SCHATTEN(-HERZ) CLASS EXTENDED CESÀRO OPERATORS ON BERGMAN SPACES IN THE UNIT BALL OF $\mathbf{C}^n$

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ABSTRACT. For a large class of weights φ , we characterize the holomorphic symbols g , for which the extended Cesàro operators T_g acting on the weighted Bergman space $A_\varphi^2(\mathbf{B})$ are in the Schatten (or Schatten-Herz) class in the unit ball of $\mathbf{C}^n$.

1. INTRODUCTION

Let $\mathbf{B}$ and $\partial\mathbf{B}$ be the open unit ball and the unit sphere in $\mathbf{C}^n$, and let v and σ be the normalized Lebesgue measure on $\mathbf{B}$ and $\partial\mathbf{B}$, respectively. When $n = 1$, we denote the unit disc by $\mathbf{D}$.

A positive continuous function φ on $[0, 1)$ is called normal if there are two constants $b > a > -1$ such that

$$(1.1) \quad \frac{\varphi(r)}{(1-r)^a} \downarrow 0, \quad \frac{\varphi(r)}{(1-r)^b} \uparrow \infty$$

as $r \rightarrow 1^-$. Given φ , we extend it to $\mathbf{B}$ by setting $\varphi(z) = \varphi(|z|)$, and we may assume that it is normalized so that $\int_{\mathbf{B}} \varphi(z) dv(z) = 1$. The space $L_\varphi^2(\mathbf{B})$ is the space of all Lebesgue measurable functions f on $\mathbf{B}$ for which

$$\|f\|_{2,\varphi} = \left\{ \int_{\mathbf{B}} |f(z)|^2 \varphi(z) dv(z) \right\}^{\frac{1}{2}} < \infty.$$

Let $H(\mathbf{B})$ be the family of all holomorphic functions on $\mathbf{B}$. The space

$$A_\varphi^2(\mathbf{B}) = L_\varphi^2(\mathbf{B}) \cap H(\mathbf{B})$$

is called the weighted Bergman space. It is a closed subspace of $L_\varphi^2(\mathbf{B})$. Hence, $A_\varphi^2(\mathbf{B})$ is a Hilbert space with the inner product

$$\langle f, g \rangle = \int_{\mathbf{B}} f(z) \overline{g(z)} \varphi(z) dv(z).$$

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For $g \in H(\mathbf{B})$, let $\Re g(z) = \sum_{j=1}^n z_j \frac{\partial g(z)}{\partial z_j}$ be the radial derivative of g . In [Hu02], the extended Cesàro operator on $A_\varphi^2(\mathbf{B})$ was defined. Precisely, given $g \in H(\mathbf{B})$, we define the extended Cesàro operator T_g with the symbol g as

$$T_g(f)(z) = \int_0^1 f(tz) \Re g(tz) \frac{dt}{t}, \quad f \in H(\mathbf{B}), \quad z \in \mathbf{B}.$$

In the case that $n = 1$,

$$T_g(f)(z) = \int_0^z f(z) g'(z) dz.$$

The operator T_g has attracted many mathematician's attention; for example, see [AS95, AC01, Hu02, Hu04, Xi04, Ta07, LS08] and the references therein. In the case of one variable, A. Aleman and A. G. Siskakis proved in [AS97] that T_g is bounded (or compact) on $A_\varphi^2(\mathbf{D})$ if and only if g is a Bloch function (or a little Bloch function). For the weight $\varphi(r) = (1-r)^\alpha$ with $\alpha > -1$ they also prove that T_g is in the Schatten class S_p with $p > 1$ if and only if g is in the Besov space and T_g does not belong to S_p with $0 < p \leq 1$ unless g is constant.

Recently in [LLP05], M. Loaiza, M. López-García and S. Pérez-Esteva introduced the concept of Schatten-Herz class Toeplitz operators. Given a finite Borel measure μ on $\mathbf{B}$, the Toeplitz operator $T(\mu)$ is defined on $A_\varphi^2(\mathbf{B})$ by

$$T(\mu)f(z) = \int_{\mathbf{B}} f(w) K_\varphi(z, w) d\mu(w),$$

where $K_\varphi(z, w)$ is the reproducing kernel of $A_\varphi^2(\mathbf{B})$. Following [LLP05], given $0 < p, q < \infty$, $T(\mu)$ is said to be of the Schatten-Herz class $S_{p,q}$ (write $T(\mu) \in S_{p,q}$) if $T(\mu\chi_k) \in S_p$ for each $k = 0, 1, 2, \dots$, and the S_p -norm sequence $\{\|T(\mu\chi_k)\|_{S_p}\}_{k=0}^\infty$ belongs to l^q , where $A_k = \{z \in \mathbf{B} : 1 - \frac{1}{2^k} \leq |z| < 1 - \frac{1}{2^{k+1}}\}$ and χ_k is the characteristic function of A_k .

It will be shown (see Theorem 2.2 below) that $T_g^* T_g$ is a Toeplitz operator on $A_\varphi^2(\mathbf{B})$, so it is natural to give the following definition.

Definition 1.1. Let $0 < p, q < \infty$ and $g \in H(\mathbf{B})$. The extended Cesàro operator T_g is said to be of the Schatten-Herz class $S_{p,q}$, denoted by $T_g \in S_{p,q}$, if $T_g^* T_g \in S_{\frac{p}{2}, \frac{q}{2}}$.

The purpose of the present work is to characterize those symbols $g \in H(\mathbf{B})$ for which the induced extended Cesàro operators $T_g \in S_p$ (or $T_g \in S_{p,q}$) on $A_\varphi^2(\mathbf{B})$. Our results are new even in the one-variable case. Here are the main results of this paper.

Theorem 1.1. Suppose $g \in H(\mathbf{B})$ and $0 < p < \infty$. Then $T_g \in S_p$ if and only if

$$\int_{\mathbf{B}} |\Re g(z)|^p (1 - |z|^2)^{p-(n+1)} dv(z) < \infty.$$

Moreover,

$$\|T_g\|_{S_p} \simeq \left(\int_{\mathbf{B}} |\Re g(z)|^p (1 - |z|^2)^{p-(n+1)} dv(z) \right)^{\frac{1}{p}}.$$

Theorem 1.2. Suppose $g \in H(\mathbf{B})$ and $0 < p, q < \infty$. Then $T_g \in S_{p,q}$ if and only if

$$\int_0^1 M_p^q(\Re g, r) (1 - r^2)^{q-(n\frac{q}{p}+1)} dr < \infty.$$

Furthermore,

$$\|T_g\|_{S_{p,q}} \simeq \left(\int_0^1 M_p^q(\Re g, r) (1-r^2)^{q-(n\frac{q}{p}+1)} dr \right)^{\frac{1}{q}}.$$

Here, $M_p(\Re g, r) = \left\{ \int_{\partial \mathbf{B}} |\Re g(r\xi)|^p d\sigma(\xi) \right\}^{\frac{1}{p}}$.

In this paper, C will always stand for positive constants whose value may change from line to line but not depend on the functions in $H(\mathbf{B})$. The expression $A \simeq B$ means $C^{-1}A \leq B \leq CA$.

2. SOME PRELIMINARIES

First, we exhibit some preliminary results, which will not only be used in the proof of the main results but also have their own interest.

Theorem 2.1. *Given a normal weight φ on $[0, 1)$, set*

$$\phi(w) = \frac{1}{|w|^{2n}} \int_{|w|}^1 \frac{\int_r^1 t^{2n-1} \varphi(t) dt}{r} dr.$$

Then the inner product on $A_\varphi^2(\mathbf{B})$ can be expressed as

$$(2.1) \quad \langle f, g \rangle = f(0)\overline{g(0)} + 4 \int_{\mathbf{B}} \Re f(w) \overline{\Re g(w)} \phi(w) dv(w).$$

Proof. If f and g are holomorphic in the unit disc $\mathbf{D} \subset \mathbf{C}$ and continuous on $\overline{\mathbf{D}}$, then

$$\int_0^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} \frac{d\theta}{2\pi} = f(0)\overline{g(0)} + 2 \int_{\mathbf{D}} f'(z) \overline{g'(z)} \log \frac{1}{|z|} dm(z).$$

Here, dm is the Lebesgue measure on $\mathbf{D}$. Then, Prop. 1.4.7 in [Ru80] implies, for $f, g \in H(\mathbf{B}) \cap C(\overline{\mathbf{B}})$, that

$$(2.2) \quad \int_{\partial \mathbf{B}} f(\xi) \overline{g(\xi)} d\sigma(\xi) = f(0)\overline{g(0)} + 2 \int_{\partial \mathbf{B}} d\sigma(\xi) \int_{\mathbf{D}} \Re f(z\xi) \overline{\Re g(z\xi)} \frac{1}{|z|^2} \log \frac{1}{|z|} dm(z).$$

Since

$$\begin{aligned} & \int_{\partial \mathbf{B}} d\sigma(\xi) \int_{\mathbf{D}} \Re f(z\xi) \overline{\Re g(z\xi)} \frac{1}{|z|^2} \log \frac{1}{|z|} dm(z) \\ &= \int_{\partial \mathbf{B}} d\sigma(\xi) \int_0^1 \log \frac{1}{r} \frac{dr}{r} \int_0^{2\pi} \Re f(re^{i\theta}\xi) \overline{\Re g(re^{i\theta}\xi)} \frac{d\theta}{\pi} \\ &= \int_0^1 \log \frac{1}{r} \frac{dr}{r} \int_{\partial \mathbf{B}} d\sigma(\xi) \int_0^{2\pi} \Re f(e^{i\theta} \cdot r\xi) \overline{\Re g(e^{i\theta} \cdot r\xi)} \frac{d\theta}{\pi} \\ &= 2 \int_0^1 \log \frac{1}{r} \frac{dr}{r} \int_{\partial \mathbf{B}} \Re f(r\xi) \overline{\Re g(r\xi)} d\sigma(\xi) \\ &= \frac{1}{n} \int_{\mathbf{B}} \Re f(z) \overline{\Re g(z)} \log \frac{1}{|z|} \frac{dv(z)}{|z|^{2n}}, \end{aligned}$$

(2.2) gives

$$\int_{\partial \mathbf{B}} f(\xi) \overline{g(\xi)} d\sigma(\xi) = f(0)\overline{g(0)} + \frac{2}{n} \int_{\mathbf{B}} \Re f(z) \overline{\Re g(z)} \log \frac{1}{|z|} \frac{dv(z)}{|z|^{2n}}.$$

This and [Ru80] yield, for $f, g \in A_\varphi^2(\mathbf{B})$,

$$\begin{aligned}
 & \int_{\mathbf{B}} f(z) \overline{g(z)} \varphi(z) dv(z) \\
 &= 2n \int_0^1 r^{2n-1} \varphi(r) dr \int_{\partial \mathbf{B}} f(r\xi) \overline{g(r\xi)} d\sigma(\xi) \\
 &= 2n \int_0^1 r^{2n-1} \varphi(r) dr \left[f(0) \overline{g(0)} + \frac{2}{n} \int_{\mathbf{B}} \Re f(rz) \Re \overline{g(rz)} \log \frac{1}{|z|} \frac{dv(z)}{|z|^{2n}} \right] \\
 (2.3) \quad &= f(0) \overline{g(0)} + 4 \int_0^1 r^{2n-1} \varphi(r) dr \int_{\mathbf{B}} \Re f(rz) \Re \overline{g(rz)} \log \frac{1}{|z|} \frac{dv(z)}{|z|^{2n}}.
 \end{aligned}$$

Taking the variable transformation $w = rz$, we have

$$\begin{aligned}
 & \int_0^1 r^{2n-1} \varphi(r) dr \int_{\mathbf{B}} \Re f(rz) \Re \overline{g(rz)} \log \frac{1}{|z|} \frac{dv(z)}{|z|^{2n}} \\
 &= \int_0^1 r^{2n-1} \varphi(r) dr \int_{|w| < r, z \in \mathbf{B}} \Re f(w) \Re \overline{g(w)} \frac{r^{2n}}{|w|^{2n}} \log \frac{r}{|w|} \frac{dv(w)}{r^{2n}} \\
 (2.4) \quad &= \int_{\mathbf{B}} \Re f(w) \Re \overline{g(w)} \frac{1}{|w|^{2n}} dv(w) \int_{|w|}^1 r^{2n-1} \varphi(r) \log \frac{r}{|w|} dr.
 \end{aligned}$$

Because

$$\begin{aligned}
 \int_{|w|}^1 r^{2n-1} \varphi(r) \log \frac{r}{|w|} dr &= \int_{|w|}^1 \log \frac{r}{|w|} \left[\frac{d}{dr} \left(- \int_r^1 t^{2n-1} \varphi(t) dt \right) \right] dr \\
 &= \int_{|w|}^1 \frac{\int_r^1 t^{2n-1} \varphi(t) dt}{r} dr,
 \end{aligned}$$

(2.4) gives

$$\begin{aligned}
 & \int_0^1 r^{2n-1} \varphi(r) dr \int_{\mathbf{B}} \Re f(rz) \Re \overline{g(rz)} \log \frac{1}{|z|} \frac{dv(z)}{|z|^{2n}} \\
 (2.5) \quad &= \int_{\mathbf{B}} \Re f(w) \Re \overline{g(w)} \frac{1}{|w|^{2n}} \left(\int_{|w|}^1 \frac{\int_r^1 t^{2n-1} \varphi(t) dt}{r} dr \right) dv(w).
 \end{aligned}$$

Putting (2.5) into (2.3), we obtain (2.1). The proof is complete. $\square$

In particular, in the very special case that $n = 1$ and $\varphi(r) = (\alpha + 1)(1 - r^2)^\alpha$ with $\alpha > -1$, the equality (2.1) is just (5.1) of [AS97].

We are now going to exhibit the relation between the extended Cesàro operator and the Toeplitz operator. Recall that, given a finite Borel measure μ on $\mathbf{B}$, the Toeplitz operator $T(\mu)$ on $A_\varphi^2(\mathbf{B})$ is defined by

$$T(\mu)f(z) = \int_{\mathbf{B}} f(w) K_\varphi(z, w) d\mu(w),$$

where $K_\varphi(z, w)$ is the reproducing kernel of $A_\varphi^2(\mathbf{B})$. Toeplitz operators have been extensively studied by many authors; see [Lu87, Zh88, Zh07, CKN07, Lu08]. We also need Bloch functions. A holomorphic function g on $\mathbf{B}$ is said to be a Bloch function if $\sup_{z \in \mathbf{B}} |\Re g(z)| (1 - |z|^2) < \infty$.

Theorem 2.2. *Suppose g is a Bloch function on $\mathbf{B}$. Then the operator $T_g^* T_g$ is a Toeplitz operator $T(\mu)$ induced by the measure $d\mu(w) = 4 |\Re g(w)|^2 \phi(w) dv(w)$.*

Proof. Since g is a Bloch function, by [Hu04] T_g is bounded on $A_\varphi^2(\mathbf{B})$; hence the adjoint T_g^* is also bounded. Notice also that $\Re(T_g f)(z) = f(z)\Re g(z)$. Then, for $f \in A_\varphi^2(\mathbf{B})$, applying Theorem 2.1 we get

$$\begin{aligned} T_g^* T_g f(z) &= \langle T_g^* T_g f, K_\varphi(\cdot, z) \rangle \\ &= \langle T_g f, T_g K_\varphi(\cdot, z) \rangle \\ &= 4 \int_{\mathbf{B}} f(w) \Re g(w) \overline{K_\varphi(w, z)} \cdot \overline{\Re g(w)} \phi(w) dv(w) \\ &= \int_{\mathbf{B}} f(w) K_\varphi(z, w) d\mu(w), \end{aligned}$$

where $d\mu = 4|\Re g(w)|^2 \phi(w) dv(w)$. The proof is complete. $\square$

3. SCHATTEN CLASS EXTENDED CESÀRO OPERATORS

In this section we will give the proof of Theorem 1.1. Before doing this we need some more preliminary results. Let H be a separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle$, and let T be a positive compact operator on H . There exists (see [Zh90] for example) an orthonormal set $\{e_k\}$ in H and a sequence $\{\lambda_k\}$ decreasing to 0 such that

$$Tx = \sum_k \lambda_k \langle x, e_k \rangle e_k$$

for all $x \in H$. The numbers λ_k are called the singular values of T . For $p > 0$, T is said to be in the Schatten class S_p if

$$\|T\|_{S_p} = \left(\sum_k \lambda_k^p \right)^{\frac{1}{p}} < \infty.$$

More generally, a compact (not necessarily positive) operator T on H belongs to S_p if the positive operator $|T| = (T^*T)^{\frac{1}{2}}$ belongs to S_p . In this case, define $\|T\|_{S_p} = \||T|\|_{S_p}$. For more details, see [Zh90].

Let $\beta(\cdot, \cdot)$ denote the Bergman metric on $\mathbf{B}$. The Bergman ball $D(z, \rho)$ with center $z \in \mathbf{B}$ and radius $\rho > 0$ is defined as $D(z, \rho) = \{w \in \mathbf{B} : \beta(z, w) < \rho\}$. Set $d\lambda(z) = \frac{dv(z)}{(1-|z|^2)^{n+1}}$. It is easy to see that

$$|D(z, \rho)| \simeq (1 - |z|^2)^{n+1}, \quad 1 - |w| \simeq 1 - |z| \text{ for } w \in D(z, \rho).$$

Given a finite positive Borel measure μ on $\mathbf{B}$ and $\rho > 0$, define the averaging function $\hat{\mu}_\rho$ by $\hat{\mu}_\rho(z) = \frac{\mu(D(z, \rho))}{\int_{D(z, \rho)} \varphi(w) dv(w)}$. Since φ is normal, by (1.1) it is easy to check that

$$\hat{\mu}_\rho(z) \simeq \frac{\mu(D(z, \rho))}{\varphi(z)(1 - |z|^2)^{n+1}}$$

(see [Hu02]). The following lemma is in [LH10].

Lemma 3.1. *Suppose μ is a finite positive Borel measure on $\mathbf{B}$ and $0 < p < \infty$. Then $T_\mu \in S_p$ if and only if $\hat{\mu}_\rho \in L^p(\lambda)$ for any (or some) $\rho > 0$. In this case,*

$$\|T_\mu\|_{S_p} \simeq \|\hat{\mu}_\rho\|_{L^p(\lambda)}.$$

Now we are in the position to give the proof of Theorem 1.1.

Proof of Theorem 1.1. We prove the sufficiency first. Suppose $0 < p < \infty$ and that $g \in H(\mathbf{B})$ satisfies

$$\int_{\mathbf{B}} |\Re g(z)|^p (1 - |z|^2)^{p-(n+1)} dv(z) < \infty.$$

This means that g is in the p -th Besov space, and hence g is in the little Bloch space. By [Hu04], T_g is compact on $A_\varphi^2(\mathbf{B})$. Applying Theorem 2.2 and Lemma 3.1, we need only prove that $\hat{\mu}_\rho \in L^{\frac{p}{2}}(\mathbf{B}, d\lambda)$ for $\mu(w) = 4|\Re g(w)|^2 \phi(w) dv(w)$. Now we fix $\rho > 0$ so that

$$(3.1) \quad \bigcup_{|z| \leq \frac{1}{2}} D(z, \rho) \subset \left\{ z : |z| \leq \frac{3}{4} \right\} \quad \text{and} \quad \bigcup_{|z| \geq \frac{1}{2}} D(z, \rho) \subset \left\{ z : |z| \geq \frac{1}{4} \right\}.$$

Since φ is integrable on the interval $[0, 1]$,

$$\phi(w) = \frac{1}{|w|^{2n}} \int_{|w|}^1 \frac{\int_r^1 t^{2n-1} \varphi(t) dt}{r} dr \leq C \frac{1}{|w|^{2n}} \log \frac{1}{|w|}.$$

For fixed $\zeta \in \partial \mathbf{B}$, set $g_\zeta(\lambda) = g(\lambda\zeta)$ for $\lambda \in \mathbf{D}$. From 6.4.4 in [Ru80] we know that $\Re g(\lambda\zeta) = \lambda g'_\zeta(\lambda)$. Hence

$$\begin{aligned} \sup_{z \in \mathbf{B}, |z| \leq \frac{3}{4}} \frac{|\Re g(z)|}{|z|} &= \sup_{\lambda \in \mathbf{D}, |\lambda| \leq \frac{3}{4}, \zeta \in \partial \mathbf{B}} \frac{|\lambda g'_\zeta(\lambda)|}{|\lambda|} = \sup_{\lambda \in \mathbf{D}, |\lambda| = \frac{3}{4}, \zeta \in \partial \mathbf{B}} |g'_\zeta(\lambda)| \\ &= C \sup_{\lambda \in \mathbf{D}, |\lambda| = \frac{3}{4}, \zeta \in \partial \mathbf{B}} |\lambda g'_\zeta(\lambda)| = C \sup_{z \in \mathbf{B}, |z| = \frac{3}{4}} |\Re g(z)|. \end{aligned}$$

Therefore, by (3.1) and the plurisubharmonicity of $|\Re g(z)|^p$,

$$\begin{aligned} \sup_{|z| \leq \frac{1}{2}} \hat{\mu}_\rho(z) &\leq C \int_{|z| \leq \frac{3}{4}} |\Re g(z)|^2 \phi(z) dv(z) \\ &\leq C \sup_{|z| \leq \frac{3}{4}} \frac{|\Re g(z)|^2}{|z|^2} \int_{|z| \leq \frac{3}{4}} \frac{1}{|z|^{2n-2}} \log \frac{1}{|z|} dv(z) \\ &\leq C \sup_{|z| = \frac{3}{4}} |\Re g(z)|^2 \\ &\leq C \left\{ \int_{\frac{1}{2} < |w| < \frac{7}{8}} |\Re g(w)|^p dv(w) \right\}^{\frac{2}{p}} \end{aligned} \quad (3.2)$$

$$(3.3) \quad \leq C \left\{ \int_{\mathbf{B}} |\Re g(w)|^p (1 - |w|^2)^{p-(n+1)} dv(w) \right\}^{\frac{2}{p}}.$$

For $\frac{1}{4} \leq r < 1$, by definition (1.1) and [Hu04],

$$\frac{\int_r^1 t^{2n-1} \varphi(t) dt}{r} \simeq \int_r^1 \varphi(t) dt \simeq \varphi(r)(1-r) \quad \text{and} \quad \varphi(w) \simeq \varphi(z) \quad \text{for } w \in D(z, \rho).$$

Then, for $|z| > \frac{1}{2}$ we have

$$\begin{aligned}
 \hat{\mu}_\rho(z)^{\frac{p}{2}} &\simeq \left\{ \frac{\int_{D(z,\rho)} |\Re g(w)|^2 \varphi(w) (1 - |w|^2)^2 dv(w)}{\int_{D(z,\rho)} \varphi(w) dv(w)} \right\}^{\frac{p}{2}} \\
 (3.4) \quad &\simeq \left\{ (1 - |z|^2)^{1-n} \int_{D(z,\rho)} |\Re g(w)|^2 dv(w) \right\}^{\frac{p}{2}} \\
 &\leq C(1 - |z|^2)^p \sup_{w \in D(z,\rho)} |\Re g(w)|^p \\
 &\leq C(1 - |z|^2)^p \frac{1}{|D(z, 2\rho)|} \int_{D(z, 2\rho)} |\Re g(w)|^p dv(w) \\
 &\leq C(1 - |z|^2)^{p-(n+1)} \int_{D(z, 2\rho)} |\Re g(w)|^p dv(w).
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 &\int_{|z| \geq \frac{1}{2}} |\hat{\mu}_\rho(z)|^{\frac{p}{2}} d\lambda(z) \\
 &\leq C \int_{|z| > \frac{1}{2}} \left(\int_{D(z, 2\rho)} |\Re g(w)|^p dv(w) \right) (1 - |z|^2)^{p-2(n+1)} dv(z) \\
 &\leq C \int_{\mathbf{B}} \left(\int_{D(z, 2\rho)} |\Re g(w)|^p dv(w) \right) (1 - |z|^2)^{p-2(n+1)} dv(z) \\
 &= C \int_{\mathbf{B}} |\Re g(w)|^p \left(\int_{\mathbf{B}} \chi_{D(z, 2\rho)}(w) (1 - |z|^2)^{p-2(n+1)} dv(z) \right) dv(w) \\
 (3.5) \quad &\leq C \int_{\mathbf{B}} |\Re g(w)|^p (1 - |w|^2)^{p-(n+1)} dv(w).
 \end{aligned}$$

Combine (3.3) and (3.5) to obtain

$$\begin{aligned}
 \int_{\mathbf{B}} |\hat{\mu}_\rho(z)|^{\frac{p}{2}} d\lambda(z) &= \left\{ \int_{|z| < \frac{1}{2}} + \int_{|z| \geq \frac{1}{2}} \right\} |\hat{\mu}_\rho(z)|^{\frac{p}{2}} d\lambda(z) \\
 &\leq C \int_{\mathbf{B}} |\Re g(z)|^p (1 - |z|^2)^{p-(n+1)} dv(z).
 \end{aligned}$$

This gives the desired estimate:

$$(3.6) \quad \|T_g\|_{S_p} \leq C \left\{ \int_{\mathbf{B}} |\Re g(z)|^p (1 - |z|^2)^{p-(n+1)} dv(z) \right\}^{\frac{1}{p}}.$$

Now we prove the necessity. Suppose $\|T_g\|_{S_p} < \infty$. We claim that

$$(3.7) \quad \hat{\mu}_\rho(z) \geq C |\Re g(z)|^2 (1 - |z|^2)^2.$$

In fact, if $|z| \leq \frac{1}{2}$ and $w \in D(z, \rho)$, it is easy to check that $\phi(w) \geq C$. Hence, for $|z| \leq \frac{1}{2}$,

$$\hat{\mu}_\rho(z) \geq \frac{C}{\varphi(z)|D(z, \rho)|} \int_{D(z, \rho)} |\Re g(w)|^2 dv(w) \geq C |\Re g(z)|^2 (1 - |z|^2)^2.$$

If $|z| > \frac{1}{2}$, by (3.4) and the plurisubharmonicity of $|\Re g(z)|^2$,

$$\hat{\mu}_\rho(z) \geq (1 - |z|^2)^{1-n} \int_{D(z, \rho)} |\Re g(w)|^2 dv(w) \geq C |\Re g(z)|^2 (1 - |z|^2)^2.$$

Then (3.7) follows from these two inequalities. Now (3.7) and Lemma 3.1 yield

$$\int_{\mathbf{B}} |\Re g(z)|^p (1 - |z|^2)^{p-(n+1)} dv(z) \leq C \int_{\mathbf{B}} |\hat{\mu}_\rho(z)|^{\frac{p}{2}} d\lambda(z) \leq C \|T_g\|_{S_p}^p.$$

Combine this and (3.6) to obtain the desired estimate:

$$\|T_g\|_{S_p} \simeq \left\{ \int_{\mathbf{B}} |\Re g(z)|^p (1 - |z|^2)^{p-(n+1)} dv(z) \right\}^{\frac{1}{p}}.$$

The proof is complete. $\square$

The Besov space B_p consists of all functions $f \in H(\mathbf{B})$ for which

$$\|f\|_{B_p} = \left\{ \int_{\mathbf{B}} |\Re g(z)|^p (1 - |z|^2)^{p-(n+1)} dv(z) \right\}^{\frac{1}{p}} < \infty.$$

It is easy to verify that B_p contains constants only if $0 < p \leq n$. In terms of the space B_p , Theorem 1.1 can be reformulated as below.

Corollary 3.1. *Suppose $g \in H(\mathbf{B})$ and $0 < p < \infty$. Then for $p > n$, $T_g \in S_p$ if and only if $g \in B_p$; and for $0 < p \leq n$, T_g does not belong to S_p unless g is constant. While for $n < p < \infty$, $\|T_g\|_{S_p} \simeq \|g\|_{B_p}$.*

4. SCHATTEN-HERZ CLASS EXTENDED CESÀRO OPERATORS

In this section we prove Theorem 1.2. First, we recall some more about Schatten-Herz class Toeplitz operators. We decompose $\mathbf{B}$ into the disjoint union of annuli $A_k = \{z \in \mathbf{B} : r_k \leq |z| < r_{k+1}\}$, where $r_k = 1 - 2^{-k}$, $k = 0, 1, 2, \dots$. For each k , set $\chi_k = \chi_{A_k}$, the characteristic function of A_k . For $0 < p, q < \infty$, the Herz space $\mathcal{K}_q^p(\lambda)$ consists of measurable function f for which

$$\|f\|_{\mathcal{K}_q^p(\lambda)} = \left\| \{ \|f\chi_k\|_{L^p(\lambda)} \} \right\|_{l^q}.$$

Given μ a finite Borel measure, let $\mu\chi_k$ stand for the restriction of μ to A_k . Given $0 < p, q < \infty$, the induced Toeplitz operator $T(\mu)$ is said to be of the Schatten-Herz class $S_{p,q}$, denoted by $T(\mu) \in S_{p,q}$, if $T(\mu\chi_k) \in S_p$ for each k and the sequence $\{\|T(\mu\chi_k)\|_{S_p}\}_{k=0}^\infty$ belongs to l^q . The $S_{p,q}$ -norm of $T(\mu)$ is defined by $\|T(\mu)\|_{S_{p,q}} = \left\| \{ \|T(\mu\chi_k)\|_{S_p} \} \right\|_{l^q}$. For more details on Herz spaces and the Schatten-Herz class Toeplitz operators, see [LLP05, CKL07, CKN07, HY99].

The following lemma comes from [LH10].

Theorem 4.1. *Suppose μ is a finite positive Borel measure on $\mathbf{B}$ and $0 < p, q < \infty$. Then $T(\mu) \in S_{p,q}$ if and only if $\hat{\mu}_\rho \in \mathcal{K}_q^p(\lambda)$ for any (or some) $\rho > 0$. Furthermore,*

$$\|T(\mu)\|_{S_{p,q}} \simeq \|\hat{\mu}_\rho\|_{\mathcal{K}_q^p(\lambda)}.$$

Proof of Theorem 1.2. By Theorem 2.2 and Theorem 4.1, $T_g \in S_{p,q}$ if and only if

$$\hat{\mu}_\rho \in \mathcal{K}_{\frac{p}{2}}^{\frac{p}{2}}(\lambda) \quad \text{and} \quad \|T_g\|_{S_{p,q}} \simeq \|\hat{\mu}_\rho\|_{\mathcal{K}_{\frac{p}{2}}^{\frac{p}{2}}(\lambda)},$$

where $\rho > 0$ is fixed and small enough. Thus, we need only prove that

$$(4.1) \quad \sum_{k=0}^{\infty} \left\{ \int_{A_k} \hat{\mu}_\rho(z)^{\frac{p}{2}} d\lambda(z) \right\}^{\frac{q}{p}} \simeq \int_0^1 M_p^q(\Re g, r) (1-r^2)^{q-(n\frac{q}{p}+1)} dr.$$

For each $k \geq 1$, by (3.7) and the fact that $M_p(\Re g, t)$ is increasing in $(0, 1)$ we have

$$\begin{aligned} \int_{A_k} \hat{\mu}_\rho(z)^{\frac{p}{2}} d\lambda(z) &\geq C \int_{A_k} |\Re g(z)|^p (1-|z|^2)^p d\lambda(z) \\ &\simeq 2^{k[(n+1)-p]} \int_{r_k}^{r_{k+1}} r^{2n-1} dr \int_{\partial \mathbf{B}} |\Re g(r\xi)|^p d\sigma(\xi) \\ &\geq C 2^{k[(n+1)-p-1]} M_p^p(\Re g, r_k), \quad k = 1, 2, \dots \end{aligned}$$

Therefore, with $(1-t) \simeq 2^{-k}$ for $t \in (r_{k-1}, r_k)$ in mind,

$$\begin{aligned} \sum_{k=0}^{\infty} \left\{ \int_{A_k} \hat{\mu}_\rho(z)^{\frac{p}{2}} d\lambda(z) \right\}^{\frac{q}{p}} &\geq C \sum_{k=1}^{\infty} 2^{\frac{qk}{p}(n-p)} M_p^q(\Re g, r_k) \\ &\geq C \sum_{k=1}^{\infty} \int_{r_{k-1}}^{r_k} (1-t)^{q-n\frac{q}{p}-1} M_p^q(\Re g, t) dt \\ (4.2) \quad &= C \int_0^1 (1-t)^{q-(n\frac{q}{p}+1)} M_p^q(\Re g, t) dt. \end{aligned}$$

This gives one direction estimate of (4.1). Now we deal with the other direction. Since

$$\inf_{|k-l| \geq 2} \beta(A_k, A_l) > C > 0,$$

we can fix $\rho > 0$ so that $\widetilde{A}_k = \bigcup_{z \in A_k} D(z, 2\rho)$ satisfies

$$\widetilde{A}_k \subseteq \{z \in \mathbf{B} : r_{k-1} \leq |z| < r_{k+2}\} = A_{k-1} \cup A_k \cup A_{k+1},$$

here $A_{-1} = A_0$. For $z \in \mathbf{B}$, with some $w_0 \in \overline{D(z, \rho)}$ we have

$$\begin{aligned} \left[\int_{D(z, \rho)} |\Re g(w)|^2 dv(w) \right]^{\frac{p}{2}} &\leq \left[|D(z, \rho)| \max_{w \in \overline{D(z, \rho)}} |\Re g(w)|^2 \right]^{\frac{p}{2}} \\ &\leq C (1-|z|^2)^{(n+1)\frac{p}{2}} |\Re g(w_0)|^p \\ &\leq C \frac{(1-|z|^2)^{(n+1)\frac{p}{2}}}{|D(w_0, \rho)|} \int_{D(w_0, \rho)} |\Re g(u)|^p dv(u) \\ (4.3) \quad &\leq C (1-|z|^2)^{(n+1)(\frac{p}{2}-1)} \int_{D(z, 2\rho)} |\Re g(w)|^p dv(w). \end{aligned}$$

For $k \geq 1$, combine (3.4) and (4.3) to obtain

$$\begin{aligned}
 & \int_{A_k} \hat{\mu}_\rho(z)^{\frac{p}{2}} dv(z) \\
 & \leq C \int_{A_k} (1 - |z|^2)^{(1-n)\frac{p}{2}} \left(\int_{D(z, \rho)} |\Re g(w)|^2 dv(w) \right)^{\frac{p}{2}} dv(z) \\
 & \leq C \int_{A_k} (1 - |z|^2)^{(1-n)\frac{p}{2} + (n+1)(\frac{p}{2}-1)} \left(\int_{D(z, 2\rho)} |\Re g(w)|^p dv(w) \right) dv(z) \\
 & \leq C \int_{A_k} (1 - |z|^2)^{p-(n+1)} \left(\int_{\widetilde{A_k}} |\Re g(w)|^p \chi_{D(z, 2\rho)}(w) dv(w) \right) dv(z) \\
 & \leq C \int_{\widetilde{A_k}} |\Re g(w)|^p \left(\int_{A_k} (1 - |z|^2)^{p-(n+1)} \chi_{D(z, 2\rho)}(w) dv(z) \right) dv(w) \\
 & \leq C \int_{\widetilde{A_k}} (1 - |w|^2)^p |\Re g(w)|^p dv(w).
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 & \left\{ \int_{A_k} \hat{\mu}_\rho(z)^{\frac{p}{2}} d\lambda(z) \right\}^{\frac{q}{p}} \\
 & \leq C(1 - r_k)^{-(n+1)\frac{q}{p}} \left\{ \int_{A_k} \hat{\mu}_\rho(z)^{\frac{p}{2}} dv(z) \right\}^{\frac{q}{p}} \\
 & \leq C(1 - r_k)^{-(n+1)\frac{q}{p}} \left\{ \int_{\widetilde{A_k}} (1 - |w|^2)^p |\Re g(w)|^p dv(w) \right\}^{\frac{q}{p}} \\
 & \leq C(1 - r_k)^{-(n+1)\frac{q}{p}} \left\{ \int_{r_{k-1}}^{r_{k+2}} (1 - r^2)^p M_p^p(\Re g, r) dr \right\}^{\frac{q}{p}} \\
 (4.4) \quad & \leq C(1 - r_k)^{-(n+1)\frac{q}{p} + (p+1)\frac{q}{p}} M_p^q(\Re g, r_{k+2}), \quad k = 1, 2, \dots
 \end{aligned}$$

Meanwhile, by (3.2) we get

$$(4.5) \quad \left\{ \int_{A_0} \hat{\mu}_\rho(z)^{\frac{p}{2}} d\lambda(z) \right\}^{\frac{q}{p}} \leq C \left\{ \int_{\frac{1}{2} \leq |z| \leq \frac{7}{8}} |\Re g(z)|^p dv(z) \right\}^{\frac{q}{p}} \leq C M_p^q(\Re g, r_3).$$

(4.4) and (4.5) yield

$$\begin{aligned}
 \sum_{k=0}^{\infty} \left\{ \int_{A_k} \hat{\mu}_\rho(z)^{\frac{p}{2}} d\lambda(z) \right\}^{\frac{q}{p}} & \leq C \sum_{k=1}^{\infty} (1 - r_k)^{q-n\frac{q}{p}} M_p^q(\Re g, r_{k+2}) \\
 & \leq C \sum_{k=1}^{\infty} \int_{r_{k+2}}^{r_{k+3}} (1 - t)^{q-n\frac{q}{p}-1} M_p^q(\Re g, t) dt \\
 & \leq C \int_0^1 (1 - t)^{q-(n\frac{q}{p}+1)} M_p^q(\Re g, t) dt.
 \end{aligned}$$

From this and (4.2) we obtain (4.1). The proof is complete. $\square$

Notice that, for $0 < p \leq n$ and $g \in H(\mathbf{B})$,

$$\int_0^1 M_p^q(\Re g, r)(1-r^2)^{q-(\frac{n}{p}+1)} dr < \infty$$

implies that g is a constant. Similar to Corollary 3.1 we have the following at once.

Corollary 4.1. *Suppose $g \in H(\mathbf{B})$ and $0 < p, q < \infty$. Then for $p > n$, $T_g \in S_{p,q}$ if and only if $\int_0^1 M_p^q(\Re g, r)(1-r^2)^{q-(\frac{n}{p}+1)} dr < \infty$; and for $0 < p \leq n$, T_g does not belong to $S_{p,q}$ unless g is a constant. While for $n < p < \infty$, $\|T_g\|_{S_{p,q}} \simeq \left\{ \int_0^1 M_p^q(\Re g, r)(1-r^2)^{q-(\frac{n}{p}+1)} dr \right\}^{\frac{1}{q}}$.*

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