AFFINE VARIETIES WITH STABLY TRIVIAL ALGEBRAIC VECTOR BUNDLES

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ABSTRACT. Let k be an algebraically closed field. For every affine variety X with $\dim X \geq 7$ we construct a smooth affine variety Y which is birationally equivalent to X and which possesses a stably trivial but not trivial algebraic vector bundle. We give some application of this fact to the cancellation problem.

1. INTRODUCTION

Let k be an algebraically closed field and let X be a smooth affine variety over k . If $\dim X \leq 2$, then there is no non-trivial and stably trivial algebraic vector bundle on it. Moreover, there are only few examples of such vector bundles in the general case (see [13], [14], [12]). Here we deal with the question of whether such algebraic vector bundles can exist on a sufficiently general affine variety. We show that for every affine variety X with $\dim X \geq 7$, there exists a smooth affine variety Y which is birationally equivalent to X and which possesses a stably trivial but not trivial algebraic vector bundle.

We apply this result to the cancellation problem. Let us recall that an affine variety X has the cancellation property (CP) if for every affine variety Y , if $X \times k \cong Y \times k$, then $X \cong Y$ (see e.g. [1], [3], [4], [5], [6], [11]).

Let X be an affine variety over k with $\dim X \geq 7$. We show that there exists a smooth affine variety X' , which is birationally equivalent to X , such that the variety $X' \times k^2$ does not have CP.

2. PRELIMINARIES

Let X be an affine variety (which is assumed to be irreducible) over k and let $R = k[X]$ be the ring of regular functions on X (by $k(X)$ we denote the field of rational functions on X). Let us recall some basic facts about algebraic vector bundles over X , which we identify with finitely generated projective R -modules. We say that an algebraic vector bundle $\mathbf{F}$ is *stably trivial* (of type t) if

$$\mathbf{F} \oplus \mathbf{E}_t = \mathbf{E}_s$$

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for some trivial vector bundles $\mathbf{E}_t$ and $\mathbf{E}_s$. Recall that a vector $(f_1, \dots, f_r) \in R^r$ is called a *unimodular row* if there exists a vector $g = (g_1, \dots, g_r) \in R^r$ such that $\sum_{i=1}^r f_i g_i = 1$. This is equivalent to the fact that $f_1, \dots, f_r$ have no common zeros on X . A unimodular row $f = (f_1, \dots, f_r) \in R^r$ determines uniquely a vector bundle $\mathbf{F}(f) = R^r/Rg \cong (R^r/Rf)^*$. Of course $\mathbf{F}(f) \oplus \mathbf{E}_1 = \mathbf{E}_r$.

3. CONSTRUCTION

We start with the following nice classical example of Raynaud (see [13] for $k = \mathbb{C}$ and [14] for an arbitrary field k):

Example 3.1. Let $n \geq 3$ and

$$R = \frac{k[x_1, \dots, x_n, y_1, \dots, y_n]}{(\sum_{i=1}^n x_i y_i - 1)}.$$

Then the stably free submodule of R^n given by the unimodular row $(x_1, \dots, x_n)$ is not free.

In the sequel the following result will be useful (for the proof see [8], Proposition 21):

Theorem 3.2. *Let X be an affine variety of dimension n and let $Z \subset X$ be a closed subvariety. Take $m \geq n$. If the mapping $f : Z \rightarrow k^m$ is finite, then there exists a finite mapping $F : X \rightarrow k^m$ such that*

$$F|_Z = f.$$

We also need the following stronger version of a result from [7]:

Theorem 3.3. *Let X be a smooth affine variety and $Z \subset X$ a smooth closed subvariety. Assume that on Z there is a stably trivial algebraic vector bundle $\mathbf{F}$ of type t . Then there is an open affine neighborhood $U \subset X$ of Z and a stably trivial vector bundle $\mathbf{F}'$ (of the same type t) on U such that*

$$\mathbf{F}'|_Z = \mathbf{F}.$$

In particular if $\mathbf{F}$ is not trivial, then neither is $\mathbf{F}'$.

Proof. Let $\mathbf{F} \oplus \mathbf{E}_t = \mathbf{E}_n$. Hence $\mathbf{F}$ is the kernel of some epimorphism

$$\phi : \mathbf{E}_n \ni (v_1, \dots, v_n) \rightarrow \left(\sum_{i=1}^n f_{1i} v_i, \dots, \sum_{i=1}^n f_{ti} v_i \right) \in \mathbf{E}_t,$$

where $f_{ij} \in k[Z]$. Let $\mathbf{a}_i = (a_{i1}, a_{i2}, \dots, a_{in}) \in \mathbf{E}_n$ be sections such that $\phi(\mathbf{a}_i) = \mathbf{e}_i$, where $\mathbf{e}_1 = (1, 0, \dots, 0), \dots, \mathbf{e}_t = (0, 0, \dots, 1)$. Let $F_{ij} \in k[X]$ and $A_{ij} \in k[X]$ be extensions of the functions f_{ij} and a_{ij} to X . Take $\mathbf{A}_i = (A_{i1}, \dots, A_{in})$, $i = 1, \dots, t$. Consider the following morphism of vector bundles on X :

$$\Phi : \mathbf{E}_n \ni (v_1, \dots, v_n) \rightarrow \left(\sum_{i=1}^n F_{1i} v_i, \dots, \sum_{i=1}^n F_{ti} v_i \right) \in \mathbf{E}_t.$$

Put

$$U = \{x \in X : \text{rank } \langle \Phi(\mathbf{A}_1), \dots, \Phi(\mathbf{A}_t) \rangle = t\}.$$

Note that $Z \subset U$. Moreover, on U we have $\ker \Phi \oplus \langle A_1, \dots, A_t \rangle \cong \ker \phi \oplus \mathbf{E}_t = \mathbf{E}_n$, i.e., $\ker \Phi$ is a stably trivial vector bundle of type t on U . Since $\Phi|_U = \phi$ we have $\ker \Phi|_U = \ker \phi$. Put $\mathbf{F}' = \ker \Phi$.

Now it is enough to show that we can shrink U to get U' affine. Indeed let $W = X \setminus U$. Then W is closed and $W \cap Z = \emptyset$. If $n = \dim X$, then $\dim W, Z \leq n - 1$. In particular, there are finite mappings $f_1 : W \rightarrow k^{n-1} \times \{1\} \subset k^n$ and $f_0 : Z \rightarrow k^{n-1} \times \{0\} \subset k^n$. Since $W \cap Z = \emptyset$ we can glue these mappings to one map $f : Z \cup W \rightarrow k^n$. By Theorem 3.2 we can extend a finite mapping f to a finite mapping $F : X \rightarrow k^n$. Now take $U' = F^{-1}(k^n \setminus k^{n-1} \times \{1\})$. $\square$

Now we are ready to prove:

Theorem 3.4. *For every affine variety X with $\dim X = n \geq 7$ there exists a smooth affine variety Y , which is birationally equivalent to X and which possesses a stably trivial but not trivial algebraic vector bundle (of type 1 and rank 2).*

Proof. We can assume that X is smooth. Take a point $x \in X$. We can assume (after shrinking X if necessary) that there are functions $\{f_1, \dots, f_n\} \subset k[X]$, which form a system of local coordinates at x and additionally $V(f_1, \dots, f_n) = \{x\}$. Let us blow up the ideal $I = (f_1, \dots, f_n)$. We obtain a quasiprojective variety $X' \subset X \times \mathbb{P}^{n-1}$ birationally equivalent to X . Moreover, X' contains a subvariety $x \times \mathbb{P}^{n-1}$. Take $Y' = X' \cap (X \times k^{n-1})$. Then Y' is an affine variety birationally equivalent to X which contains a subvariety isomorphic to k^{n-1} . Since $n \geq 7$ we can embed into Y' a sphere $Z = \{(x, y) \in k^6 : \sum_{i=1}^3 x_i y_i = 1\}$. By Example 3.1 on Z there is a stably trivial but non-trivial algebraic vector bundle $\mathbf{F}$ (of type 1 and rank 2). By Theorem 3.3 we can extend this bundle to an open affine neighborhood $Y \subset Y'$ of Z . Of course Y is birationally equivalent to Y' , hence also birationally equivalent to X . $\square$

We also have a more direct application of the Raynaud example:

Proposition 3.5. *Let $X = \{(z, x, y) \in k^m \times k^n \times k^n : g(z_1, \dots, z_m) = \sum_{i=1}^n x_i y_i\}$, where g is a non-zero polynomial. Assume that $m > 0$ and $n > 2$. Then the algebraic vector bundle $\mathbf{F}$ with a smooth base $Y = X \setminus \{g = 0\}$ given by the unimodular row $(x_1, \dots, x_n)$ is stably trivial but non-trivial.*

Proof. It is easy to check that Y is a smooth variety. Now choose $z = z_0$ such that $g(z_0) \neq 0$. Then the bundle $\mathbf{F}$ restricted to the variety $X \cap (\{z_0\} \times k^n \times k^n)$ is the Raynaud vector bundle, hence it is non-trivial. Hence $\mathbf{F}$ is non-trivial, too. $\square$

Remark 3.6. Let us note that for $m = 1$ and $g = z(z - 1)$ we obtain the Mohan-Kumar-Nori example; see [12].

4. APPLICATION TO THE CANCELLATION PROBLEM

In this section we apply our result to the cancellation problem. Let us recall the definition of a k -uniruled variety. First recall that a *polynomial curve* in X is the image of the affine line $A^1(k)$ under a non-constant morphism $\phi : A^1(k) \rightarrow X$. Moreover, a family $\mathcal{F}$ of polynomial curves on X is called *bounded* if there exists an embedding $i : X \subset k^N$ and a natural number D such that every curve $\Gamma_a \in \mathcal{F}$ has degree less than or equal to D in k^N . It is easy to see that the definition of bounded family does not depend on an embedding $i : X \rightarrow k^N$.

Now we have the following:

Definition 4.1 (see [9]). An affine variety X is said to be *k -uniruled* if it is of dimension ≥ 1 and there is a bounded family $\mathcal{F}$ of polynomial curves such that for every point $x \in X$ there is a curve $l_x \in \mathcal{F}$ going through x .

The k -uniruled varieties can be characterized in a following way (see [9]):

Proposition 4.2. *Let $X \subset k^N$ be an irreducible affine variety of dimension ≥ 1 . The following conditions are equivalent:*

- 1) *there is a bounded family $\mathcal{F}$ of affine polynomial curves, such that for every point $x \in X$ there is a curve $l_x \in \mathcal{F}$ going through x ;*
- 2) *there is an open, non-empty subset $U \subset X$ and a bounded family $\mathcal{F}$ of affine polynomial curves such that for every point $x \in U$ there is a curve $l_x \in \mathcal{F}$ going through x ;*
- 3) *there is an affine variety W of dimension $\dim X - 1$ and a polynomial dominant mapping $\phi : W \times k \rightarrow X$.*

We have the following important examples of non- k -uniruled varieties (see [7]):

Proposition 4.3. *Let $h \in k[x_1, \dots, x_n]$ be a non-constant polynomial. The variety*

$$X(h) = \{x \in k^n : h(x) \neq 0, 1 + x_i(h(x) - 1) \neq 0 \text{ for } i = 1, \dots, n\}$$

is not k -uniruled.

Proof. Let $\phi : k \rightarrow X(h)$ be a regular mapping. Thus $\phi = (\phi_1(t), \dots, \phi_n(t))$, where ϕ_i are polynomials. Moreover, $h \circ \phi \neq 0$ for every t , which implies that $h \circ \phi$ is a non-zero constant. Similarly, $1 + \phi_i(t)(h \circ \phi(t) - 1)$ is a constant. Consequently, either $h \circ \phi(t) = 1$ or all ϕ_i are constant. This means that outside the hypersurface $\{x \in k^n : h(x) = 1\}$ there are no polynomial curves in $X(h)$. $\square$

In the sequel we need the following theorem, which was proved in [7]:

Theorem 4.4. *Let X be a smooth affine non- k -uniruled variety with a stably trivial (of type 1) but non-trivial algebraic vector bundle $\mathbf{F}$ on it. If $\text{rank } \mathbf{F} = r$, then the variety $X \times k^r$ fails CP.*

Remark 4.5. In [7] the definition of k -uniruledness is slightly different than in our paper; however it is easy to check that all proofs in [7] work in our situation as well.

Now we are in a position to prove:

Theorem 4.6. *Let X be an affine variety of with $\dim X \geq 7$. There exists a smooth affine variety X' which is birationally equivalent to X such that the variety $X' \times k^2$ does not have CP.*

Proof. By Theorem 3.4 there exists a smooth affine variety Y which is birationally equivalent to X and which possesses a stably trivial but non-trivial algebraic vector bundle $\mathbf{F}$ of type 1 and rank 2. Moreover, the bundle $\mathbf{F}$ is not trivial on some proper closed subset Z of dimension smaller than $n = \dim Y$.

We show that we can find a Zariski open affine neighborhood U of Z which is non- k -uniruled. Indeed, there is a finite mapping $f : Z \rightarrow k^{n-1} \times \{1\}$. By Theorem 3.2 we can extend this mapping to a finite map $F : Y \rightarrow k^n$.

Consider the variety

$$X(x_n) = \{x \in k^n : x_n \neq 0, 1 + x_i(x_n - 1) \neq 0 \text{ for } i = 1, \dots, n\}.$$

By Proposition 4.3 it is not k -uniruled. This also implies that the variety $X' = F^{-1}(X(x_n))$ [which is birationally equivalent to Y , hence also to X] is not k -uniruled. By construction it contains Z ; in particular the vector bundle $\mathbf{F}' = \mathbf{F}|_{X'}$ is not trivial on X' . Now by Theorem 4.4 the variety $X' \times k^2$ does not have CP. $\square$

Corollary 4.7. *Let K be a field which is finitely generated over k . Assume that $\text{trdeg}(K/k) = n \geq 7$. Then there exists a smooth affine variety W without CP, such that*

$$k(W) = K(x, y),$$

where x, y are elements algebraically independent over K .

Proof. Let $K = k(a_1, \dots, a_m)$. Take $X = \text{Spec}(k[a_1, \dots, a_m])$ and use Theorem 4.6. $\square$

Remark 4.8. It is worth noting that Theorem 4.6 is in some sense the best possible. Indeed, if we start with a variety X , which has a non-uniruled projective closure, then every birational modification X' of X is non- k -uniruled. In particular, X' and $X' \times k$ do have CP (for details see [2]).

REFERENCES

- [1] Danielewski, W., On a cancellation problem and automorphism groups of affine algebraic varieties, preprint, Warsaw (1989).
- [2] Dryło, R., Non-uniruledness and the cancellation problem. II, *Ann. Polon. Math.*, **92**, 41-48 (2007). MR2318509 (2008c:14079)
- [3] Dubouloz, A., Additive group actions on Danielewski varieties and the cancellation problem, *Math. Z.*, **255**, 77-93 (2007). MR2262722 (2007i:14066)
- [4] Fieseler, K., On complex affine surfaces with $\mathbb{C}_+$ actions, *Comment. Math. Helvetici*, **69**, 5-27 (1989). MR1259603 (95b:14027)
- [5] Finston, D., Maubach, S., The automorphism group of certain factorial threefolds and a cancellation problem, *Israel J. Math.*, **163**, 369-381 (2008). MR2391136 (2009a:14078)
- [6] Fujita, T., On Zariski problem, *Proc. Japan Acad.*, **55**, 106-110 (1979). MR531454 (80j:14029)
- [7] Jelonek, Z., On the cancellation problem, *Math. Ann.*, **344**, 769-778 (2009). MR2507623
- [8] Jelonek, Z., The set of points at which the polynomial mapping is not proper, *Ann. Polon. Math.*, **58**, 259-266 (1993). MR1244397 (94i:14018)
- [9] Jelonek, Z., Lasoń, M., The set of fixed points of a unipotent group, *J. Algebra*, **322**, 2180-2185 (2009). MR2542841
- [10] Lam, T. Y., *Serre Conjecture*, Springer-Verlag, Berlin-Heidelberg, 1978. MR0485842 (58:5644)
- [11] Miyanishi, M., Sugie, T., Affine surfaces containing cylinderlike open sets, *J. Math. Kyoto Univ.*, **20**, 11-42 (1980). MR564667 (81h:14020)
- [12] Mohan Kumar, N., Stably free modules, *Amer. J. of Math.*, **107**, 1439-1443 (1985). MR815767 (87c:13014)
- [13] Raynaud, M., Modules projectifs universels, *Invent. Math.*, **6**, 1-26 (1968). MR0236164 (38:4462)
- [14] Swan, R.G., Vector bundles, projective modules and the K -theory of spheres, *Algebraic topology and algebraic K-theory* (Princeton, NJ, 1983), 432-522, Ann. of Math. Stud., 113, Princeton Univ. Press, Princeton, NJ, 1987. MR921488 (89f:18009)

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