ON PSEUDO-RIEMANNIAN LIE ALGEBRAS

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ABSTRACT. In this paper, we show that $\mathfrak{g}\mathfrak{g} \subset \mathfrak{g}$ if $\mathfrak{g}$ is a pseudo-Riemannian Lie algebra with $C(\mathfrak{g}) \neq 0$. Then we show that $\mathfrak{g}\mathfrak{g} \subset \mathfrak{g}$ when $\dim \mathfrak{g} = 4$, which leads to the classification of the pseudo-Riemannian Lie algebras in dimension 4.

1. INTRODUCTION

A Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ is called a pseudo-Riemannian Lie algebra if there is a bilinear product $(u, v) \mapsto uv$ such that, for any $u, v, w \in \mathfrak{g}$,

$$(1.1) \quad uv - vu = [u, v],$$

$$(1.2) \quad [uv, w] + [u, wv] = 0$$

and a non-degenerate symmetric bilinear form $(\cdot, \cdot)$ exists on $\mathfrak{g}$ such that

$$(1.3) \quad (uv, w) + (v, uw) = 0.$$

Furthermore, $\mathfrak{g}$ is called a Riemann-Lie algebra if $(\cdot, \cdot)$ is positive definite.

The notion of pseudo-Riemannian Lie algebras was introduced by M. Boucetta in [1]. They are strongly related to pseudo-Riemannian Poisson manifolds [2]. A pseudo-Riemannian Poisson manifold is a Poisson manifold (P, π) endowed with a pseudo-Riemannian metric $(\cdot, \cdot)$ such that the couple $(\pi, (\cdot, \cdot))$ is compatible in the sense that the Poisson tensor π is parallel with respect to the Levi-Civita contravariant connection associated to $(\pi, (\cdot, \cdot))$. The notion of pseudo-Riemannian Lie algebra appeared when one looked for the pseudo-Riemannian metrics compatible with the canonical Poisson structure on the dual of a Lie algebra. Indeed, the dual space $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ carries a Riemannian metric compatible with the linear Poisson structure if and only if the Lie algebra $\mathfrak{g}$ carries a structure of pseudo-Riemannian Lie algebra. For more details see [1, 2, 6, 9].

Moreover in [3], Boucetta has shown that the Lie algebra of a Lie group G carries a structure of Riemann-Lie algebra if and only if G carries a flat left invariant Riemannian metric. There are many studies about the geometric structures on a Lie group G with a left invariant Riemannian metric g . For example, Milnor [8]

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considered the left invariant Riemannian connection ∇ associated to g and gave a detailed structure theory on the flat metrics.

In this paper, we consider the problem of the classification of pseudo-Riemannian Lie algebras in dimension 4. We show first that if $\mathfrak{g}$ is a pseudo-Riemannian Lie algebra with $C(\mathfrak{g}) \neq 0$, then $\mathfrak{g}\mathfrak{g} \subset \mathfrak{g}$. On the other hand, we show that $\mathfrak{g}\mathfrak{g} \subset \mathfrak{g}$ if $\mathfrak{g}$ is a 4-dimensional pseudo-Riemannian Lie algebra. The proof is based on a result of [5] (a pseudo-Riemannian Lie algebra must be solvable) and the classification of solvable 4-dimensional Lie algebras [4]. Finally, the classification of 4-dimensional pseudo-Riemannian Lie algebras is deduced by a direct computation.

2. BASIC FACTS ON PSEUDO-RIEMANNIAN LIE ALGEBRAS

In this section, we will list some definitions and lemmas on pseudo-Riemannian Lie algebras.

Lemma 2.1 ([5]). *Any Lie algebra over a field of characteristic 0 with a product satisfying the identities (1.1) and (1.2) is solvable.*

Definition 2.2. A pseudo-Riemannian Lie algebra $\mathfrak{g}$ is called trivial if $xy = 0$ for any $x, y \in \mathfrak{g}$. A vector subspace H of $\mathfrak{g}$ is called an ideal if $xy, yx \in H$ for any $x \in H$ and $y \in \mathfrak{g}$.

Let $\mathfrak{g}$ be a pseudo-Riemannian Lie algebra and let $C(\mathfrak{g}) = \{a \in \mathfrak{g} \mid ax = xa, \text{ for any } x \in \mathfrak{g}\}$ be the center of $\mathfrak{g}$. Then we have

Lemma 2.3. $C(\mathfrak{g})$ is an ideal of $\mathfrak{g}$, and $xy = 0$ for any $x, y \in C(\mathfrak{g})$.

Proof. For any $a \in C(\mathfrak{g})$, $x, y \in \mathfrak{g}$, one has by (1.2)

$$[ax, y] + [a, yx] = [ax, y] = 0,$$

which implies that $ax = xa \in C(\mathfrak{g})$. For any $x, y \in C(\mathfrak{g})$ and $z \in \mathfrak{g}$,

$$(xy, z) = -(y, xz) = -(y, zx) = (zy, x) = (yz, x) = -(z, yx) = -(xy, z).$$

It follows that $(xy, z) = 0$ for any $z \in \mathfrak{g}$. Thus $xy = 0$. $\square$

Lemma 2.4. For any $x, y \in \mathfrak{g}$, $xy + yx \in (C(\mathfrak{g}))^\perp$.

Proof. For any $x, y \in \mathfrak{g}$, $z \in C(\mathfrak{g})$,

$$(z, xy + yx) = (z, xy) + (z, yx) = -(xz, y) - (yz, x) = -(zx, y) - (x, zy) = 0.$$

Namely, $xy + yx \in (C(\mathfrak{g}))^\perp$. $\square$

For any $x \in C(\mathfrak{g})$, $y \in C(\mathfrak{g})^\perp$, $z \in \mathfrak{g}$, we have

$$(x, yz) = -(yx, z) = -(xy, z) = (y, xz) = 0 = (y, zx) = (zy, x).$$

So $yz, zy \in C(\mathfrak{g})^\perp$. Thus we have

Lemma 2.5. $C(\mathfrak{g})^\perp$ is an ideal of $\mathfrak{g}$. If the restriction of the bilinear form to $C(\mathfrak{g})$ is non-degenerate (say, if $\mathfrak{g}$ is a Riemann-Lie algebra), then $[\mathfrak{g}, \mathfrak{g}] \subseteq C(\mathfrak{g})^\perp$ and

$$\mathfrak{g} = C(\mathfrak{g}) \oplus C(\mathfrak{g})^\perp.$$

Let $\mathfrak{g}\mathfrak{g} = \langle \{xy \mid x, y \in \mathfrak{g}\} \rangle$ and let $Z_R(\mathfrak{g}) = \{x \in \mathfrak{g} \mid yx = 0, \text{ for any } y \in \mathfrak{g}\}$. Then we have

Lemma 2.6. $\mathfrak{g}\mathfrak{g}$ is an ideal of $\mathfrak{g}$ and $(\mathfrak{g}\mathfrak{g})^\perp = Z_R(\mathfrak{g})$.

Proof. The first assertion is trivial. For the second one, for any $x, y, z \in \mathfrak{g}$, $x \in (\mathfrak{g}\mathfrak{g})^\perp \Leftrightarrow (x, yz) = 0 \Leftrightarrow (yx, z) = 0 \Leftrightarrow yx = 0 \Leftrightarrow x \in Z_R(\mathfrak{g})$. $\square$

3. PSEUDO-RIEMANNIAN LIE ALGEBRAS $\mathfrak{g}$ WITH $\mathfrak{g}\mathfrak{g} \subset \mathfrak{g}$

Theorem 3.1. *Let $\mathfrak{g}$ be a pseudo-Riemannian Lie algebra with $C(\mathfrak{g}) \neq 0$. Then $\mathfrak{g}\mathfrak{g} \subset \mathfrak{g}$.*

Proof. By Lemma 2.1, $\mathfrak{g}$ is solvable. Then there exists a positive integer m such that $\mathfrak{g}^{(m)} = 0$. Let n be the minimal m such that $\mathfrak{g}^{(m)} = 0$. Namely

$$\mathfrak{g} = \mathfrak{g}^{(0)} \supset \mathfrak{g}^{(1)} \supset \cdots \supset \mathfrak{g}^{(n)} = 0.$$

For any $x, y \in \mathfrak{g}$,

$$xy = \frac{(xy + yx)}{2} + \frac{[x, y]}{2}.$$

Assume that $\mathfrak{g} = \mathfrak{g}\mathfrak{g}$. In terms of Lemma 2.4, we have

$$\mathfrak{g} = \mathfrak{g}\mathfrak{g} \subseteq C(\mathfrak{g})^\perp + [\mathfrak{g}, \mathfrak{g}] = C(\mathfrak{g})^\perp + \mathfrak{g}^{(1)} \subseteq \mathfrak{g}.$$

It follows that $\mathfrak{g} = C(\mathfrak{g})^\perp + \mathfrak{g}^{(1)}$. Thus by Lemma 2.5,

$$\mathfrak{g} = \mathfrak{g}\mathfrak{g} = (C(\mathfrak{g})^\perp + \mathfrak{g}^{(1)})(C(\mathfrak{g})^\perp + \mathfrak{g}^{(1)}) \subseteq C(\mathfrak{g})^\perp + \mathfrak{g}^{(1)}\mathfrak{g}^{(1)} \subseteq \mathfrak{g}.$$

Namely, $\mathfrak{g} = C(\mathfrak{g})^\perp + \mathfrak{g}^{(1)}\mathfrak{g}^{(1)}$. For any $x, y \in \mathfrak{g}^{(1)}$,

$$xy = \frac{(xy + yx)}{2} + \frac{[x, y]}{2} \in C(\mathfrak{g})^\perp + \mathfrak{g}^{(2)},$$

which implies that

$$\mathfrak{g} = C(\mathfrak{g})^\perp + \mathfrak{g}^{(1)}\mathfrak{g}^{(1)} = C(\mathfrak{g})^\perp + \mathfrak{g}^{(2)}.$$

By induction, we must have

$$\mathfrak{g} = C(\mathfrak{g})^\perp + \mathfrak{g}^{(n)} = C(\mathfrak{g})^\perp.$$

Since $\dim \mathfrak{g} = \dim C(\mathfrak{g})^\perp + \dim C(\mathfrak{g})$, we have that $C(\mathfrak{g}) = 0$, a contradiction. $\square$

In the following, we will show that $\mathfrak{g}\mathfrak{g} \subset \mathfrak{g}$ when $\mathfrak{g}$ is a pseudo-Riemannian Lie algebra in dimension 4. To do this, we need to extend the real number field to the complex number field.

Let $\mathfrak{g}_{\mathbb{C}}$ be a Lie algebra in dimension 4 over the complex number field with a bilinear product satisfying the identities (1.1) and (1.2) and a non-degenerate symmetric bilinear form satisfying the identity (1.3). By Lemma 2.1, $\mathfrak{g}_{\mathbb{C}}$ is solvable. Thus $\dim[\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}] \leq 3$ and $[\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]$ is nilpotent.

Proposition 3.2. *If $\dim[\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}] = 3$, then $[\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]$ is not abelian.*

Proof. Assume that $[\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]$ is abelian. Let $d \in \mathfrak{g}_{\mathbb{C}}$ and $d \notin [\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]$. Then there exists $z \in [\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]$ such that $[d, z] = az$, where $a \neq 0$. Since $[dd, z] = -[d, zd] \in \langle z \rangle$, we have that $zd \in \langle d, z \rangle$. Thus $dz \in \langle d, z \rangle$. Assume that $dz = t_1d + t_2z$ where $t_1, t_2 \in \mathbb{C}$. Then for any $u \in [\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]$, $[dz, u] = -[d, uz] = -[d, zu] = [du, z] \in \langle z \rangle$. Namely $t_1[d, u] \in \langle z \rangle$ for any $u \in [\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]$. Then $t_1 = 0$ since $\dim[\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}] = 3$. That is, $dz, zd \in \langle z \rangle$. For any $u \in [\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]$, $uz = zu \in \langle d \rangle$ by $[d, uz] = -[dz, u] = 0$. Similar to the above discussion, $uz = zu = 0$. Furthermore, $du, ud \in [\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]$ since $[du, z] = -[d, zu] = 0$ and $[ud, z] = -[u, dz] = 0$. Also we have that $[dd, z] = -[d, zd] = -a(zd)$. Let $dd = bd + u_0$, where $b \in \mathbb{C}$ and $u_0 \in [\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]$. For any $u, v \in [\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]$, $uv \in [\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]$ since $[uv, z] = -[u, zv] = 0$. Thus $uv = 0$ by $[d, uv] = -[dv, u] = 0$. Since $[dd, u] = b[d, u] = -[d, ud]$, we have that $[d, bu + ud] = 0$. Namely $ud = -bu$. If $b \neq 0$, then $(ud, d) = 0$ implies that $(d, [\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]) = 0$. But $(d, dd) = 0$, that is, $(d, \mathfrak{g}_{\mathbb{C}}) = 0$. So $b = 0$. Hence $ud = 0$. Replacing d by

$d - u_1$ for some $u_1 \in [\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]$ satisfying $du_1 = u_0$ if necessary, we can assume that $u_0 = 0$. So $dd = 0$. Thus $\mathfrak{g}_{\mathbb{C}}\mathfrak{g}_{\mathbb{C}} = [\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]$ and $Z_R(\mathfrak{g}_{\mathbb{C}}) = \langle d \rangle$. We have that $(z, z) = 0$ since $(dz, z) = 0$. Let $z^\perp = \langle d, y, z \rangle$ where $y \in [\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]$. Thus $(y, y) \neq 0$. Then there exists $x \in [\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]$ such that $(x, y) = (x, x) = 0$ and $(x, z) = 1$. Since $(dy, z) = -(y, dz) = 0$, we have that $dy \in \langle y, z \rangle$. But $(dy, y) = 0$ so $dy \in \langle z \rangle$. This contradicts the fact that $\dim[\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}] = 3$. $\square$

Theorem 3.3. $\dim[\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}] \leq 2$.

Proof. Assume that $\dim[\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}] = 3$. By Proposition 3.2, $[\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]$ must be Heisenberg. Then there exists a basis x, y, z of $[\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]$ such that $[x, y] = z$. Let $d \in \mathfrak{g}_{\mathbb{C}}$ and $d \notin [\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]$. Assume that $\text{ad}_d(x, y, z) = (x, y, z)(a_{ij})$. Since $[d, z] = [[d, x], y] + [x, [d, y]] \in \langle z \rangle$, we have that $a_{13} = a_{23} = 0$ and $a_{11} + a_{22} = a_{33}$. Note that $\text{Im ad}_d + \langle z \rangle = \langle x, y, z \rangle$, so we can assume that $x, y \in \text{Im ad}_d$.

I) $a_{33} = 0$. Then $a_{11} = -a_{22} \neq 0$, so ad_d is diagonalizable. Assume that (a_{ij}) is a diagonal matrix, that is, $[d, x] = a_{11}x = ax$, $[d, y] = -ay$. We can assume that $a = 1$. Then $\langle z \rangle = C(\mathfrak{g}_{\mathbb{C}})$. Thus $C(\mathfrak{g}_{\mathbb{C}})^\perp$ is a 3-dimensional ideal, hence a Lie-ideal. So $C(\mathfrak{g}_{\mathbb{C}})^\perp = \langle x, y, z \rangle$ and $(z, z) = 0$. But $(dd, z) = -(d, dz) = -(d, zd) = 0$. Together with Lemma 2.5 ($C(\mathfrak{g}_{\mathbb{C}})^\perp$ is an ideal), we have that $\mathfrak{g}_{\mathbb{C}}\mathfrak{g}_{\mathbb{C}} = [\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}] = C(\mathfrak{g}_{\mathbb{C}})^\perp$. Namely $C(\mathfrak{g}_{\mathbb{C}}) = Z_R(\mathfrak{g}_{\mathbb{C}})$. For any $u \in [\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]$ and $v \in \mathfrak{g}_{\mathbb{C}}$, $uv \in \langle z \rangle$ since $[d, uv] = -[dv, u] \in \langle z \rangle$. Thus $[dv, u] = 0$, i.e., $dv \in \langle z \rangle$. But then

$$0 = [xd, d] + [x, dd] = [dx - [d, x], d] + [x, dd] = -[x, d] + [x, dd] = x + [x, dd] \neq 0.$$

II) $a_{33} \neq 0$. Then (a_{ij}) is non-degenerate. Let $a, b, c = a_{33}$ be the eigenvalues of (a_{ij}) . Thus $a + b = a_{11} + a_{22} = a_{33} = c \neq 0$; namely $a \neq 0$ and $b \neq 0$. Thus we can assume

$$[d, x] = ax + ey, [d, y] = by, [d, z] = cz,$$

where $e = 0$ or 1 , $c = a + b$. In any case, we have that $\text{Im ad}_x \cap \text{Im ad}_y = \langle z \rangle$. Therefore for any $u \in \mathfrak{g}_{\mathbb{C}}$, $xu, yu \in [\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]$ since $[xu, y] = -[x, yu] \in \text{Im ad}_x \cap \text{Im ad}_y$. Hence $ux, uy \in [\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]$. By $[zd, x] = -[z, xd] \in \langle z \rangle$, zd and then $dz \in [\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]$. Furthermore for any $u, v \in [\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]$, $uv \in \langle z \rangle$ since $[d, uv] = -[dv, u] \in \langle z \rangle$. It follows that $zu = uz = 0$ since $[d, zu] = -[du, z] = 0$ for any $u \in [\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]$. Since $[zd, u] = -[z, ud] = 0$ for any $u \in [\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]$, we have that $zd, dz \in \langle z \rangle$. By $[dd, y] = -[d, yd] \in \text{Im ad}_y$, we have that $yd \in \langle y, z \rangle$. Thus $dy \in \langle y, z \rangle$. It follows that $yy = 0$ since $[d, yy] = -[dy, y] = 0$. Then we have

$$\begin{aligned} (z, y) &= ([x, y], y) = (x, yy) = 0; & b(y, y) &= ([d, y], y) = (yd, y) = 0 \Rightarrow (y, y) = 0; \\ c(z, z) &= ([d, z], z) = (d, zz) = 0 \Rightarrow (z, z) = 0; & (z, x) &= ([x, y], x) = -(y, xx) = 0. \end{aligned}$$

So $(d, z) \neq 0$. Since $(zd, d) = 0$, we have that $zd = 0$. It follows that $dz = [d, z] = cz$. Since $(dd, z) = -(d, dz) = -(d, cz) \neq 0$, we have that $dd = -cd + u_0$, where $u_0 \in [\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]$. Thus $c = 0$ since $[dd, z] = -[d, zd] = 0$. This is a contradiction. That is, we must have $\dim[\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}] \leq 2$. $\square$

Based on the classification of 4-dimensional complex Lie algebras [4], the solvable Lie algebra $\mathfrak{g}$ with $C(\mathfrak{g}) = 0$ such that $\dim[\mathfrak{g}, \mathfrak{g}] \leq 2$ must be one of the following two cases:

- (1) there is a basis x, y, z, w of $\mathfrak{g}$ such that $[x, z] = z, [x, w] = w, [y, z] = w$, or
- (2) there is a basis x, y, z, w of $\mathfrak{g}$ such that $[x, y] = x, [z, w] = z$.

Theorem 3.4. *There is no basis x, y, z, w of $\mathfrak{g}_{\mathbb{C}}$ that satisfies $[x, z] = z, [x, w] = w, [y, z] = w$.*

Proof. Assume that there is a basis x, y, z, w of $\mathfrak{g}_{\mathbb{C}}$ satisfying $[x, z] = z, [x, w] = w, [y, z] = w$. For any $t_1, t_2 \in \mathfrak{g}_{\mathbb{C}}$ and $s \in \langle y, w \rangle$, $[t_1, st_2] = -[t_1 t_2, s] \in \langle y, w \rangle$. It follows that $st_2 \in \langle y, w \rangle$. Thus $t_2 s = [t_2, s] + st_2 \in \langle y, w \rangle$. Namely $\langle y, w \rangle$ is an ideal. Furthermore for any $s, t \in \langle y, w \rangle$ and $u \in \mathfrak{g}_{\mathbb{C}}$, $st \in C(\mathfrak{g}_{\mathbb{C}})$ since $[u, st] = -[ut, s] = 0$. Namely $st = 0$. Then we have

$$\begin{aligned}(w, w) &= ([x, w], w) = (xw - wx, w) = -(wx, w) = (x, ww) = 0, \\ (w, y) &= (xw - wx, y) = (xw, y) = -(w, xy) = -(w, yx) = (yw, x) = 0.\end{aligned}$$

Assume that $(y, y) \neq 0$. Then we have that $yu, uy, wu, uw \in \langle w \rangle$. By $[xz, w] = -[x, wz] = -[x, zw] = [xw, z] = 0$, $zw = wz = 0$ and $xz \in \langle y, z, w \rangle$. Hence $zx \in \langle y, z, w \rangle$. Also $zz \in \langle y, z, w \rangle$ since $[zz, w] = -[z, wz] = 0$. Since $[x, zy] = -[xy, z] = -[yx, z] = 0 = [y, zx]$, we have that $zx \in \langle y, w \rangle$, $zy = 0$ and $yz = [y, z] = w$. It follows that $(w, z) = (yz, z) = 0$ and $(w, x) \neq 0$. Also we have that $(z, y) = (xz - zx, y) = 0$ and $zx \in \langle w \rangle$ since $(zx, y) = -(x, zy) = 0$. Thus $xx \in \langle y, z, w \rangle$ since $[xx, z] = -[x, zx] \in \langle w \rangle$. Therefore $\mathfrak{g}_{\mathbb{C}}\mathfrak{g}_{\mathbb{C}} \subseteq \langle y, z, w \rangle$. It follows that $w \in \langle y, z, w \rangle^{\perp} \subseteq Z_R(\mathfrak{g}_{\mathbb{C}})$. Then $0 = (xx, w) = -(x, xw) = -(x, xw - wx) = -(x, w) \neq 0$. That is, $(y, y) = 0$.

Since $(xw - wx, z) = (w, z) = (w, xz - zx) = -(xw, z) + (zw, x) = -(xw, z) + (wz, x) = -(xw, z) - (z, wx) = -(xw + wx, z)$, we have that $(xw, z) = 0$. Hence $(w, xz) = 0$.

Assume that $(z, w) = 0$. Then $(y, z) \neq 0$, $(x, w) \neq 0$ and $w \in [\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]^{\perp}$. The identity $(xw, z) = 0$ implies $xw \in \langle w \rangle$, and $(wx, x) = 0$ implies $wx = 0$. Then $xw = w$. Since $[xx, w] = -[x, wx] = 0$, we have that $xx \in \langle y, z, w \rangle$, which contradicts $(xx, w) = -(x, xw) = -(x, w) \neq 0$; namely $(z, w) \neq 0$.

So we can assume that $(z, w) = 1$ and $(z, z) = 0$. Replacing y by $y - (y, z)w$ and x by $x - (y, z)z$, we can assume that $(y, z) = 0$. It follows that $(x, y) \neq 0$. Furthermore we can assume that $(x, z) = 0$ by replacing x by $x - (x, z)w$. Then $y \in [\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]^{\perp}$.

We have that $zz \in \langle z, w \rangle^{\perp}$ since $(zz, z) = 0$ and $(zz, w) = -(z, zw) = -(z, wz) = 0$. Similarly $zw \in \langle z, w \rangle^{\perp}$. Also we have that $yz \in \langle y \rangle$ by $(yz, z) = 0$ and $zw \in \langle y \rangle$ by $(zw, w) = (zw, z) = 0$. Since $(xw, z) = 0$, we have that $xw \in \langle y \rangle$ and $(x, wz) = -(wx, z) = (w, z) \neq 0$. So $zw = wz \neq 0$. Let $zw = ay$, where $a \neq 0$. By $[xw, z] = -[x, zw] = 0$, $xw = 0$. Then $wx = [w, x] = -w$. Thus $(w, x) = -(wx, x) = 0$ and $x \in [\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]^{\perp}$. Furthermore $(xx, u) = 0$ for any $u \in \mathfrak{g}_{\mathbb{C}}$. It follows that $xx = 0$ and $xy = yx = 0$. It is easy to check $(xz, u) = 0$ for any $u \in \mathfrak{g}_{\mathbb{C}}$; namely, $xz = 0$. Thus $zx = -z$. But $z = [zx, x] = -[z, xx] = 0$, which is absurd. Then the theorem follows. $\square$

Theorem 3.5. *If there is a basis x, y, z, w of $\mathfrak{g}_{\mathbb{C}}$ such that $[x, y] = x, [z, w] = z$, then $\mathfrak{g}_{\mathbb{C}}\mathfrak{g}_{\mathbb{C}} \subset \mathfrak{g}_{\mathbb{C}}$.*

Proof. It is clear that $\mathfrak{g}_{\mathbb{C}} = \langle x, y \rangle \oplus \langle z, w \rangle$, where $\langle x, y \rangle$ and $\langle z, w \rangle$ are Lie-ideals. For any $t_1, t_2 \in \langle x, y \rangle$ and $s_1 \in \langle z, w \rangle$,

$$[t_1 t_2, s_1] = -[t_1, s_1 t_2] \in \langle x, y \rangle \cap \langle z, w \rangle = 0.$$

It follows that $t_1 t_2 \in \langle x, y \rangle$ and $s_1 t_2 \in \langle z, w \rangle$. For any $s_2 \in \langle z, w \rangle$,

$$[t_2 s_1, s_2] = -[t_2, s_2 s_1] \in \langle x, y \rangle \cap \langle z, w \rangle = 0.$$

It follows that $s_2s_1 \in \langle z, w \rangle$ and $t_2s_1 \in \langle x, y \rangle$. Since $[s_1, t_2] = 0$, we have that $s_1t_2 = t_2s_1 = 0$. That is, $\langle x, y \rangle$ and $\langle z, w \rangle$ are ideals.

Assume that $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}\mathfrak{g}_{\mathbb{C}}$. Thus $Z_R(\mathfrak{g}_{\mathbb{C}}) = 0$. Then $\langle x, y \rangle$ and $\langle z, w \rangle$ are non-degenerate. In fact, let $s \in \langle x, y \rangle$ satisfy $(s, t) = 0$ for any $t \in \langle x, y \rangle$. Then there exists $t_1 \in \langle x, y \rangle$ such that $t_1s \neq 0$ since $Z_R(\mathfrak{g}_{\mathbb{C}}) = 0$. By the non-degeneracy of $(\ , \)$, there exists $t_2 \in \mathfrak{g}_{\mathbb{C}}$ such that $(t_1s, t_2) \neq 0$. It follows that $(s, t_1t_2) \neq 0$. Thus we have $s = 0$ by the choice of s . That is, $\langle x, y \rangle$ is non-degenerate. Similarly, $\langle z, w \rangle$ is non-degenerate. It is easy to see that $s_1s_2 = 0$ and $t_1t_2 = 0$ for any $s_1, s_2 \in \langle x, y \rangle$ and $t_1t_2 \in \langle z, w \rangle$. Then $\mathfrak{g}_{\mathbb{C}}\mathfrak{g}_{\mathbb{C}} = 0$. This contradicts the assumption. Therefore we have $\mathfrak{g}_{\mathbb{C}} \neq \mathfrak{g}_{\mathbb{C}}\mathfrak{g}_{\mathbb{C}}$. $\square$

Theorem 3.6. $\mathfrak{g}_{\mathbb{C}}\mathfrak{g}_{\mathbb{C}} \subset \mathfrak{g}_{\mathbb{C}}$.

Proof. Clearly, $\mathfrak{g}_{\mathbb{C}}$ is solvable and Theorem 3.1 holds for $\mathfrak{g}_{\mathbb{C}}$. Thus this theorem follows from Proposition 3.2 and Theorems 3.3, 3.4 and 3.5 based on the classification of 4-dimensional complex Lie algebras [4]. $\square$

Corollary 3.7. Let $\mathfrak{g}$ be a pseudo-Riemannian Lie algebra in dimension 4. Then $\mathfrak{g}\mathfrak{g} \subset \mathfrak{g}$.

Proof. Assume that $\mathfrak{g}\mathfrak{g} = \mathfrak{g}$. The complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$ is a complex Lie algebra in dimension 4 with a bilinear product satisfying the identities (1.1) and (1.2) and a non-degenerate symmetric bilinear form satisfying the identity (1.3). By Theorem 3.6, $\mathfrak{g}_{\mathbb{C}}\mathfrak{g}_{\mathbb{C}} \subset \mathfrak{g}_{\mathbb{C}}$. This contradicts the assumption. Therefore $\mathfrak{g}\mathfrak{g} \subset \mathfrak{g}$. $\square$

4. THE CLASSIFICATION IN DIMENSION 4

Let $\mathfrak{g}$ be a pseudo-Riemannian Lie algebra of dimension 4. By Corollary 3.7 and Lemma 2.6, we know that $\dim Z_R(\mathfrak{g}) = \dim(\mathfrak{g}\mathfrak{g})^{\perp} > 0$.

4.1. $Z_R(\mathfrak{g})$ is isotropic. That is, $(x, y) = 0$ for any $x, y \in Z_R(\mathfrak{g})$. Then $Z_R(\mathfrak{g}) \subseteq Z_R(\mathfrak{g})^{\perp} = \mathfrak{g}\mathfrak{g}$. It follows that $\dim Z_R(\mathfrak{g}) = 1$ or $\dim Z_R(\mathfrak{g}) = 2$ since $\dim Z_R(\mathfrak{g}) + \dim \mathfrak{g}\mathfrak{g} = 4$.

- (1) If $\dim Z_R(\mathfrak{g}) = 1$, then $\dim \mathfrak{g}\mathfrak{g} = 3$. Hence there exists a basis x, y, z, w of $\mathfrak{g}$ such that $(x, w) = 1, (y, y) = d_1, (z, z) = d_2$, where $x \in Z_R(\mathfrak{g}), x, y, z \in \mathfrak{g}\mathfrak{g}$ and $d_1d_2 \neq 0$. (Here, only the non-zero forms are given; likewise below.) By the identity (1.3), suppose that the multiplication table in $\mathfrak{g}$ is given as follows $(\alpha_i, \beta_i, \lambda_i \in \mathbb{R}, i = 1, 2, 3, 4)$:

	x	y	z	w
x	0	$-d_1\alpha_1x + \beta_1z$	$-d_2\lambda_1x - \frac{d_2\beta_1y}{d_1}$	$\alpha_1y + \lambda_1z$
y	0	$-d_1\alpha_2x + \beta_2z$	$-d_2\lambda_2x - \frac{d_2\beta_2y}{d_1}$	$\alpha_2y + \lambda_2z$
z	0	$-d_1\alpha_3x + \beta_3z$	$-d_2\lambda_3x - \frac{d_2\beta_3y}{d_1}$	$\alpha_3y + \lambda_3z$
w	0	$-d_1\alpha_4x + \beta_4z$	$-d_2\lambda_4x - \frac{d_2\beta_4y}{d_1}$	$\alpha_4y + \lambda_4z$

By the above multiplication table and the identity (1.2), we get the relations which $\alpha_i, \beta_i, \lambda_i$ must satisfy. We describe just the results here and omit the detailed calculation.

If $\beta_1 \neq 0$, the multiplication table must be

	x	y	z	w
x	0	$-d_1\alpha_1x + \beta_1z$	$-\frac{d_2(d_1\lambda_1x + \beta_1y)}{d_1}$	$\alpha_1y + \lambda_1z$
y	0	$-\frac{d_1\lambda_1(-d_1\alpha_1x + \beta_1z)}{\beta_1}$	$\frac{d_2\lambda_1(d_1\lambda_1x + \beta_1y)}{\beta_1}$	$-\frac{d_1\lambda_1(\alpha_1y + \lambda_1z)}{\beta_1}$
z	0	$\frac{d_1\alpha_1(-d_1\alpha_1x + \beta_1z)}{\beta_1}$	$-\frac{d_2\alpha_1(d_1\lambda_1x + \beta_1y)}{\beta_1}$	$\frac{d_1\alpha_1(\alpha_1y + \lambda_1z)}{\beta_1}$
w	0	$\frac{\beta_4(-d_1\alpha_1x + \beta_1z)}{\beta_1}$	$-\frac{d_2\beta_4(d_1\lambda_1x + \beta_1y)}{d_1\beta_1}$	$\frac{\beta_4(\alpha_1y + \lambda_1z)}{\beta_1}$

It is easy to see that $\dim \mathfrak{gg} \leq 2$ from the above table. Similarly if $\beta_1 = 0$, we also get $\dim \mathfrak{gg} \leq 2$. This contradicts the fact that $\dim \mathfrak{gg} = 3$, so this case is impossible.

- (2) If $\dim Z_R(\mathfrak{g}) = 2$, then $\dim \mathfrak{gg} = 2$ and $Z_R(\mathfrak{g}) = \mathfrak{gg}$. Then there exists a basis x, y, z, w of $\mathfrak{g}$ such that $(x, w) = (y, z) = 1$, where $x, y \in Z_R(\mathfrak{g})$. Similar to the above calculation, we get the multiplication table ($\alpha, \beta \in \mathbb{R}$):

	x	y	z	w
x	0	0	0	0
y	0	0	0	0
z	0	0	αx	$-\alpha y$
w	0	0	βx	$-\beta y$

If $\alpha\beta \neq 0$, then $\mathfrak{g}$ is a pseudo-Riemannian Lie algebra of dimension 4 with $\dim \mathfrak{gg} = 2$.

4.2. $Z_R(\mathfrak{g})$ is **nonisotropic**. That is, $(x, y) \neq 0$ for some $x, y \in Z_R(\mathfrak{g})$. Then there exists an element $x \in Z_R(\mathfrak{g})$ such that $(x, x) \neq 0$. It follows from the identity (1.3) that $x^\perp = \{y \in \mathfrak{g} | (x, y) = 0\}$ is an ideal of $\mathfrak{g}$; namely $x^\perp$ is a pseudo-Riemannian Lie algebra of dimension 3, which has been classified in [1].

- (1) $x^\perp$ is abelian. Then there exists a basis x, y, z, w of $\mathfrak{g}$ such that $(x, x) \neq 0, (y, y) = d_1, (z, z) = d_2, (w, w) = d_3$, where $d_1d_2d_3 \neq 0$. The multiplication table is ($\alpha, \beta, \gamma \in \mathbb{R}$):

	x	y	z	w
x	0	$\alpha z + \beta w$	$-\frac{d_2\alpha y}{d_1} + \gamma w$	$-\frac{d_3\beta y}{d_1} - \frac{d_3\gamma z}{d_2}$
y	0	0	0	0
z	0	0	0	0
w	0	0	0	0

- (2) $x^\perp$ is a Heisenberg Lie algebra. Then there exists a basis x, y, z, w of $\mathfrak{g}$ such that $(x, x) \neq 0, (y, w) = 1, (z, z) = d$, where $d \neq 0$. The multiplication table is ($\alpha \in \mathbb{R}$):

	x	y	z	w
x	0	αz	$-d\alpha w$	0
y	0	$-\frac{1}{d}z$	w	0
z	0	0	0	0
w	0	0	0	0

- (3) $x^\perp$ is the centerless 3-dimensional Lie algebra. Then there exists a basis x, y, z, w of $\mathfrak{g}$ such that $(x, x) \neq 0, (y, y) = a, (z, z) = b, (w, w) = -bc$, where $abc \neq 0$. The multiplication table is ($\beta \in \mathbb{R}$):

	x	y	z	w
x	0	0	βw	βcz
y	0	0	w	cz
z	0	0	0	0
w	0	0	0	0

In summary, we have:

Theorem 4.1. *Pseudo-Riemannian Lie algebras in dimension 4 are one of the following cases:*

- (1) *There exists a basis x, y, z, w such that $(x, w) = (y, z) = 1$, and the non-zero Lie brackets are given by $[z, w] = -\beta x - \alpha y$, where $\alpha, \beta \in \mathbb{R}$ and $\alpha\beta \neq 0$.*
- (2) *There exists a basis x, y, z, w such that $(x, x) = d, (y, y) = d_1, (z, z) = d_2, (w, w) = d_3$, and the Lie brackets are given by $[x, y] = \alpha z + \beta w, [x, z] = -\frac{d_2\alpha}{d_1}y + \gamma w, [x, w] = -\frac{d_3\beta}{d_1}y - \frac{d_3\gamma}{d_2}z$, where $\alpha, \beta, \gamma, d, d_1, d_2, d_3 \in \mathbb{R}$ and $dd_1d_2d_3 \neq 0$.*
- (3) *There exists a basis x, y, z, w such that $(x, x) = b, (y, w) = 1, (z, z) = d$, and the Lie brackets are given by $[x, y] = \alpha z, [x, z] = -d\alpha w, [y, z] = w$, where $b, d, \alpha \in \mathbb{R}$ and $bd \neq 0$.*
- (4) *There exists a basis x, y, z, w such that $(x, x) = d, (y, y) = a, (z, z) = b, (w, w) = -bc$, and the Lie brackets are given by $[x, z] = \beta w, [x, w] = \beta cz, [y, z] = w, [y, w] = cz$, where $a, b, c, d, \beta \in \mathbb{R}$ and $abcd \neq 0$.*

5. FURTHER DISCUSSION

For a pseudo-Riemannian Lie algebra $\mathfrak{g}$ with $C(\mathfrak{g}) \neq 0, \mathfrak{gg} \subset \mathfrak{g}$. Furthermore, $\mathfrak{gg} \subset \mathfrak{g}$ if $\dim \mathfrak{g} = 4$. But the proof is based on Proposition 3.2 and Theorems 3.1, 3.3, 3.4, 3.5 and 3.6 (Theorems 3.3 and 3.4 are given in [7] without proofs). Actually, we have a conjecture:

Conjecture 5.1. *Let $\mathfrak{g}$ be a pseudo-Riemannian Lie algebra. Then $\mathfrak{gg} \subset \mathfrak{g}$.*

If the conjecture holds, in theory, we could give the classification of pseudo-Riemannian Lie algebras in higher dimensions by the method given in Section 4. Also there are some difficulties in detailed calculation.

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