YANG-TYPE INEQUALITIES FOR WEIGHTED EIGENVALUES OF A SECOND ORDER UNIFORMLY ELLIPTIC OPERATOR WITH A NONNEGATIVE POTENTIAL

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(Communicated by Matthew J. Gursky)

ABSTRACT. In this paper, we investigate the Dirichlet weighted eigenvalue problem of a second order uniformly elliptic operator with a nonnegative potential on a bounded domain $\Omega \subset \mathbb{R}^n$. First, we prove a general inequality of eigenvalues for this problem. Then, by using this general inequality, we obtain Yang-type inequalities which give universal upper bounds for eigenvalues. An explicit estimate for the gaps of any two consecutive eigenvalues is also derived. Our results contain and extend the previous results for eigenvalues of the Laplacian, the Schrödinger operator and the second order elliptic operator on a bounded domain $\Omega \subset \mathbb{R}^n$.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with piecewise smooth boundary $\partial\Omega$. Assume that $u(x)$ is a solution of the following Dirichlet eigenvalue problem corresponding to the eigenvalue Λ :

$$(1.1) \quad \begin{cases} - \sum_{i,j=1}^n D_i(a_{ij}(x)D_j u) = \Lambda u, & \text{in } \Omega \subset \mathbb{R}^n, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where $D_i = \frac{\partial}{\partial x_i}$, $a_{ij}(x) \in C^{1,\alpha}(\overline{\Omega})$, $0 < \alpha < 1$, and $a_{ij}(x) = a_{ji}(x)$. Moreover, we assume that the coefficients $a_{ij}(x)$ satisfy the uniform ellipticity assumption: for any $\xi = (\xi_1, \dots, \xi_n)$ on every point $x \in \overline{\Omega}$, there is a constant $\varsigma > 0$ such that

$$(1.2) \quad \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq \varsigma|\xi|^2,$$

where $|\xi| = (\xi_1^2 + \dots + \xi_n^2)^{\frac{1}{2}}$ denotes the Euclidean norm of ξ .

A prototype of a second order elliptic operator $\sum_{i,j=1}^n D_i(a_{ij}(x)D_j)$ is the Laplacian Δ . When $a_{ij}(x) = \delta_{ij}$, problem (1.1) becomes the classical Dirichlet Laplacian

Received by the editors August 22, 2009, and, in revised form, November 18, 2009.

2010 *Mathematics Subject Classification.* Primary 35P15, 58C40; Secondary 58J50.

Key words and phrases. Eigenvalue, universal inequality, elliptic operator, Laplacian, Schrödinger operator.

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problem (also called the fixed membrane problem):

$$(1.3) \quad \begin{cases} -\Delta u = \lambda u, & \text{in } \Omega \subset \mathbb{R}^n, \\ u|_{\partial\Omega} = 0. \end{cases}$$

Let us recall some previous results for eigenvalues of problem (1.3). In 1956, Payne, Pólya and Weinberger [14] considered problem (1.3) and established an important universal inequality for $\Omega \subset \mathbb{R}^2$, which was extended to $\Omega \subset \mathbb{R}^n$ as the PPW inequality

$$(1.4) \quad \lambda_{k+1} - \lambda_k \leq \frac{4}{nk} \sum_{t=1}^k \lambda_t.$$

The results of Hile and Protter [8], and Yang [19] are two remarkable contributions to the improvements of the PPW inequality. In 1980, Hile and Protter [8] proved

$$(1.5) \quad \sum_{t=1}^k \frac{\lambda_t}{\lambda_{k+1} - \lambda_t} \geq \frac{nk}{4},$$

which is called the HP inequality (cf. [9]). In 1991, Yang [19] obtained Yang's first inequality,

$$(1.6) \quad \sum_{t=1}^k (\lambda_{k+1} - \lambda_t)^2 \leq \frac{4}{n} \sum_{t=1}^k (\lambda_{k+1} - \lambda_t) \lambda_t.$$

For Yang's original proof, we also refer to [6]. Utilizing the Cauchy-Schwarz inequality, one can obtain a weaker inequality, namely Yang's second inequality,

$$(1.7) \quad \lambda_{k+1} \leq \left(1 + \frac{4}{n}\right) \frac{1}{k} \sum_{t=1}^k \lambda_t.$$

Up to now, Yang's first inequality has been the sharpest result for problem (1.3). In [7], Harrell and Stubbe gave a new proof of Yang's inequalities based on commutator algebra. One can find more discussions about inequalities (1.4)-(1.7) in [1, 2] of Ashbaugh. In 2008, Chen and Cheng [3] obtained (1.6) for a bounded domain in an n -dimensional complete Riemannian manifold. Some remarkable results for problem (1.3) were also derived on some types of manifolds (see [4, 5, 10, 11, 17, 20]).

The spectrum of problem (1.1) has been discussed from some angles, but fewer universal inequalities of eigenvalues have been derived. The previous main result was obtained by Qian and Chen [15]. They proved

$$(1.8) \quad \sum_{t=1}^k \frac{\Lambda_t}{\Lambda_{k+1} - \Lambda_t} \geq \frac{\varsigma n^2 k}{4} (\max_{x \in \overline{\Omega}} \text{tr} A(x))^{-1},$$

where $\text{tr} A(x)$ denotes the trace of the matrix $A(x) = (a_{ij}(x))_{n \times n}$. When $a_{ij}(x) = \delta_{ij}$, it follows that $\text{tr} A(x) = n$. Therefore, (1.8) corresponds to (1.5).

In this paper, we investigate the Dirichlet weighted eigenvalue problem of a second order uniformly elliptic operator with a non-negative potential:

$$(1.9) \quad \begin{cases} -\sum_{i,j=1}^n D_i(a_{ij}(x)D_j u) + a_0(x)u = \Lambda \rho u, & \text{in } \Omega \subset \mathbb{R}^n, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where $a_0(x)$ is a non-negative function on $\overline{\Omega}$ and ρ is a positive function continuous on $\overline{\Omega}$. Problem (1.9) is significant in the theory of PDE, the applications of mechanics and physics. In general, the weight function ρ denotes the density. Hence, weighted estimates for eigenvalues have many applications. For example, it is intelligent to make weighted estimates in filtering and identification problems (see [12, 13]).

When $a_{ij}(x) = \delta_{ij}$, problem (1.9) becomes the following weighted eigenvalue problem of the Schrödinger operator:

$$(1.10) \quad \begin{cases} -\Delta u + a_0(x)u = \mu\rho u, & \text{in } \Omega \subset \mathbb{R}^n, \\ u|_{\partial\Omega} = 0. \end{cases}$$

In [2], Ashbaugh obtained Yang's inequalities

$$(1.11) \quad \sum_{t=1}^k (\mu_{k+1} - \mu_t)^2 \leq \frac{4\sigma}{n\tau} \sum_{t=1}^k (\mu_{k+1} - \mu_t)\mu_t$$

and

$$(1.12) \quad \mu_{k+1} \leq \left(1 + \frac{4\sigma}{n\tau}\right) \frac{1}{k} \sum_{t=1}^k \mu_t.$$

For more results of problem (1.10) on the manifolds, we refer the reader to [16, 18].

From the above survey, we find that problem (1.9) synthesizes the eigenvalue problems of various kinds of differential operators, such as the Laplacian, the Schrödinger operator and the second order elliptic operator. It is a very interesting problem.

The goal of this paper is to obtain Yang-type inequalities for the Dirichlet weighted eigenvalues problem (1.9) of a second order uniformly elliptic operator. In Section 2, we prove a general inequality of eigenvalues. Then, by using this general inequality, we derive Yang-type inequalities for problem (1.9) in Theorem 3.1 and Corollary 3.4. As corollaries of Theorem 3.1, some universal inequalities and bounds of eigenvalues are also derived. For instance, when the potential $a_0(x) = 0$ and the weight function $\rho = 1$, problem (1.9) becomes problem (1.1). As applications of Theorem 3.1, we consider problem (1.1) in Section 4. A series of explicit estimates for problem (1.1) are derived in Corollaries 4.1–4.4. Moreover, our result (4.1) in Corollary 4.1 is sharper than (1.8) derived by Qian and Chen [15] for problem (1.1).

2. A GENERAL INEQUALITY

In this section, we shall prove a general inequality for eigenvalues of problem (1.9).

Theorem 2.1. *Let u_t be the t -th weighted orthonormal eigenfunction of problem (1.9) corresponding to the eigenvalue Λ_t , $t = 1, 2, \dots, k$. Namely, u_t satisfies*

$$(2.1) \quad \begin{cases} -\sum_{i,j=1}^n D_i(a_{ij}(x)D_j u_t) + a_0(x)u_t = \Lambda_t \rho u_t, & \text{in } \Omega \subset \mathbb{R}^n, \\ u_t|_{\partial\Omega} = 0, \\ \int_{\Omega} \rho u_t u_s = \delta_{ts}. \end{cases}$$

Then, we have

$$(2.2) \quad \sum_{t=1}^k (\Lambda_{k+1} - \Lambda_t)^2 \int_{\Omega} u_t^2 \leq \delta \sum_{t=1}^k (\Lambda_{k+1} - \Lambda_t)^2 \int_{\Omega} a_{ll}(x) u_t^2 + \frac{1}{\delta} \sum_{t=1}^k (\Lambda_{k+1} - \Lambda_t) \int_{\Omega} \frac{1}{\rho} |D_l u_t|^2,$$

where the constant $\delta > 0$.

Proof. Let $x = (x_1, \dots, x_n)$ be the Cartesian coordinate functions of $\mathbb{R}^n$. We define the trial functions φ_{tl} by

$$(2.3) \quad \varphi_{tl} = x_l u_t - \sum_{s=1}^k b_{ts}^l u_s, \quad \text{for } t = 1, \dots, k \text{ and } l = 1, \dots, n,$$

where

$$(2.4) \quad b_{ts}^l = \int_{\Omega} \rho x_l u_t u_s = b_{st}^l.$$

Then, it is easy to check that φ_{tl} is weighted orthogonal with u_s , namely

$$(2.5) \quad \int_{\Omega} \rho \varphi_{tl} u_s = 0, \quad \text{for } t, s = 1, \dots, k.$$

Hence, it is obvious that

$$(2.6) \quad \int_{\Omega} \rho \varphi_{tl} x_l u_t = \int_{\Omega} \rho \varphi_{tl}^2.$$

By direct calculations, one can get

$$(2.7) \quad \begin{aligned} \sum_{i,j=1}^n D_i [a_{ij}(x) D_j (x_l u_t)] &= \sum_{i=1}^n D_i (a_{il}(x) u_t) + \sum_{i,j=1}^n D_i (a_{ij}(x) x_l D_j u_t) \\ &= \sum_{i=1}^n [D_i (a_{il}(x) u_t) + a_{il}(x) D_i u_t] + x_l \sum_{i,j=1}^n D_i (a_{ij}(x) D_j u_t). \end{aligned}$$

This yields

$$(2.8) \quad \begin{aligned} &\sum_{i,j=1}^n D_i (a_{ij}(x) D_j \varphi_{tl}) - a_0(x) \varphi_{tl} \\ &= \sum_{i,j=1}^n D_i [a_{ij}(x) D_j (x_l u_t)] - \sum_{s=1}^k b_{ts}^l \sum_{i,j=1}^n D_i (a_{ij}(x) D_j u_s) - a_0(x) \varphi_{tl} \\ &= \sum_{i=1}^n [D_i (a_{il}(x) u_t) + a_{il}(x) D_i u_t] - x_l \left[- \sum_{i,j=1}^n D_i (a_{ij}(x) D_j u_t) + a_0(x) u_t \right] \\ &\quad + \sum_{s=1}^k b_{ts}^l \left[- \sum_{i,j=1}^n D_i (a_{ij}(x) D_j u_s) + a_0(x) u_s \right] \\ &= \sum_{i=1}^n [D_i (a_{il}(x) u_t) + a_{il}(x) D_i u_t] - x_l \Lambda_t \rho u_t + \sum_{s=1}^k b_{ts}^l \Lambda_s \rho u_s. \end{aligned}$$

Hence, from (2.5), (2.6) and (2.8), we have

$$\begin{aligned}
 (2.9) \quad & \int_{\Omega} \varphi_{tl} \left[- \sum_{i,j=1}^n D_i(a_{ij}(x) D_j \varphi_{tl}) + a_0(x) \varphi_{tl} \right] \\
 &= - \int_{\Omega} \varphi_{tl} \sum_{i=1}^n [D_i(a_{il}(x) u_t) + a_{il}(x) D_i u_t] + \Lambda_t \int_{\Omega} \rho \varphi_{tl}^2.
 \end{aligned}$$

Substituting (2.9) into the Rayleigh-Ritz formula in the variation method

$$(2.10) \quad \Lambda_{k+1} \leq \frac{\int_{\Omega} \varphi_{tl} \left[- \sum_{i,j=1}^n D_i(a_{ij}(x) D_j \varphi_{tl}) + a_0(x) \varphi_{tl} \right]}{\int_{\Omega} \rho \varphi_{tl}^2},$$

we obtain

$$\begin{aligned}
 (2.11) \quad & (\Lambda_{k+1} - \Lambda_t) \int_{\Omega} \rho \varphi_{tl}^2 \\
 & \leq - \int_{\Omega} \varphi_{tl} \sum_{i=1}^n [D_i(a_{il}(x) u_t) + a_{il}(x) D_i u_t] \\
 &= - \int_{\Omega} x_l u_t \sum_{i=1}^n [D_i(a_{il}(x) u_t) + a_{il}(x) D_i u_t] + \sum_{s=1}^k b_{ts}^l c_{ts}^l,
 \end{aligned}$$

where

$$c_{ts}^l = \int_{\Omega} u_s \sum_{i=1}^n [D_i(a_{il}(x) u_t) + a_{il}(x) D_i u_t] = -c_{st}^l.$$

Using integration by parts and utilizing (2.1), we have

$$\begin{aligned}
 (2.12) \quad & \Lambda_t b_{ts}^l = \int_{\Omega} x_l u_s \left[- \sum_{i,j=1}^n D_i(a_{ij}(x) D_j u_t) + a_0(x) u_t \right] \\
 &= - \int_{\Omega} u_t \sum_{i,j=1}^n D_j[a_{ij}(x) D_i(x_l u_s)] + \int_{\Omega} a_0(x) x_l u_t u_s \\
 &= - \int_{\Omega} u_t x_l \sum_{i,j=1}^n D_j(a_{ij}(x) D_i u_s) - \int_{\Omega} u_t \sum_{i=1}^n a_{il}(x) D_i u_s \\
 &\quad - \int_{\Omega} u_t \sum_{i=1}^n D_i(a_{il}(x) u_s) + \int_{\Omega} a_0(x) x_l u_t u_s \\
 &= \int_{\Omega} u_t x_l \Lambda_s \rho u_s - \int_{\Omega} u_t \sum_{i=1}^n [a_{il}(x) D_i u_s + D_i(a_{il}(x) u_s)] \\
 &= \Lambda_s b_{ts}^l - c_{st}^l.
 \end{aligned}$$

This yields

$$(2.13) \quad c_{ts}^l = (\Lambda_t - \Lambda_s) b_{ts}^l.$$

Moreover, we have

$$\begin{aligned}
 & \int_{\Omega} x_l u_t \sum_{i=1}^n [D_i(a_{il}(x)u_t) + a_{il}(x)D_i u_t] \\
 &= \int_{\Omega} x_l u_t \sum_{i=1}^n a_{il}(x)D_i u_t - \int_{\Omega} \sum_{i=1}^n a_{il}(x)u_t D_i(x_l u_t) \\
 (2.14) \quad &= \int_{\Omega} x_l u_t \sum_{i=1}^n a_{il}(x)D_i u_t - \int_{\Omega} x_l u_t \sum_{i=1}^n a_{il}(x)D_i u_t - \int_{\Omega} a_{ll}(x)u_t^2 \\
 &= - \int_{\Omega} a_{ll}(x)u_t^2.
 \end{aligned}$$

Substituting (2.13) and (2.14) into (2.11), we have

$$(2.15) \quad (\Lambda_{k+1} - \Lambda_t) \int_{\Omega} \rho \varphi_{tl}^2 \leq \int_{\Omega} a_{ll}(x)u_t^2 + \sum_{s=1}^k (\Lambda_t - \Lambda_s)(b_{ts}^l)^2.$$

It is not hard to see that

$$(2.16) \quad \int_{\Omega} x_l u_t D_l u_t = -\frac{1}{2} \int_{\Omega} u_t^2.$$

Hence, using integration by parts again, we have

$$(2.17) \quad -2 \int_{\Omega} \varphi_{tl} D_l u_t = -2 \int_{\Omega} x_l u_t D_l u_t + 2 \sum_{s=1}^k b_{ts}^l d_{ts}^l = \int_{\Omega} u_t^2 + 2 \sum_{s=1}^k b_{ts}^l d_{ts}^l,$$

where

$$d_{ts}^l = \int_{\Omega} u_s D_l u_t = -d_{st}^l.$$

On the other hand, we have

$$\begin{aligned}
 & -2(\Lambda_{k+1} - \Lambda_t)^2 \int_{\Omega} \varphi_{tl} D_l u_t \\
 &= -2(\Lambda_{k+1} - \Lambda_t)^2 \int_{\Omega} \sqrt{\rho} \varphi_{tl} \left(\frac{1}{\sqrt{\rho}} D_l u_t - \sqrt{\rho} \sum_{s=1}^k d_{ts}^l u_s \right) \\
 &\leq \delta (\Lambda_{k+1} - \Lambda_t)^3 \int_{\Omega} \rho \varphi_{tl}^2 + \frac{\Lambda_{k+1} - \Lambda_t}{\delta} \int_{\Omega} \left(\frac{1}{\sqrt{\rho}} D_l u_t - \sqrt{\rho} \sum_{s=1}^k d_{ts}^l u_s \right)^2 \\
 (2.18) \quad &= \frac{\Lambda_{k+1} - \Lambda_t}{\delta} \left[\int_{\Omega} \frac{1}{\rho} |D_l u_t|^2 - 2 \sum_{s=1}^k d_{ts}^l \int_{\Omega} u_s D_l u_t + \sum_{s,q=1}^k d_{ts}^l d_{tq}^l \int_{\Omega} \rho u_s u_q \right] \\
 &\quad + \delta (\Lambda_{k+1} - \Lambda_t)^3 \int_{\Omega} \rho \varphi_{tl}^2 \\
 &= \delta (\Lambda_{k+1} - \Lambda_t)^3 \int_{\Omega} \rho \varphi_{tl}^2 + \frac{\Lambda_{k+1} - \Lambda_t}{\delta} \left[\int_{\Omega} \frac{1}{\rho} |D_l u_t|^2 - \sum_{s=1}^k (d_{ts}^l)^2 \right],
 \end{aligned}$$

where $\delta > 0$ is a constant to be determined in the proof of Theorem 3.1. Substituting (2.15) into (2.18), we have

$$\begin{aligned}
 & -2(\Lambda_{k+1} - \Lambda_t)^2 \int_{\Omega} \varphi_{tl} D_l u_t \\
 (2.19) \quad & \leq \delta(\Lambda_{k+1} - \Lambda_t)^2 \left[\int_{\Omega} a_{ll}(x) u_t^2 + \sum_{s=1}^k (\Lambda_t - \Lambda_s) (b_{ts}^l)^2 \right] \\
 & + \frac{\Lambda_{k+1} - \Lambda_t}{\delta} \left[\int_{\Omega} \frac{1}{\rho} |D_l u_t|^2 - \sum_{s=1}^k (d_{ts}^l)^2 \right].
 \end{aligned}$$

Combining (2.17) and (2.19) and taking the sum on t from 1 to k , we have

$$\begin{aligned}
 & \sum_{t=1}^k (\Lambda_{k+1} - \Lambda_t)^2 \int_{\Omega} u_t^2 + 2 \sum_{t,s=1}^k (\Lambda_{k+1} - \Lambda_t)^2 b_{ts}^l d_{ts}^l \\
 (2.20) \quad & \leq \delta \sum_{t=1}^k (\Lambda_{k+1} - \Lambda_t)^2 \int_{\Omega} a_{ll}(x) u_t^2 + \delta \sum_{t,s=1}^k (\Lambda_{k+1} - \Lambda_t)^2 (\Lambda_t - \Lambda_s) (b_{ts}^l)^2 \\
 & + \frac{1}{\delta} \sum_{t=1}^k (\Lambda_{k+1} - \Lambda_t) \int_{\Omega} \frac{1}{\rho} |D_l u_t|^2 - \frac{1}{\delta} \sum_{t,s=1}^k (\Lambda_{k+1} - \Lambda_t) (d_{ts}^l)^2.
 \end{aligned}$$

Then, we can eliminate the unwanted terms in both sides of (2.20) by using the following equalities:

$$\sum_{t,s=1}^k (\Lambda_{k+1} - \Lambda_t)^2 b_{ts}^l d_{ts}^l = - \sum_{t,s=1}^k (\Lambda_{k+1} - \Lambda_t) (\Lambda_t - \Lambda_s) b_{ts}^l d_{ts}^l$$

and

$$\sum_{t,s=1}^k (\Lambda_{k+1} - \Lambda_t)^2 (\Lambda_t - \Lambda_s) (b_{ts}^l)^2 = - \sum_{t,s=1}^k (\Lambda_{k+1} - \Lambda_t) (\Lambda_t - \Lambda_s)^2 (b_{ts}^l)^2.$$

Therefore, (2.2) is true. $\square$

3. YANG-TYPE INEQUALITIES FOR PROBLEM (1.9)

In this section, we derive Yang-type inequalities for eigenvalues of problem (1.9) by utilizing the general inequality (2.2).

Theorem 3.1. *Let Λ_t be the t -th eigenvalue of problem (1.9). Set*

$$\sigma = \left(\min_{x \in \Omega} \rho(x) \right)^{-1}, \quad \tau = \left(\max_{x \in \Omega} \rho(x) \right)^{-1}.$$

Assume that $a_{ij}(x)$ satisfies the uniform ellipticity assumption (1.2). Denote by $A(x)$ an $n \times n$ positive definite matrix with components $a_{ij}(x)$ and by $\text{tr} A(x)$ the trace of the matrix $A(x) = (a_{ij}(x))_{n \times n}$. Then, we have

$$(3.1) \quad \sum_{t=1}^k (\Gamma_{k+1} - \Gamma_t)^2 \leq \frac{4\sigma^2}{n^2 \tau^2 \zeta} \max_{x \in \Omega} \text{tr} A(x) \sum_{t=1}^k (\Gamma_{k+1} - \Gamma_t) \Gamma_t,$$

where $\Gamma_t = \Lambda_t - \tau \min_{x \in \Omega} a_0(x)$.

Proof. Taking the sum on l from 1 to n in (2.2), we have

$$(3.2) \quad \begin{aligned} n \sum_{t=1}^k (\Lambda_{k+1} - \Lambda_t)^2 \int_{\Omega} u_t^2 &\leq \delta \sum_{t=1}^k (\Lambda_{k+1} - \Lambda_t)^2 \int_{\Omega} \sum_{l=1}^n a_{ll}(x) u_t^2 \\ &\quad + \frac{1}{\delta} \sum_{t=1}^k (\Lambda_{k+1} - \Lambda_t) \int_{\Omega} \frac{1}{\rho} \sum_{l=1}^n |D_l u_t|^2. \end{aligned}$$

According to the assumptions of Theorem 3.1, it is not hard to find that

$$(3.3) \quad 0 < \tau = \tau \int_{\Omega} \rho u_t^2 \leq \int_{\Omega} u_t^2 = \int_{\Omega} \rho u_t^2 \frac{1}{\rho} \leq \sigma \int_{\Omega} \rho u_t^2 = \sigma,$$

$$(3.4) \quad \sum_{l=1}^n \int_{\Omega} a_{ll}(x) u_t^2 = \int_{\Omega} \operatorname{tr} A(x) u_t^2 \leq \sigma \max_{x \in \overline{\Omega}} \operatorname{tr} A(x).$$

Noticing the uniform ellipticity assumption (1.2), we have

$$\begin{aligned} \Lambda_t &= \int_{\Omega} u_t \left[- \sum_{i,j} D_i (a_{ij} D_j u_t) + a_0(x) u_t \right] = \int_{\Omega} \sum_{i,j} a_{ij} D_i u_t D_j u_t + \int_{\Omega} a_0(x) u_t^2 \\ &\geq \varsigma \int_{\Omega} |Du_t|^2 + \tau \min_{x \in \overline{\Omega}} a_0(x). \end{aligned}$$

This yields

$$(3.5) \quad \int_{\Omega} |Du_t|^2 \leq \frac{1}{\varsigma} (\Lambda_t - \tau \min_{x \in \overline{\Omega}} a_0(x)).$$

Therefore, we have

$$(3.6) \quad \int_{\Omega} \frac{1}{\rho} \sum_{l=1}^n |D_l u_t|^2 = \int_{\Omega} \frac{1}{\rho} |Du_t|^2 \leq \frac{\sigma}{\varsigma} (\Lambda_t - \tau \min_{x \in \overline{\Omega}} a_0(x)).$$

Combining (3.3)–(3.6), we obtain

$$(3.7) \quad \begin{aligned} n\tau \sum_{t=1}^k (\Lambda_{k+1} - \Lambda_t)^2 &\leq \delta \sigma \max_{x \in \overline{\Omega}} \operatorname{tr} A(x) \sum_{t=1}^k (\Lambda_{k+1} - \Lambda_t)^2 \\ &\quad + \frac{\sigma}{\delta \varsigma} \sum_{t=1}^k (\Lambda_{k+1} - \Lambda_t) (\Lambda_t - \tau \min_{x \in \overline{\Omega}} a_0(x)). \end{aligned}$$

Then, putting

$$\delta = \left[\sum_{t=1}^k (\Lambda_{k+1} - \Lambda_t) (\Lambda_t - \tau \min_{x \in \overline{\Omega}} a_0(x)) \right]^{\frac{1}{2}} \left[\varsigma \max_{x \in \overline{\Omega}} \operatorname{tr} A(x) \sum_{t=1}^k (\Lambda_{k+1} - \Lambda_t)^2 \right]^{-\frac{1}{2}}$$

in (3.7), we have

$$\begin{aligned} &n\tau \sum_{t=1}^k (\Lambda_{k+1} - \Lambda_t)^2 \\ &\leq 2\sigma \varsigma^{-\frac{1}{2}} \left[\sum_{t=1}^k (\Lambda_{k+1} - \Lambda_t)^2 \right]^{\frac{1}{2}} \left[\max_{x \in \overline{\Omega}} \operatorname{tr} A(x) \sum_{t=1}^k (\Lambda_{k+1} - \Lambda_t) (\Lambda_t - \tau \min_{x \in \overline{\Omega}} a_0(x)) \right]^{\frac{1}{2}}. \end{aligned}$$

Deleting $\left[\sum_{t=1}^k (\Lambda_{k+1} - \Lambda_t)^2 \right]^{\frac{1}{2}}$ on both sides of the above inequality yields

$$(3.8) \quad \sum_{t=1}^k (\Lambda_{k+1} - \Lambda_t)^2 \leq \frac{4\sigma^2}{n^2\tau^2\zeta} \max_{x \in \Omega} \text{tr} A(x) \sum_{t=1}^k (\Lambda_{k+1} - \Lambda_t) \left(\Lambda_t - \tau \min_{x \in \Omega} a_0(x) \right).$$

Then, putting $\Gamma_t = \Lambda_t - \tau \min_{x \in \Omega} a_0(x)$ in (3.8), we derive (3.1). $\square$

Remark 3.1. We find that the estimate (3.1) depends on the trace of the positive definite matrix $A(x)$. It is well known that $\text{tr} A(x) = \sum_{l=1}^n \nu_l(x)$, where $\nu_l(x)$ is the eigenvalue of the matrix $A(x)$ at x , for $l = 1, \dots, n$. Therefore, it is interesting that the coefficient $\max_{x \in \Omega} \text{tr} A(x)$ can also be replaced by $\max_{x \in \Omega} \sum_{l=1}^n \nu_l(x)$ or $n \cdot \max_{x \in \Omega} \nu_0(x)$, where $\nu_0(x)$ is the the maximal eigenvalue of $A(x)$ at x .

Since $\Gamma_t = \Lambda_t - \tau \min_{x \in \Omega} a_0(x) \leq \Lambda_t$, we can obtain a weaker inequality:

Corollary 3.2. *Under the same assumptions as Theorem 3.1, we have*

$$(3.9) \quad \sum_{t=1}^k (\Lambda_{k+1} - \Lambda_t)^2 \leq \frac{4\sigma^2}{n^2\tau^2\zeta} \max_{x \in \Omega} \text{tr} A(x) \sum_{t=1}^k (\Lambda_{k+1} - \Lambda_t) \Lambda_t.$$

Noticing that (3.1) is a quadratic inequality of Γ_{k+1} , we can obtain the following two corollaries which give universal upper bounds of the $(k+1)$ -th eigenvalue Λ_{k+1} .

Corollary 3.3. *Under the same assumptions as Theorem 3.1, we have*

$$(3.10) \quad \begin{aligned} \Gamma_{k+1} \leq & \left(1 + \frac{2\sigma^2}{n^2\tau^2\zeta} \max_{x \in \Omega} \text{tr} A(x) \right) \frac{1}{k} \sum_{t=1}^k \Gamma_t + \left\{ \left[\frac{2\sigma^2}{n^2\tau^2\zeta} \max_{x \in \Omega} \text{tr} A(x) \frac{1}{k} \sum_{t=1}^k \Gamma_t \right]^2 \right. \\ & \left. - \left(1 + \frac{4\sigma^2}{n^2\tau^2\zeta} \max_{x \in \Omega} \text{tr} A(x) \right) \frac{1}{k} \sum_{s=1}^k \left(\Gamma_s - \frac{1}{k} \sum_{t=1}^k \Gamma_t \right)^2 \right\}^{\frac{1}{2}}. \end{aligned}$$

Corollary 3.4. *Under the same assumptions as Theorem 3.1, we have*

$$(3.11) \quad \Gamma_{k+1} \leq \left(1 + \frac{4\sigma^2}{n^2\tau^2\zeta} \max_{x \in \Omega} \text{tr} A(x) \right) \frac{1}{k} \sum_{t=1}^k \Gamma_t.$$

Remark 3.2. The inequalities (3.1) and (3.11) are Yang-type. In fact, it is easy to find that (3.1) and (3.11) respectively correspond to Yang's first inequality (1.6) and Yang's second inequality (1.7).

At the same time, an explicit universal estimate for the gaps of any two consecutive eigenvalues of problem (1.9) can be obtained.

Corollary 3.5. *Under the same assumptions as Theorem 3.1, we have*

$$(3.12) \quad \begin{aligned} \Lambda_{k+1} - \Lambda_k \leq & 2 \left\{ \left[\frac{2\sigma^2}{n^2\tau^2\zeta} \max_{x \in \Omega} \text{tr} A(x) \left(\frac{1}{k} \sum_{t=1}^k \Lambda_t - \tau \min_{x \in \Omega} a_0(x) \right) \right]^2 \right. \\ & \left. - \left(1 + \frac{4\sigma^2}{n^2\tau^2\zeta} \max_{x \in \Omega} \text{tr} A(x) \right) \frac{1}{k} \sum_{s=1}^k \left(\Lambda_s - \frac{1}{k} \sum_{t=1}^k \Lambda_t \right)^2 \right\}^{\frac{1}{2}}. \end{aligned}$$

4. APPLICATIONS OF THEOREM 3.1

In this section, we want to give some applications of Theorem 3.1. One of the reasons for the importance of the estimate (3.1) is that we can easily get some estimates for the special cases of problem (1.9). For example, when the potential $a_0(x) = 0$ and the weight function $\rho = 1$, problem (1.9) becomes problem (1.1). Here, as corollaries of Theorem 3.1, we only give the estimates for eigenvalues of problem (1.1). Of course, the reader can also derive some estimates for other special cases of problem (1.9) from Theorem 3.1.

Corollary 4.1. *Let Λ_t be the t -th eigenvalue of problem (1.1). Assume that σ, τ and $A(x)$ satisfy the same assumptions as Theorem 3.1. Then, we have*

$$(4.1) \quad \sum_{t=1}^k (\Lambda_{k+1} - \Lambda_t)^2 \leq \frac{4}{n^2 \zeta} \max_{x \in \Omega} \operatorname{tr} A(x) \sum_{t=1}^k (\Lambda_{k+1} - \Lambda_t) \Lambda_t.$$

Corollary 4.2. *Under the same assumptions as Corollary 4.1, we have*

$$(4.2) \quad \begin{aligned} \Lambda_{k+1} \leq & \left(1 + \frac{2}{n^2 \zeta} \max_{x \in \Omega} \operatorname{tr} A(x)\right) \frac{1}{k} \sum_{t=1}^k \Lambda_t + \left\{ \left[\frac{2}{n^2 \zeta} \max_{x \in \Omega} \operatorname{tr} A(x) \frac{1}{k} \sum_{t=1}^k \Lambda_t \right]^2 \right. \\ & \left. - \left(1 + \frac{4}{n^2 \zeta} \max_{x \in \Omega} \operatorname{tr} A(x)\right) \frac{1}{k} \sum_{s=1}^k (\Lambda_s - \frac{1}{k} \sum_{t=1}^k \Lambda_t)^2 \right\}^{\frac{1}{2}}. \end{aligned}$$

Corollary 4.3. *Under the same assumptions as Corollary 4.1, we have*

$$(4.3) \quad \Lambda_{k+1} \leq \left(1 + \frac{4}{n^2 \zeta} \max_{x \in \Omega} \operatorname{tr} A(x)\right) \frac{1}{k} \sum_{t=1}^k \Lambda_t.$$

Remark 4.1. When $a_{ij}(x) = \delta_{ij}$, (4.1) and (4.3) respectively become Yang's first inequality (1.6) and Yang's second inequality (1.7). Therefore, our result (4.1) is sharper than (1.8) derived by Qian and Chen [15] for problem (1.1).

Corollary 4.4. *Under the same assumptions as Corollary 4.1, we have an estimate for the gaps of any two consecutive eigenvalues of problem (1.1):*

$$(4.4) \quad \begin{aligned} \Lambda_{k+1} - \Lambda_k \leq & 2 \left\{ \left[\frac{2}{n^2 \zeta} \max_{x \in \Omega} \operatorname{tr} A(x) \frac{1}{k} \sum_{t=1}^k \Lambda_t \right]^2 \right. \\ & \left. - \left(1 + \frac{4}{n^2 \zeta} \max_{x \in \Omega} \operatorname{tr} A(x)\right) \frac{1}{k} \sum_{s=1}^k (\Lambda_s - \frac{1}{k} \sum_{t=1}^k \Lambda_t)^2 \right\}^{\frac{1}{2}}. \end{aligned}$$

Remark 4.2. We want to point out that Γ_t is not the t -th eigenvalue of problem (1.9) in Theorem 3.1. If we substitute $\Gamma_t = \Lambda_t - \tau \min_{x \in \Omega} a_0(x)$ into the inequalities in Theorem 3.1 and Corollaries 3.3 - 3.5, their actual forms are a little complicated. But for problem (1.1), the situation is different. The above estimates in Corollaries 4.1 - 4.4 are very explicit.

ACKNOWLEDGEMENT

The author would like to express his gratitude to the referee for valuable suggestions.

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