STABLE HYPERSURFACES WITH CONSTANT SCALAR CURVATURE

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ABSTRACT. We obtain some nonexistence results for complete noncompact stable hypersurfaces with nonnegative constant scalar curvature in Euclidean spaces. As a special case we prove that there is no complete noncompact strongly stable hypersurface M in $\mathbb{R}^4$ with zero scalar curvature S_2 , nonzero Gauss-Kronecker curvature and finite total curvature (i.e. $\int_M |A|^3 < +\infty$).

1. INTRODUCTION

In this paper we study the complete noncompact stable hypersurfaces with constant scalar curvature in Euclidean spaces. It has been proved by Cheng and Yau [CY] that any complete noncompact hypersurfaces in the Euclidean space with constant scalar curvature and nonnegative sectional curvature must be a generalized cylinder. Note that the assumption of nonnegative sectional curvature is a strong condition for hypersurfaces in the Euclidean space with zero scalar curvature since the hypersurface has to be flat in this case. Let M^n be a complete orientable Riemannian manifold and let $x : M^n \rightarrow \mathbb{R}^{n+1}$ be an isometric immersion into the Euclidean space $\mathbb{R}^{n+1}$ with constant scalar curvature. We can choose a global unit normal vector field N , and the Riemannian connections ∇ and $\tilde{\nabla}$ of M and $\mathbb{R}^{n+1}$, respectively, are related by

$$\tilde{\nabla}_X Y = \nabla_X Y + \langle A(X), Y \rangle N,$$

where A is the second fundamental form of the immersion, defined by

$$A(X) = -\tilde{\nabla}_X N.$$

Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of A . The r -mean curvature of the immersion in a point p is defined by

$$H_r = \frac{1}{\binom{n}{r}} \sum_{i_1 < \dots < i_r} \lambda_{i_1} \dots \lambda_{i_r} = \frac{1}{\binom{n}{r}} S_r,$$

where S_r is the r -symmetric function of the $\lambda_1, \dots, \lambda_n$, $H_0 = 1$ and $H_r = 0$, for all $r \geq n + 1$. For $r = 1$, $H_1 = H$ is the mean curvature of the immersion, in the case $r = 2$, H_2 is the normalized scalar curvature, and for $r = n$, H_n is the Gauss-Kronecker curvature.

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It is well-known that hypersurfaces with constant scalar curvature are critical points for a geometric variational problem, namely, that associated to the functional

$$(1) \quad \mathcal{A}_1(M) = \int_M S_1 dM,$$

under compactly supported variations that preserve volume. Let M be a hypersurface in the Euclidean space with constant scalar curvature. Following [AdCE], when the scalar curvature is zero, we say that a regular domain $D \subset M$ is *stable* if the critical point is such that $(\frac{d^2 \mathcal{A}_1}{dt^2})_{t=0} \geq 0$ for all variations with compact support in D , and when the scalar curvature is nonzero, we say that a regular domain $D \subset M$ is *strongly stable* if the critical point is such that $(\frac{d^2 \mathcal{A}_1}{dt^2})_{t=0} \geq 0$, for all variations with compact support in D . It is natural to study the global properties of hypersurfaces in the Euclidean space with constant scalar curvature. For example, we have the following open question (see 4.3 in [AdCE]).

Question 1.1. Is there any stable complete hypersurface M in $\mathbb{R}^4$ with zero scalar curvature and nonzero Gauss-Kronecker curvature?

We have a partial answer to Question 1.1.

Theorem A (see Theorem 3.1). *There is no complete noncompact stable hypersurface M in $\mathbb{R}^{n+1}$ with zero scalar curvature S_2 and 3-mean curvature $S_3 \neq 0$ satisfying*

$$(2) \quad \lim_{R \rightarrow +\infty} \frac{\int_{B_R} S_1^3}{R^2} = 0,$$

where B_R is the geodesic ball in M .

When $S_2 = 0$, $S_1^2 = |A|^2$ we have

Corollary B. *There is no complete noncompact stable hypersurface M in $\mathbb{R}^4$ with zero scalar curvature S_2 , nonzero Gauss-Kronecker curvature and finite total curvature (i.e. $\int_M |A|^3 < +\infty$).*

We remark that Shen and Zhu (see [SZ]) proved that a complete stable minimal n -dimensional hypersurface in $\mathbb{R}^{n+1}$ with finite total curvature must be a hyperplane. The above corollary can be seen as a similar result in dimension 3 for hypersurfaces with zero scalar curvature.

We also prove the following result for hypersurfaces with positive constant scalar curvature in Euclidean space.

Theorem C (see Theorem 3.2). *There is no complete immersed strongly stable hypersurface $M^n \rightarrow \mathbb{R}^{n+1}$, $n \geq 3$, with positive constant scalar curvature and polynomial growth of 1-volume; that is*

$$\lim_{R \rightarrow \infty} \frac{\int_{B_R} S_1 dM}{R^n} < \infty,$$

where B_R is a geodesic ball of radius R of M^n .

As a consequence of the properties of a graph with constant scalar curvature, we have the following corollary:

Corollary D (see Corollary 4.1). *Any entire graph on $\mathbb{R}^n$ with nonnegative constant scalar curvature must have zero scalar curvature.*

This can be compared with a result of Chern [Ch], which says any entire graph on $\mathbb{R}^n$ with constant mean curvature must be minimal. It has been known by a result of X. Cheng in [Che] (see also [ENR]) that any complete noncompact stable hypersurface in $\mathbb{R}^{n+1}$ with constant mean curvature must be minimal if $n < 5$. It is natural to ask that any complete noncompact stable hypersurface in $\mathbb{R}^{n+1}$ with nonnegative constant scalar curvature must have zero scalar curvature.

It should be remarked that Chern [Ch] proved that there is no entire graph on $\mathbb{R}^n$ with Ricci curvature less than a negative constant. We don't know whether there exists an entire graph on $\mathbb{R}^n$ with constant negative scalar curvature.

The rest of this paper is organized as follows: we include some results and definitions which will be used in the proof of our theorems in Section 2. The proof of main results are given in Section 3, and Section 4 is an appendix in which we prove some stability properties for graphs with constant scalar curvature in the Euclidean space.

2. SOME STABILITY AND INDEX PROPERTIES FOR HYPERSURFACES WITH $S_2 = \text{const.}$

We introduce the r 'th Newton transformation, $P_r : T_p M \rightarrow T_p M$, which is defined inductively by

$$\begin{aligned} P_0 &= I, \\ P_r &= S_r I - A \circ P_{r-1}, \quad r \geq 1. \end{aligned}$$

The following formulas are useful in the proof (see [Re], Lemma 2.1):

$$\begin{aligned} (3) \quad \text{trace}(P_r) &= (n-r)S_r, \\ (4) \quad \text{trace}(A \circ P_r) &= (r+1)S_{r+1}, \\ (5) \quad \text{trace}(A^2 \circ P_r) &= S_1 S_{r+1} - (r+2)S_{r+2}. \end{aligned}$$

From [AdCC] we have the second variation formula for hypersurfaces in a space form of constant curvature c , $\mathbb{Q}_c^{n+1}$, with constant 2-mean curvature:

$$(6) \quad \frac{d^2 \mathcal{A}_1}{dt^2} \Big|_{t=0} = \int_D \langle P_1(\nabla f), \nabla f \rangle dM - \int_D (S_1 S_2 - 3S_3 + c(n-1)S_1) f^2 dM, \quad \forall f \in C_c^\infty(D).$$

Definition 2.1. When $S_2 = 0$ and $c = 0$, M is stable if and only if

$$(7) \quad \int_M \langle P_1(\nabla f), \nabla f \rangle dM \geq -3 \int_M S_3 f^2 dM,$$

for any $f \in C_c^\infty(M)$. One can see that if $P_1 \equiv 0$, then $S_3 = 0$ and M is stable. When $S_2 = \text{const.} \neq 0$, M is stable if and only if

$$\int_D \langle P_1(\nabla f), \nabla f \rangle dM \geq \int_D (S_1 S_2 - 3S_3 + c(n-1)S_1) f^2 dM,$$

for all $f \in C_c^\infty(M)$ and $\int_M f dM = 0$. We say that M is strongly stable if and only if the above inequality holds for all $f \in C_c^\infty(M)$.

Similar to minimal hypersurface we can also define the index I for hypersurfaces with constant scalar curvature. Given a relatively compact domain $\Omega \subset M$, we denote by $\text{Ind}^1(\Omega)$ the number of linearly independent normal deformations with support on Ω that decrease $\mathcal{A}_1$. The *index* of the immersion is defined as

$$(8) \quad \text{Ind}^1(M) := \sup\{\text{Ind}^1(\Omega) \mid \Omega \subset M, \Omega \text{ relatively compact}\}.$$

M is *strongly stable* if $\text{Ind}^1(M) = 0$. The following result has been shown in [El].

Lemma 2.1. *Let $M^n \rightarrow \mathbb{Q}_c^{n+1}$ be a noncompact hypersurface with $S_2 = \text{const.} > 0$. If M has finite index, then there exists a compact set $K \subset M$ such that $M \setminus K$ is strongly stable.*

For hypersurfaces with constant mean curvature, do Carmo and Zhou [dCZ] proved that

Theorem 2.1. *Let $x : M^n \rightarrow \overline{M}^{n+1}$ be an isometric immersion with constant mean curvature H . Assume M has subexponential volume growth and finite index. Then there exists a constant R_0 such that*

$$H \leq -\overline{\text{Ric}}_{M \setminus B_{R_0}}(N),$$

where N is a smooth normal vector field along M and $\overline{\text{Ric}}(N)$ is the Ricci curvature of $\overline{M}$ in the normal vector N .

The technique in [dCZ] was generalized by Elbert [El] to prove the following result:

Theorem 2.2. *Let $x : M^n \rightarrow \mathcal{Q}(c)^{n+1}$ be an isometric immersion with $S_2 = \text{const.} > 0$. Assume that $\text{Ind}^1 M < \infty$ and that the 1-volume of M is infinite and has polynomial growth. Then c is negative and*

$$S_2 \leq -c.$$

In particular, this theorem implies that when $c = 0$ the hypersurfaces in the above theorem must have nonpositive scalar curvature.

3. PROOF OF THE THEOREMS

When $S_2 = 0$ we know that $|S_1|^2 = |A|^2$. Thus, if $S_3 \neq 0$, we have that $|A|^2 > 0$. Hence $S_1 \neq 0$ and we can choose an orientation such that P_1 is semi-positive definite. Since

$$\begin{aligned} |\sqrt{P_1}A|^2 &= \text{trace}(A^2 \circ P_1) \\ &= -3S_3, \end{aligned}$$

then, when $c = 0$, M is stable if

$$(9) \quad \int_M \langle P_1(\nabla f), \nabla f \rangle dM \geq \int_M |\sqrt{P_1}A|^2 f^2 dM,$$

for any $f \in C_c^\infty(M)$.

When $S_2 = 0$, we have the following inequality, which is essentially due to Cheng and Yau [CY] (see also Lemma 4.1 in [AdCC] or Lemma 3.2 in [Li]):

$$(10) \quad |\nabla A|^2 - |\nabla S_1|^2 \geq 0.$$

In the following lemma, we characterize the equality case in some special case.

Lemma 3.1. *Let $M^n (n \geq 3)$ be a nonflat connected immersed 1-minimal hypersurface in $\mathbb{R}^{n+1}$. If $|\nabla A|^2 = |\nabla S_1|^2$ holds on all nonvanishing points of $|A|$ in M , then each component of M with $|A| \neq 0$ must be a cylinder over a curve.*

Proof. To prove our lemma we recall the computations in [SSY]. Choose a frame at p so that the second fundamental form is diagonalized. We have $|A|^2 = \sum_i h_{ii}^2$ and

$$\begin{aligned}
 & \sum_{i,j,k} h_{ijk}^2 - |\nabla|A||^2 \\
 &= [(\sum_{i,j} h_{ij}^2)(\sum_{s,t,k} h_{stk}^2) - \sum_k (\sum_{i,j} h_{ij} h_{ijk})^2] (\sum_{i,j} h_{ij}^2)^{-1} \\
 &= \frac{1}{2} \sum_{i,j,k,s,t} (h_{ij} h_{stk} - h_{st} h_{ijk})^2 |A|^{-2} \\
 &= \frac{1}{2} \left[\sum_{i,k,s,t} (h_{ii} h_{stk} - h_{st} h_{iik})^2 + \sum_s h_{ss}^2 (\sum_k \sum_{i \neq j} h_{ijk}^2) \right] |A|^{-2} \\
 (11) \quad &= \frac{1}{2} \left[\sum_{i,k,s} (h_{ii} h_{ssk} - h_{ss} h_{iik})^2 \right] |A|^{-2} + 2 \sum_{i \neq j} h_{ij}^2 + \sum_{i \neq j, j \neq k, i \neq k} h_{ijk}^2.
 \end{aligned}$$

The right hand side of the above equation is nonnegative and zero if and only if all terms on the right hand side of the last equation are zero. Suppose $x : M \rightarrow \mathbb{R}^{n+1}$ is the 1-minimal immersion. Since M is not a hyperplane, then $|A|$ is a nonnegative continuous function which does not vanish identically. Let p be such a point such that $|A|(p) > 0$. Then $|A| > 0$ in a connected open set U containing p . The equality in (10) implies

$$\begin{aligned}
 h_{jji} &= 0, \text{ for all } j \neq i, \\
 h_{ijk} &= 0, \text{ for all } j \neq i, j \neq k, k \neq i, \\
 h_{ii} h_{ssk} &= h_{ss} h_{iik}, \text{ for all } i, s, k.
 \end{aligned}$$

So we have $h_{jjj} = 0$, for all $j \neq i$, and from the last equation we claim at most one i such that $h_{iii} \neq 0$. Otherwise, without the loss of generality we assume $h_{111} \neq 0$ and $h_{222} \neq 0$; we have $h_{11} h_{22k} = h_{22} h_{11k}$ for all k . This implies $h_{11} = h_{22} = 0$ by choosing $k = 1, 2$. Using the third formula again we have $h_{jj} h_{111} = h_{11} h_{jj1}$ for $j = 3, \dots, n$. Hence $h_{jj} = 0$ for all $j = 3, \dots, n$, which contradicts $|A| \neq 0$.

We now assume $h_{111} \neq 0$; by continuity we can also assume $h_{11} \neq 0$. From the last line of the above equation, we have $h_{11} h_{ss1} = h_{ss} h_{111}$ for $s \neq 1$. Hence $h_{ss} = 0$ for all $s \neq 1$. This implies that M is a cylinder over a curve. $\square$

We are now ready to prove

Theorem 3.1. *There is no complete noncompact stable hypersurface in $\mathbb{R}^{n+1}$ with $S_2 = 0$ and $S_3 \neq 0$ satisfying*

$$\lim_{R \rightarrow +\infty} \frac{\int_{B_R} S_1^3}{R^2} = 0.$$

Proof. Assume for the sake of contradiction that there is such a hypersurface M . From Lemma 3.7 in [AdCC], we have

$$(12) \quad L_1 S_1 = |\nabla A|^2 - |\nabla S_1|^2 + 3 S_1 S_3.$$

Since for any $\phi \in C_c^\infty(M)$,

$$\begin{aligned} & \int_M \langle P_1(\nabla(\phi S_1)), \nabla(\phi S_1) \rangle dM \\ &= \int_M \langle P_1((\nabla\phi)S_1 + \phi\nabla S_1), (\nabla\phi)S_1 + \phi\nabla S_1 \rangle dM \\ &= \int_M \langle P_1(\nabla\phi), \nabla\phi \rangle S_1^2 dM + 2 \int_M \langle P_1(\nabla\phi), \nabla S_1 \rangle \phi S_1 dM \\ &+ \int_M \phi^2 \langle P_1(\nabla S_1), \nabla S_1 \rangle dM, \end{aligned}$$

using (12) we have

$$\begin{aligned} & \int_M \phi^2 S_1 (|\nabla A|^2 - |\nabla S_1|^2) dM = \int_M (L_1 S_1 - 3S_1 S_3) \phi^2 S_1 dM \\ &= - \int_M \langle P_1(\nabla S_1), \nabla(\phi^2 S_1) \rangle dM - \int_M 3S_3 \phi^2 S_1^2 dM \\ &= - \int_M \phi^2 \langle P_1(\nabla S_1), \nabla S_1 \rangle dM - 2 \int_M \langle P_1(\nabla\phi), \nabla S_1 \rangle \phi S_1 dM \\ &\quad - \int_M 3S_3 \phi^2 S_1^2 dM \\ &= - \int_M \langle P_1(\nabla(\phi S_1)), \nabla(\phi S_1) \rangle dM + \int_M \langle P_1(\nabla\phi), \nabla(\phi) \rangle S_1^2 dM \\ &\quad - \int_M 3S_3 \phi^2 S_1^2 dM \\ &\leq \int_M \langle P_1(\nabla\phi), \nabla\phi \rangle S_1^2 dM \\ &\leq (n-1) \int_M |\nabla\phi|^2 S_1^3 dM, \end{aligned}$$

for any $\phi \in C_c^\infty(M)$. Here we have used the stability inequality (7) in the seventh line and we use the following consequence of (3) in the last inequality:

$$(13) \quad (n-1)S_1 |\nabla\phi|^2 \geq \langle P_1(\nabla\phi), \nabla\phi \rangle.$$

We can choose ϕ as

$$\phi(x) = \begin{cases} \frac{2R-r(x)}{R}, & \text{on } B_{2R} \setminus B_R; \\ 1, & \text{on } B_R; \\ 0, & \text{on } M \setminus B_{2R}. \end{cases}$$

Thus from the choice of ϕ we have $S_1(|\nabla A|^2 - |\nabla S_1|^2) \equiv 0$. Therefore the ellipticity of L_1 implies $L_1 S_1 = 3S_1 S_3$. From Lemma 3.1, M must be a cylinder over a curve, which contradicts $S_3 \neq 0$. The proof is complete. $\square$

The following lemma is of some independent interest, and we include it here since its second part is useful in the proof of Theorem 3.2.

Lemma 3.2. *Let M be a complete immersed hypersurface in $\mathcal{Q}_c^{n+1}$ with positive constant scalar curvature $S_2 > -\frac{n(n-1)}{2}c$ and $S_1 \neq 0$.*

1) If M is strongly stable outside a compact subset, then either M has finite 1-volume or

$$\lim_{R \rightarrow +\infty} \frac{1}{R^2} \int_{B_R} S_1 = +\infty.$$

2) If M is strongly stable, then

$$\lim_{R \rightarrow +\infty} \frac{1}{R^2} \int_{B_R} S_1 = +\infty.$$

In particular M has infinite 1-volume.

Proof. We can assume that there exists a geodesic ball $B_{R_0} \subset M$ such that $M \setminus B_{R_0}$ is strongly stable. That is,

$$(14) \quad \int_M (S_1 S_2 - 3S_3 + c(n-1)S_1) f^2 dM \leq \int_M \langle P_1(\nabla f), \nabla f \rangle dM,$$

for all $f \in \mathcal{C}_c^\infty(M \setminus B_{R_0})$.

Now, since $S_2 > 0$, we have (see [AdCR], p. 392)

$$H_1 H_2 \geq H_3$$

and

$$H_1 \geq H_2^{1/2}.$$

By using $S_1 = nH_1$, $S_2 = \frac{n(n-1)}{2}H_2$ and $S_3 = \frac{n(n-1)(n-2)}{6}H_3$, we obtain that

$$\frac{(n-2)}{n} S_1 S_2 \geq 3S_3;$$

that is,

$$(15) \quad -3S_3 \geq -\frac{(n-2)}{n} S_1 S_2.$$

We also have that

$$\frac{S_1}{n} \geq \left(\frac{2S_2}{n(n-1)} \right)^{1/2},$$

which implies

$$(16) \quad S_1 \geq \left(\frac{2n}{n-1} \right)^{1/2} S_2^{1/2}.$$

By using inequality (15) in (14), we obtain that

$$\int_M \left(S_1 S_2 - \frac{n-2}{n} S_1 S_2 + c(n-1)S_1 \right) f^2 dM \leq \int_M \langle P_1(\nabla f), \nabla f \rangle dM;$$

that is,

$$\int_M \left(S_2 + \frac{n(n-1)c}{2} \right) S_1 f^2 dM \leq \frac{n}{2} \int_M \langle P_1(\nabla f), \nabla f \rangle dM.$$

By using (13), we obtain that

$$(n-1) \int_M S_1 |\nabla f|^2 dM \geq \int_M \langle P_1(\nabla f), \nabla f \rangle dM.$$

Therefore, there exists a constant $C > 0$ such that

$$(17) \quad \int_M S_1 |\nabla f|^2 dM \geq C \int_M S_1 f^2 dM.$$

1) When M is strongly stable outside B_{R_0} we can choose f as

$$f(x) = \begin{cases} r(x) - R_0, & \text{on } B_{R_0+1} \setminus B_{R_0}; \\ 1, & \text{on } B_{R+R_0+1} \setminus B_{R_0+1}; \\ \frac{2R+R_0+1-r(x)}{R}, & \text{on } B_{2R+R_0+1} \setminus B_{R+R_0+1}; \\ 0, & \text{on } M \setminus B_{2R+R_0+1}, \end{cases}$$

where $r(x)$ is the distance function to a fixed point. Then

$$\frac{1}{R^2} \int_{B_{2R+R_0+1} \setminus B_{R+R_0+1}} S_1 dM + \int_{B_{R_0+1} \setminus B_{R_0}} S_1 dM \geq C \int_{B_{R+R_0+1} \setminus B_{R_0+1}} S_1 dM.$$

If the 1-volume is infinite, we can choose R large enough such that

$$C \int_{B_{R+R_0+1} \setminus B_{R_0+1}} S_1 dM > \int_{B_{R_0+1} \setminus B_{R_0}} S_1 dM;$$

hence

$$\lim_{R \rightarrow +\infty} \frac{1}{R^2} \int_{B_{2R+R_0+1} \setminus B_{R+R_0+1}} S_1 dM = +\infty.$$

2) When M is strongly stable we can choose a simpler test function f as

$$f(x) = \begin{cases} 1, & \text{on } B_R; \\ \frac{2R-r(x)}{R}, & \text{on } B_{2R} \setminus B_R; \\ 0, & \text{on } M \setminus B_{2R}, \end{cases}$$

which implies that when $S_1 \neq 0$,

$$\lim_{R \rightarrow +\infty} \frac{1}{R^2} \int_{B_{2R}} S_1 dM = +\infty.$$

The proof is complete. $\square$

Theorem 3.2. *There is no complete immersed strongly stable hypersurface $M^n \rightarrow \mathbb{R}^{n+1}$, $n \geq 3$, with positive constant scalar curvature and polynomial growth of 1-volume; that is,*

$$\lim_{R \rightarrow \infty} \frac{\int_{B_R} S_1 dM}{R^n} < \infty,$$

where B_R is a geodesic ball of radius R of M^n .

Proof. Suppose that M is a completely immersed strongly stable hypersurface $M^n \rightarrow \mathbb{R}^{n+1}$, $n \geq 3$, with positive constant scalar curvature. From Theorem 2.2 it suffices to show that the 1-volume $\int_M S_1 dM$ is infinite, which is part (2) of Lemma 3.2. $\square$

4. GRAPHS WITH $S_2 = \text{const.}$ IN EUCLIDEAN SPACE

In this section we include some stability properties and estimates for entire graphs on $\mathbb{R}^n$ which may be known to experts but may not be easy to find a reference for. Using these facts we give the proof of Corollary 4.1. Let M^n be a hypersurface of $\mathbb{R}^{n+1}$ given by a graph of a function $u : \mathbb{R}^n \rightarrow \mathbb{R}$ of class $\mathcal{C}^\infty(\mathbb{R}^n)$. For such hypersurfaces we have:

Proposition 4.1. *Let M^n be a graph of a function $u : \mathbb{R}^n \rightarrow \mathbb{R}$ of class $\mathcal{C}^\infty(\mathbb{R}^n)$. Then*

- (1) *If $S_2 = 0$ and S_1 does not change sign on M , then M^n is a stable hypersurface.*
- (2) *If M has $S_2 = C > 0$, then M^n is strongly stable.*

Proof. Consider $f : M \rightarrow \mathbb{R}$ a $\mathcal{C}^\infty$ function with compact support and let $W = \sqrt{1 + |\nabla u|^2}$. In order to calculate $\langle P_1(\nabla f), \nabla f \rangle$, write $g = fW$. Thus

$$\begin{aligned} \langle P_1(\nabla f), \nabla f \rangle &= \langle P_1(\nabla(\frac{g}{W})), \nabla(\frac{g}{W}) \rangle \\ &= \langle P_1(g\nabla\frac{1}{W} + \nabla g\frac{1}{W}), g\nabla(\frac{1}{W}) + \frac{1}{W}\nabla g \rangle \\ &= \langle gP_1(\nabla\frac{1}{W}) + \frac{1}{W}P_1(\nabla g), g\nabla(\frac{1}{W}) + \frac{1}{W}\nabla g \rangle \\ &= g^2\langle P_1(\nabla\frac{1}{W}), \nabla\frac{1}{W} \rangle + \frac{g}{W}\langle P_1(\nabla\frac{1}{W}), \nabla g \rangle \\ &\quad + \frac{g}{W}\langle P_1(\nabla g), \nabla\frac{1}{W} \rangle + \frac{1}{W^2}\langle P_1(\nabla g), \nabla g \rangle. \end{aligned}$$

By using the fact that P_1 is selfadjoint, we have

(18)

$$\langle P_1(\nabla f), \nabla f \rangle = g^2\langle P_1(\nabla\frac{1}{W}), \nabla\frac{1}{W} \rangle + 2\frac{g}{W}\langle P_1(\nabla\frac{1}{W}), \nabla g \rangle + \frac{1}{W^2}\langle P_1(\nabla g), \nabla g \rangle.$$

On the other hand, if $\{e_1, \dots, e_n\}$ is a geodesic frame along M ,

$$\begin{aligned} \operatorname{div}(fgP_1(\nabla\frac{1}{W})) &= \sum_{i=1}^n \langle \nabla_{e_i}(fgP_1(\nabla\frac{1}{W})), e_i \rangle \\ &= \sum_{i=1}^n \langle fg_iP_1(\nabla\frac{1}{W}) + f_i gP_1(\nabla\frac{1}{W}) + fg\nabla_{e_i}(P_1(\nabla\frac{1}{W})), e_i \rangle \\ &= \sum_{i=1}^n \{fg_i\langle P_1(\nabla\frac{1}{W}), e_i \rangle + f_i g\langle P_1(\nabla\frac{1}{W}), e_i \rangle + fg\langle \nabla_{e_i}(P_1(\nabla\frac{1}{W})), e_i \rangle\}. \end{aligned}$$

Since $f = \frac{g}{W}$, we get

$$f_i = g_i \frac{1}{W} + g \left(\frac{1}{W} \right)_i ;$$

that is,

$$\begin{aligned} gf_i &= gg_i \frac{1}{W} + g^2 \left(\frac{1}{W} \right)_i \\ &= fg_i + g^2 \left(\frac{1}{W} \right)_i. \end{aligned}$$

Hence,

$$\begin{aligned}
& \operatorname{div}(fgP_1(\nabla \frac{1}{W})) \\
&= \sum_{i=1}^n \{fg_i \langle P_1(\nabla \frac{1}{W}), e_i \rangle + (fg_i + g^2 \left(\frac{1}{W}\right)_i) \langle P_1(\nabla \frac{1}{W}), e_i \rangle\} + fgL_1(\frac{1}{W}) \\
&= \sum_{i=1}^n \{2fg_i \langle P_1(\nabla \frac{1}{W}), e_i \rangle + g^2 \left(\frac{1}{W}\right)_i \langle P_1(\nabla \frac{1}{W}), e_i \rangle\} + fgL_1(\frac{1}{W}) \\
&= 2f \langle P_1(\nabla \frac{1}{W}), \nabla g \rangle + g^2 \langle P_1(\nabla \frac{1}{W}), \nabla(\frac{1}{W}) \rangle + fgL_1(\frac{1}{W}) \\
&= 2\frac{g}{W} \langle \nabla \frac{1}{W}, P_1(\nabla g) \rangle + g^2 \langle P_1(\nabla \frac{1}{W}), \nabla(\frac{1}{W}) \rangle + f^2WL_1(\frac{1}{W}).
\end{aligned}$$

Thus,

$$(19) \quad 2\frac{g}{W} \langle \nabla \frac{1}{W}, P_1(\nabla g) \rangle = \operatorname{div}(fgP_1(\nabla \frac{1}{W})) - g^2 \langle P_1(\nabla \frac{1}{W}), \nabla(\frac{1}{W}) \rangle - f^2WL_1(\frac{1}{W}).$$

Now, by using (19) in equation (18), we get

$$\langle P_1(\nabla f), \nabla f \rangle = \operatorname{div}(fgP_1(\nabla \frac{1}{W})) - f^2WL_1(\frac{1}{W}) + \frac{1}{W^2} \langle P_1(\nabla g), \nabla g \rangle.$$

Now, the divergence theorem implies that

$$\int_M \langle P_1(\nabla f), \nabla f \rangle dM = - \int_M f^2WL_1(\frac{1}{W}) dM + \int_M \frac{1}{W^2} \langle P_1(\nabla g), \nabla g \rangle dM.$$

Choose the orientation of M in such way that $S_1 \geq 0$. Since $S_1^2 - |A|^2 = 2S_2 \geq 0$, we obtain that $S_1 \geq |A|$. Thus, $\langle P_1(\nabla g), \nabla g \rangle = S_1|\nabla g|^2 - \langle A\nabla g, \nabla g \rangle \geq (S_1 - |A|)|\nabla g|^2 \geq 0$, which implies that

$$(20) \quad \int_M \langle P_1(\nabla f), \nabla f \rangle dM \geq - \int_M f^2WL_1(\frac{1}{W}) dM.$$

When S_2 is constant, we will use the following formula proved by Reilly (see [Re], Proposition C):

$$L_1(\frac{1}{W}) = L_1(\langle N, e_{n+1} \rangle) = -(S_1S_2 - 3S_3) \langle N, e_{n+1} \rangle = -(S_1S_2 - 3S_3) \frac{1}{W},$$

where N is the normal vector of M and $e_{n+1} = (0, \dots, 0, \pm 1)$, according to our choice of the orientation of M .

Thus,

$$\int_M \langle P_1(\nabla f), \nabla f \rangle dM \geq \int_M (S_1S_2 - 3S_3) f^2 dM$$

for all functions f with compact support. Hence M is stable if $S_2 = 0$ and strongly stable in the case $S_2 \neq 0$. $\square$

Remark 4.1. We would like to remark that the operator L_1 need not be elliptic in the above proof.

Proposition 4.2. *Let M^n be a graph of a function $u : \mathbb{R}^n \rightarrow \mathbb{R}$ of class $C^\infty(\mathbb{R}^n)$, with $S_1 \geq 0$. Let B_R be a geodesic ball of radius R in M . Then*

$$\int_{B_{\theta R}} S_1 dM \leq \frac{C(n)}{1-\theta} R^n,$$

where $C(n)$ and θ are constants, with $0 < \theta < 1$. In particular, $\int_M S_1 dM$ has polynomial growth.

Proof. Let $f : M \rightarrow \mathbb{R}$ be a function in $\mathcal{C}_0^\infty(M)$, that is, a smooth function with compact support. Observe that

$$\operatorname{div} \left(f \frac{\nabla u}{W} \right) = f \operatorname{div} \left(\frac{\nabla u}{W} \right) + \left\langle \nabla f, \frac{\nabla u}{W} \right\rangle,$$

where $W = \sqrt{1 + |\nabla u|^2}$. By using the fact that S_1 is given by $S_1 = \operatorname{div} \left(\frac{\nabla u}{W} \right)$, we have that

$$(21) \quad \int_M f S_1 dM = \int_M f \operatorname{div} \left(\frac{\nabla u}{W} \right) dM = - \int_M \left\langle \nabla f, \frac{\nabla u}{W} \right\rangle dM.$$

Now, choose a family of geodesic balls B_R that exhausts M . Fix θ , with $0 < \theta < 1$, and let $f : M \rightarrow \mathbb{R}$ be a continuous function that is one on $B_{\theta R}$, zero outside B_R and linear on $B_R \setminus B_{\theta R}$. Therefore, from equation (21) we obtain

$$\int_{B_{\theta R}} S_1 dM \leq \int_{B_R} f S_1 dM \leq \int_{B_R} \left| \left\langle \frac{\nabla u}{W}, \nabla f \right\rangle \right| dM.$$

By using Cauchy-Schwarz inequality and the fact that $\frac{|\nabla u|}{W} \leq 1$, we get that

$$\int_{B_{\theta R}} S_1 dM \leq \int_{B_R} |\nabla f| dM \leq \int_{B_R \setminus B_{\theta R}} \frac{1}{(1-\theta)R} dM \leq \frac{1}{(1-\theta)R} \operatorname{vol}(B_R).$$

We observe that since M is a graph, if $\Omega_R = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid -R \leq x_{n+1} \leq R; \sqrt{x_1^2 + \dots + x_n^2} \leq R\}$, then

$$\operatorname{vol}(B_R) \leq \int_{\Omega_R} 1 dx_1 \dots dx_{n+1} = C(n) R^{n+1}.$$

Hence,

$$\int_{B_{\theta R}} S_1 dM \leq \frac{1}{(1-\theta)R} \operatorname{vol}(B_R) = \frac{C(n)}{1-\theta} R^n. \quad \square$$

We have the following corollary of Theorem 3.2.

Corollary 4.1. *Any entire graph on $\mathbb{R}^n$ with nonnegative constant scalar curvature must have zero scalar curvature.*

Proof. Suppose for sake of contradiction that there exists an entire graph with $S_2 = \text{const.} > 0$. Such a graph is strongly stable; and if $S_2 > 0$, we get that $S_1^2 = |A|^2 + 2S_2 > 0$, we obtain that S_1 does not change sign, and we can choose the orientation in such a way that $S_1 > 0$. Thus the graph has polynomial growth of the 1-volume. Thus we have a contradiction with Theorem 3.2. Therefore it follows that $S_2 = 0$. $\square$

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