GENERIC PROPERTIES OF CRITICAL POINTS OF THE SCALAR CURVATURE FOR A RIEMANNIAN MANIFOLD

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ABSTRACT. Given (M, g) a smooth compact Riemannian N -manifold, we prove that for generic Riemannian metric g the critical points of the scalar curvature are nondegenerate.

1. INTRODUCTION

Let M be a connected compact C^∞ manifold of dimension $N \geq 2$ without boundary. Let $\mathcal{M}^k$ be the set of all C^k Riemannian metrics on M . Any $g \in \mathcal{M}^k$ determines the scalar curvature S_g of (M, g) . Our goal is to prove that for generic Riemannian metric g the critical points of the scalar curvature S_g are nondegenerate. More precisely, we will prove the following result.

Theorem 1.1. *The set*

$$\mathcal{A} := \{g \in \mathcal{M}^k : \text{all the critical points of } S_g \text{ are nondegenerate}\}$$

is an open dense subset of $\mathcal{M}^k$.

We could apply Theorem 1.1 to the study of the following problem:

$$(1.1) \quad \begin{cases} -\varepsilon^2 \Delta_g u + u = u^p & \text{in } M, \\ u > 0 & \text{in } M, \\ u \in H_g^1(M), \end{cases}$$

where $p > 1$ if $N = 2$, $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$ and ε is a positive parameter. Here $H_g^1(M)$ is the completion of $C^\infty(M)$ with respect to

$$\|u\|_g^2 := \int_M |\nabla_g u|^2 d\mu_g + \int_M u^2 d\mu_g.$$

It is well known that problem (1.1) has a mountain pass solution u_ε . In [2], the authors proved that u_ε exhibits a spike layer by showing that its peak point converges to a maximum point of the scalar curvature S_g as ε goes to zero.

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Recently there have been some results on the effect of the topology and the geometry of the manifold M on the number of solutions of problem (1.1). In [1] the authors showed that multiplicity of solutions to (1.1) depends on the topological properties of the manifold M . More precisely, they show that problem (1.1) has at least $\text{cat}(M) + 1$ nontrivial solutions provided ε is small enough. Here $\text{cat}(M)$ denotes the Lusternik-Schnirelmann category of M . In [11] the author obtains the same result for a more general nonlinearity. In [5] it is shown that the number of solutions is influenced by the topology of a suitable subset of M depending on the geometry of M .

Successively, in [6] the authors pointed out the role of the geometry in finding solutions to problem (1.1). In fact, they showed that for any C^1 -stable critical point of the scalar curvature it is possible to construct a single peak solution whose peak approaches such a point as ε goes to zero. Moreover, in [7] the authors constructed K -peaks positive or nodal solutions, whose positive and negative peaks approach K different C^1 -stable critical points of the scalar curvature as ε goes to zero. As far as it concerns the existence of nodal solutions, we also quote [4], where the authors prove the existence of antisymmetric sign changing solutions to (1.1).

We say that ξ_0 is a C^1 -stable critical point of a function $f \in C^1(M, \mathbb{R})$ if for any $\rho > 0$ there exists $\delta > 0$ such that if $h \in C^1(M, \mathbb{R})$ with

$$\max_{\xi \in B_g(\xi_0, \rho)} [|f(\xi) - h(\xi)| + |\nabla_g f(\xi) - \nabla_g h(\xi)|] \leq \delta,$$

then h has at least one critical point $\xi \in B_g(\xi_0, \rho)$.

In particular, if ξ_0 is a strict local minimum point or a strict local maximum point of f or it is a nondegenerate critical point of f , then ξ_0 is a C^1 -stable critical point of f .

Recently, in [3] the authors proved that if the scalar curvature has an isolated local minimum point ξ_0 , for any integer K there exists a solution to (1.1) whose K peaks collapse to ξ_0 as ε goes to zero.

All the previous results require a sort of nondegeneracy of critical points of the scalar curvature of the manifold (M, g) . Theorem 1.1 allows us to claim that for a generic metric g , all the critical points of the scalar curvature are nondegenerate. If ν is the number of the critical points of the scalar curvature, by [6, 7] we deduce that problem (1.1) has, as ε goes to zero, ν solutions with one peak, $\nu(\nu - 1)/2$ solutions with two peaks, $\dots$, one solution with ν peaks.

Finally, since the global minimum point of the scalar curvature is nondegenerate and in particular is isolated, by [3] we deduce that the number of solutions to problem (1.1) goes to ∞ as ε goes to zero.

The paper is organized as follows. In Section 2 we formulate the problem and prove the main result, using some technical results proved in Section 3.

2. FORMULATION OF THE PROBLEM AND PROOF OF THE MAIN RESULT

We denote by $\mathcal{S}^k$ the space of all C^k symmetric covariant 2-tensors on M . $\mathcal{S}^k$ is a Banach space equipped with the norm $\|\cdot\|_k$ defined in the following way. We fix a finite covering $\{V_\alpha\}_{\alpha \in L}$ of M such that the closure of V_α is contained U_α , where $\{U_\alpha, \psi_\alpha\}$ is the open coordinate neighbourhood. If $h \in \mathcal{S}^k$ we denote by h_{ij} the

components of h with respect to the coordinates $(x_1, \dots, x_N)$ on V_α . We define

$$\|h\|_k := \sum_{\alpha \in L} \sum_{|\beta| \leq k} \sum_{i,j=1}^N \sup_{\psi_\alpha(V_\alpha)} \frac{\partial^\beta h_{ij}}{\partial x_1^{\beta_1} \dots \partial x_N^{\beta_N}}.$$

The set $\mathcal{M}^k$ of all C^k Riemannian metrics on M is an open set of $\mathcal{S}^k$.

In the following we will assume $k \geq 4$.

Given $\hat{g} \in \mathcal{M}^k$, it is possible to define an atlas on M whose charts are $(B_{\hat{g}}(\xi, R), \varphi^{-1})$, where $\varphi : B(0, R) \rightarrow B_{\hat{g}}(\xi, R)$. Here $B_{\hat{g}}(\xi, R) \subset M$ is the ball centered at ξ with radius R given by the metric $\hat{g}$ and $B(0, R) \subset \mathbb{R}^N$ is the ball centered at 0 with radius R in the euclidean space $\mathbb{R}^N$. Let $\mathcal{B}_\rho := \{h \in \mathcal{S}^k : \|h\|_k < \rho\}$ be the ball centered at 0 with radius ρ in $\mathcal{S}^k$.

For any $\xi \in M$ and $h \in \mathcal{B}_\rho$, with ρ small enough so that $\hat{g} + h \in \mathcal{M}^k$, we consider the scalar curvature $S_{\hat{g}+h}(\xi)$ of $(M, \hat{g} + h)$ at the point $\xi \in M$. Given $\xi_0 \in M$ and the chart $(B_{\hat{g}}(\xi_0, R), \varphi^{-1})$ we set

$$(2.1) \quad \tilde{S}_{\hat{g}+h}(x) := S_{\hat{g}+h}(\varphi(x)), \quad x \in B(0, R), \quad h \in \mathcal{B}_\rho.$$

Now, we introduce the C^1 -map $F : \mathcal{B}_\rho \times B(0, R) \subset \mathcal{S}^k \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ defined by

$$(2.2) \quad F(h, x) := \nabla \tilde{S}_{\hat{g}+h}(x).$$

We shall apply to the map F an abstract transversality theorem (see [8, 9, 10]). We recall it (see Theorem 1.1 of [9]) in the following.

Theorem 2.1. *Let X, Y, Z be three real Banach spaces and let $U \subset X$, $V \subset Y$ be open subsets. Let $F : V \times U \rightarrow Z$ be a C^k -map with $k \geq 1$ such that*

- (i) *for any $y \in V$, $F(y, \cdot) : x \rightarrow F(y, x)$ is a Fredholm map of index l with $l \leq k$;*
- (ii) *z_0 is a regular value of F ; that is, the operator $F'(y_0, x_0) : Y \times X \rightarrow Z$ is onto at any point (y_0, x_0) such that $F(y_0, x_0) = z_0$;*
- (iii) *the set of $x \in U$ such that $F(y, x) = z_0$ with y in a compact set of X is relatively compact in U .*

Then the set $\{y \in V : z_0 \text{ is a regular value of } F(y, \cdot)\}$ is a dense open subset of V .

By Theorem 2.1 we obtain the following result, which is crucial to deduce Theorem 1.1.

Theorem 2.2. *For any $\hat{g} \in \mathcal{M}^k$ the set*

$$\{h \in \mathcal{S}^k : \text{all the critical points of } S_{\hat{g}+h} \text{ are nondegenerate}\}$$

is an open dense subset of a suitable ball centered at 0 with radius ρ in $\mathcal{S}^k$.

Proof. We are going to apply Theorem 2.1 to the map F defined in (2.2). In this case we have $X = Z = \mathbb{R}^N$ and $Y = \mathcal{S}^k$. We choose $z_0 = 0$. Since X is a finite dimensional space, it is easy to check that for any $h \in \mathcal{B}_\rho$ the map $x \rightarrow F(h, x)$ is Fredholm of index 0, and so assumption (i) holds. Moreover, assumption (iii) immediately follows again by the fact that X is a finite dimensional space. Assumption (ii) is verified in Lemma 3.4.

Finally, we are in a position to apply Theorem 2.1, and we get that the set

$$\begin{aligned} \Theta(\xi_0, \rho) &:= \{h \in \mathcal{B}_\rho : F'_x(h, x) : \mathbb{R}^N \rightarrow \mathbb{R}^N \text{ is invertible at any point} \\ &\quad (h, x) \text{ such that } F(h, x) = 0\} \\ (2.3) \quad &= \{h \in \mathcal{B}_\rho : \text{the critical points of } S_{\hat{g}+h} \text{ in } B_{\hat{g}}(\xi_0, R) \text{ are nondegenerate}\} \end{aligned}$$

is an open dense subset of $\mathcal{B}_\rho$.

Now, since M is compact, there exists a finite covering $\{B_{\hat{g}}(\xi_i, R)\}_{i=1, \dots, \nu}$ of M , where $\xi_1, \dots, \xi_\nu \in M$. For any index i there exists an open dense subset $\Theta(\xi_i, \rho)$ (see (2.3)) of $\mathcal{B}_\rho$ such that the critical points of $S_{\hat{g}+h}$ in $B_{\hat{g}}(\xi_i, R)$ are nondegenerate for any $h \in \Theta(\xi_i, \rho)$. Let $\Theta(\rho) := \bigcap_{i=1, \dots, \nu} \Theta(\xi_i, \rho)$. It is immediate that $\Theta(\rho)$ is an open dense subset of $\mathcal{B}_\rho$ such that the critical points of $S_{\hat{g}+h}$ in M are nondegenerate for any $h \in \Theta(\rho)$. $\square$

Proof of Theorem 1.1. First of all, we prove that $\mathcal{A}$ is an open set. If $\hat{g} \in \mathcal{A}$ the critical points of $S_{\hat{g}}$ are in a finite number $\xi_1, \dots, \xi_\nu$. Let us consider the chart $(B_{\hat{g}}(\xi_1), \varphi_1^{-1})$ and set $\tilde{S}_{\hat{g}+h}(x) := S_{\hat{g}+h}(\varphi_1(x))$. We introduce the C^2 map $F : \mathcal{B}_\rho \times B(0, R) \rightarrow \mathbb{R}^N$ defined by $F(h, x) := \nabla \tilde{S}_{\hat{g}+h}(x)$. We have $F(0, 0) = 0$. Moreover $F'_x(0, 0) : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is an isomorphism. Then by the implicit function theorem it is easy to deduce that locally there exists a unique $x = x(h)$ such that $F(h, x(h)) = 0$. Since the critical points of $S_{\hat{g}}$ are in a finite number, it is easy to check that choosing ρ small enough for any $h \in \mathcal{B}_\rho$ has only ν critical points. They are nondegenerate because F is a C^2 -function.

The density of $\mathcal{A}$ follows by Theorem 2.2. $\square$

3. TECHNICAL LEMMAS

Let us recall the definition of the scalar curvature $S_g(\xi)$ of the metric g at the point ξ . Let $\xi_0 \in M$ be fixed. Given a coordinate system, the scalar curvature in a point $\xi(x)$ belonging to $B_g(\xi_0, R)$ can be expressed as follows:

$$(3.1) \quad \tilde{S}_g(x) = S_g(\xi(x)) = g^{ab} (\partial_c \Gamma_{ab}^c - \partial_b \Gamma_{ac}^c + \Gamma_{ab}^c \Gamma_{cd}^d - \Gamma_{ac}^d \Gamma_{bd}^c).$$

Here Γ_{ij}^l denotes the Christoffel symbol with respect to g , namely

$$(3.2) \quad \Gamma_{ij}^l = \Gamma_{ij}^l(g, x) = \frac{1}{2} g^{lm} (\partial_j g_{mi} + \partial_i g_{mj} - \partial_m g_{ij}).$$

We are using the Einstein summation convention; i.e. when an index variable appears twice in a single term, once in an upper (superscript) and once in a lower (subscript) position, it implies that we are summing over all of its possible values.

We are going to compute the first and second derivatives of $\Gamma_{ij}^l(g, x)$ with respect to x .

It is useful to introduce the following notation. Let

$$(3.3) \quad G_{ijk} = G_{ijk}(g, x) = \partial_i g_{kj} + \partial_j g_{ki} - \partial_k g_{ij}.$$

By (3.2) and (3.3) we deduce

$$(3.4) \quad \Gamma_{ij}^l(g, x) = \frac{1}{2} g^{lk} G_{ijk}(g, x).$$

Lemma 3.1. *It holds that for any $x \in B(0, R)$ and for any $h \in \mathcal{B}_\rho$*

$$\partial_\alpha \Gamma_{ij}^l(g, x) = \frac{1}{2} \partial_\alpha g^{lk} G_{ijk}(g, x) + \frac{1}{2} g^{lk} \partial_\alpha G_{ijk}(g, x),$$

$$\begin{aligned} \partial_{\alpha\beta}^2 \Gamma_{ij}^l(g, x) &= \frac{1}{2} \partial_{\alpha\beta}^2 g^{lk} G_{ijk}(g, x) + \frac{1}{2} g^{lk} \partial_{\alpha\beta}^2 G_{ijk}(g, x) \\ &+ \partial_\alpha g^{lk} \partial_\beta G_{ijk}(g, x) + \partial_\beta g^{lk} \partial_\alpha G_{ijk}(g, x), \end{aligned}$$

$$D_g \Gamma_{ij}^l(g, x)[h] = \frac{1}{2} g^{lk} G_{ijk}(h, x) - \frac{1}{2} g^{ls} h_{st} g^{tk} G_{ijk}(g, x),$$

$$\begin{aligned} \partial_\alpha D_g \Gamma_{ij}^l(g, x)[h] &= \frac{1}{2} \partial_\alpha g^{lk} G_{ijk}(h, x) + \frac{1}{2} g^{lk} \partial_\alpha G_{ijk}(h, x) \\ &- \frac{1}{2} \partial_\alpha (g^{ls} h_{st} g^{tk}) G_{ijk}(g, x) - \frac{1}{2} g^{ls} h_{st} g^{tk} \partial_\alpha G_{ijk}(g, x) \\ &= D_g \partial_\alpha \Gamma_{ij}^l(g, x)[h], \end{aligned}$$

$$\begin{aligned} \partial_{\alpha\beta}^2 D_g \Gamma_{ij}^l(g, x)[h] &= \frac{1}{2} \partial_{\alpha\beta}^2 g^{lk} G_{ijk}(h, x) + \frac{1}{2} \partial_\alpha g^{lk} \partial_\beta G_{ijk}(h, x) \\ &+ \frac{1}{2} \partial_\beta g^{lk} \partial_\alpha G_{ijk}(h, x) + \frac{1}{2} g^{lk} \partial_{\alpha\beta}^2 G_{ijk}(h, x) \\ &- \frac{1}{2} \partial_{\alpha\beta}^2 (g^{ls} h_{st} g^{tk}) G_{ijk}(g, x) - \frac{1}{2} \partial_\alpha (g^{ls} h_{st} g^{tk}) \partial_\beta G_{ijk}(g, x) \\ &- \frac{1}{2} g^{ls} h_{st} g^{tk} \partial_{\alpha\beta}^2 G_{ijk}(g, x) - \frac{1}{2} \partial_\beta (g^{ls} h_{st} g^{tk}) \partial_\alpha G_{ijk}(g, x) \\ &= D_g \partial_{\alpha\beta}^2 \Gamma_{ij}^l(g, x)[h]. \end{aligned}$$

Now, we can evaluate the derivative of the scalar curvature.

Lemma 3.2. *It holds that for any $x \in B(0, R)$ and for any $h \in \mathcal{B}_\rho$*

$$\begin{aligned} \partial_\alpha \tilde{S}_g(x) &= \partial_\alpha g^{ab} (\partial_c \Gamma_{ab}^c - \partial_b \Gamma_{ac}^c + \Gamma_{ab}^c \Gamma_{cd}^d - \Gamma_{ac}^d \Gamma_{bd}^c) \\ (3.5) \quad &+ g^{ab} (\partial_{\alpha c}^2 \Gamma_{ab}^c - \partial_{\alpha b}^2 \Gamma_{ac}^c + \partial_\alpha \Gamma_{ab}^c \Gamma_{cd}^d + \Gamma_{ab}^c \partial_\alpha \Gamma_{cd}^d - \partial_\alpha \Gamma_{ac}^d \Gamma_{bd}^c - \Gamma_{ac}^d \partial_\alpha \Gamma_{bd}^c) \end{aligned}$$

and

$$\begin{aligned} \partial_\alpha D_g \tilde{S}_g(x)[h] &= D_g \partial_\alpha \tilde{S}_g(x)[h] \\ &= \partial_\alpha (g^{at} h_{ts} g^{sb}) (\partial_c \Gamma_{ab}^c - \partial_b \Gamma_{ac}^c + \Gamma_{ab}^c \Gamma_{cd}^d - \Gamma_{ac}^d \Gamma_{bd}^c) \\ &+ g^{ab} (\partial_{\alpha c}^2 D_g \Gamma_{ab}^c[h] - \partial_{\alpha b}^2 D_g \Gamma_{ac}^c[h] + \partial_\alpha D_g \Gamma_{ab}^c[h] \Gamma_{cd}^d + \partial_\alpha \Gamma_{ab}^c D_g \Gamma_{cd}^d[h] \\ &+ D_g \Gamma_{ab}^c[h] \partial_\alpha \Gamma_{cd}^d + \Gamma_{ab}^c \partial_\alpha D_g \Gamma_{cd}^d[h] - \partial_\alpha D_g \Gamma_{ac}^d[h] \Gamma_{bd}^c - \partial_\alpha \Gamma_{ac}^d D_g \Gamma_{bd}^c[h] \\ (3.6) \quad &- D_g \Gamma_{ac}^d[h] \partial_\alpha \Gamma_{bd}^c - \Gamma_{ac}^d \partial_\alpha D_g \Gamma_{bd}^c[h]). \end{aligned}$$

Given the metric $g = \hat{g} + h$ with $h \in \mathcal{B}_\rho$ and a point $\xi \in B_g(\xi_0, R)$, let us consider the local normal coordinates on the Riemannian manifold (M, g) given by the exponential map $\exp_\xi(z)$. Let g_ξ be the metric g read in $B(0, R)$ through the

normal coordinates. In particular

$$g_{\xi}^{ij}(0) = g_{\xi ij}(0) = \delta_{ij} \quad \text{and} \quad \frac{\partial g_{\xi}^{ij}}{z_k}(0) = \frac{\partial g_{\xi ij}}{z_k}(0) = 0,$$

which implies $\Gamma_{ij}^k(g_{\xi}, 0) = 0$ for any indexes i, j and k . Here $g_{\xi}^{ij}(z)$ is the inverse matrix of $g_{\xi ij}(z)$ and δ_{ij} is the Kronecker symbol.

Let us evaluate the derivatives of Lemma 3.1 at the point ξ using the normal coordinates.

Lemma 3.3. *It holds that*

$$(3.7) \quad \partial_{\alpha} \Gamma_{ij}^l(g_{\xi}, 0) = \frac{1}{2} \partial_{\alpha} G_{ijl}(g_{\xi}, 0)$$

and

$$(3.8) \quad \partial_{\alpha\beta}^2 \Gamma_{ij}^l(g_{\xi}, 0) = \frac{1}{2} \partial_{\alpha\beta}^2 G_{ijl}(g_{\xi}, 0).$$

Moreover

$$(3.9) \quad D_g \Gamma_{ij}^l(g_{\xi}, 0)[h] = \frac{1}{2} G_{ijl}(h, 0),$$

$$(3.10) \quad \partial_{\alpha} D_g \Gamma_{ij}^l(g_{\xi}, 0)[h] = \frac{1}{2} \partial_{\alpha} G_{ijl}(h, 0) - \frac{1}{2} h_{lk}(0) \partial_{\alpha} G_{ijk}(g_{\xi}, 0),$$

and

$$(3.11) \quad \begin{aligned} \partial_{\alpha\beta}^2 D_g \Gamma_{ij}^l(g)[h] &= \frac{1}{2} \partial_{\alpha\beta}^2 g^{lk}(0) G_{ijk}(h, 0) + \frac{1}{2} \partial_{\alpha\beta}^2 G_{ijl}(h, 0) \\ &\quad - \frac{1}{2} \sum_k \partial_{\alpha} h_{lk}(0) \partial_{\beta} G_{ijk}(g_{\xi}, 0) - \frac{1}{2} \sum_k \partial_{\beta} h_{lk}(0) \partial_{\alpha} G_{ijk}(g_{\xi}, 0) \\ &\quad - \frac{1}{2} \sum_k h_{lk}(0) \partial_{\alpha\beta}^2 G_{ijk}(g_{\xi}, 0). \end{aligned}$$

Lemma 3.4. *The map $(h, x) \rightarrow F'_h(\tilde{h}, \tilde{x})[h] + F'_x(\tilde{h}, \tilde{x})x$ is onto on $\mathbb{R}^N$ for any $(\tilde{h}, \tilde{x})$ such that $F(\tilde{h}, \tilde{x}) = 0$.*

Proof. Let $\hat{g} + h$ with $h \in \mathcal{B}_{\rho} \subset \mathcal{S}^k$ with $k \geq 4$. By (3.5) of Lemma 3.2 we have that the function $F(h, x) = \nabla_x \tilde{S}_{\hat{g}+h}(x)$ defined in (2.2) is of class C^2 . Let $(\tilde{h}, \tilde{x})$ be such that $F(\tilde{h}, \tilde{x}) = 0$. By Lemma 3.2 we get

$$(3.12) \quad F'_h(\tilde{h}, \tilde{x})[h] = \left(D_h \partial_1 \tilde{S}_{\hat{g}+\tilde{h}}(\tilde{x})[h], \dots, D_h \partial_N \tilde{S}_{\hat{g}+\tilde{h}}(\tilde{x})[h] \right).$$

We will prove that the map $F'_h(\tilde{h}, \tilde{x}) : \mathcal{S}^k \rightarrow \mathbb{R}^N$ is onto. More precisely, we are going to show that given $e_1, \dots, e_N$ the canonical base in $\mathbb{R}^N$, for any $i = 1, \dots, N$

$$(3.13) \quad \text{there exists } h \in \mathcal{S}^k \text{ such that } F'_h(\tilde{h}, \tilde{x})[h] = e_i.$$

The claim will follow immediately.

We point out that the ontoness of the map $h \rightarrow F'_h(\tilde{h}, \tilde{x})[h]$ is invariant with respect to a change of variable $x = \psi(z)$, where ψ is a diffeomorphism. Therefore, to show (3.13), we calculate $D_h \partial_{\alpha} \tilde{S}_{\hat{g}+\tilde{h}}(\tilde{x})[h]$ by choosing the normal coordinates on the Riemannian manifold $(M, \hat{g} + \tilde{h})$ given by the exponential map $\exp_{\tilde{\xi}}(z)$, where $\tilde{\xi}$ corresponds to $\tilde{x}$.

We choose $h \in \mathcal{S}^k$ such that the map $z \rightarrow h_{ij}(\exp_\xi(z))$, with its first and second derivatives, is vanishing at the point 0, for any indexes i and j . By Lemma 3.3, taking into account the definition of G_{ijk} given in (3.3), we deduce that

$$(3.14) \quad \partial_\alpha \Gamma_{ij}^l(g_\xi, 0) = D_g \Gamma_{ij}^l(g_\xi, 0)[h] = \partial_\alpha D_g \Gamma_{ij}^l(g_\xi, 0)[h] = 0$$

and

$$(3.15) \quad \partial_{\alpha\beta}^2 \Gamma_{ij}^l(g_\xi, 0)[h] = \frac{1}{2} \partial_{\alpha\beta}^2 G_{ijl}(g_\xi, 0), \quad \partial_{\alpha\beta}^2 D_g \Gamma_{ij}^l(g_\xi, 0)[h] = \frac{1}{2} \partial_{\alpha\beta}^2 G_{ijl}(h, 0).$$

Then by (3.6), (3.14) and (3.15) we deduce

$$(3.16) \quad \begin{aligned} \partial_\alpha D_g \tilde{S}_g(x)[h] &= \partial_{\alpha c}^2 D_g \Gamma_{aa}^c[h] - \partial_{\alpha a}^2 D_g \Gamma_{ac}^c[h] \\ &= \frac{1}{2} \sum_{a,c} \{ \partial_{\alpha c}^2 G_{aac}(g_\xi, 0)[h] - \partial_{\alpha a}^2 G_{acc}(g_\xi, 0)[h] \} \\ &= \sum_{a,c} \{ \partial_{\alpha ac}^3 h_{ac}(0) - \partial_{\alpha aa}^3 h_{cc}(0) \}. \end{aligned}$$

Let us prove (3.13) when $i = 1$. Let us choose $h \in \mathcal{S}^k$ such that

$$h_{11}(z) = -\frac{1}{2} z_1 z_2^2 \quad \text{and} \quad h_{ij}(z) = 0 \quad \text{for all the other indexes.}$$

An easy computation shows that

$$\sum_{a,c} \{ \partial_{1ac}^3 h_{ac}(0) - \partial_{1aa}^3 h_{cc}(0) \} = -\partial_{122}^3 h_{11}(0) = 1$$

and

$$\sum_{a,c} \{ \partial_{\alpha ac}^3 h_{ac}(0) - \partial_{\alpha aa}^3 h_{cc}(0) \} = 0 \quad \text{if} \quad \alpha \neq 1.$$

Finally, by (3.12) and (3.16) we deduce that $F'_h(\tilde{h}, \tilde{x})[h] = e_1$. We argue in a similar way to prove (3.13) when $i = 2, \dots, N$. This concludes the proof. $\square$

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