)
- [12] Y. Kawahigashi, Quantum Galois correspondence for subfactors, J. Func. Anal., 167, 481-497, 1999. MR1716205 (2000g:46090)
- [13] H. F. Kreimer, M. Takeuchi, Hopf algebras and Galois extensions of an algebra, Indiana Univ. Math. J., 30, 675-692, 1981. MR625597 (83h:16015)
- [14] G. Mason, The quantum double of a finite group and its role in conformal field theory, London Mathematical Society Lecture Notes, 212, 405-417, Cambridge University Press, Cambridge, 1995. MR1337285 (97a:11067)
- [15] F. V. Oystaeyen, Galois-type correspondences for Hopf-Galois extensions, K -theory, 8, 257-169, 1994. MR1291021 (95f:16044)
- [16] D. E. Radford, Minimal quasi-triangular Hopf algebras, J. Alg., 157, 285-315, 1993. MR1220770 (94c:16052)
- [17] K. Szlachanyi, P. Vecsernyes, Quantum symmetry and braided group statistics in G -spin models, Comm. Math. Phys., 156, 127-168, 1993. MR1234107 (94m:81063)
- [18] A. Van Daele, Y. H. Zhang, Galois theory for multiplier Hopf algebras with integrals, Algebras and Representation Theory 2, 83-106, 1999. MR1688472 (2000f:16050)
- [19] T. Yanai, Galois correspondence theorem for Hopf algebra actions, Contemporary Mathematics, 376, 393-411, AMS, Providence, RI, 2005. MR2147038 (2006c:16072)
- [20] S. C. Zhang, Y. Z. Zhang, Hopf Galois extension braided tensor categories, arXiv:math.RA/0309448, 7, 19, April 2006.

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