SOME REMARKS ON HARTOGS' EXTENSION LEMMA

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Dedicated to Professor J. M. Méndez on the occasion of his 60th birthday

ABSTRACT. Motivated by a result and a question by E. M. Chirka we consider the Hartogs' extension property for some connected sets in $\mathbb{C}^2$ of the form $K = \Sigma \cup (\partial\Delta \times \overline{\Delta})$, where Σ is a possibly nonconnected compact subset of $\overline{\Delta} \times \overline{\Delta} \subset \mathbb{C}^2$.

1. INTRODUCTION

Let $\Delta(0, 1) = \Delta \subseteq \mathbb{C}$ be the unit disc and let f be a holomorphic function in a connected neighbourhood of the set $K = (\overline{\Delta} \times \{0\}) \cup (\partial\Delta \times \overline{\Delta})$ in $\mathbb{C}^2$. It is a well-known result by Hartogs [10] that f extends holomorphically to a neighbourhood of $\overline{\Delta} \times \overline{\Delta}$. It is also classical, and an easy consequence of the Continuity Principle, that *Hartogs' lemma* holds if in K the set $\overline{\Delta} \times \{0\}$ is replaced by the graph of a holomorphic function $\varphi : \Delta \rightarrow \Delta$ or even a finitely sheeted analytic variety $\Sigma \subseteq \overline{\Delta} \times \overline{\Delta}$ over Δ .

In [4] (see also [1, 5, 9]) Chirka, using an idea from [8], proved that the same result holds if φ is merely continuous. Although the higher dimensional version of Hartogs' lemma still holds, the higher dimensional analog of Chirka's theorem in which one considers the graph of two or more continuous functions on $\overline{\Delta}$ in general does not hold. Rosay [11] constructed two smooth functions $\varphi_j : \overline{\Delta} \rightarrow \overline{\Delta}$, $j = 1, 2$, such that there exists a connected pseudoconvex neighbourhood of the set $K = \{(z, \varphi_1(z), \varphi_2(z)) ; z \in \overline{\Delta}\} \cup (\partial\Delta \times \overline{\Delta}^2)$ which does not contain $\overline{\Delta}^3$. For a result in a positive direction, see [2], but the class of functions φ_j for which the extension property holds is quite restrictive. Let us mention that these results are also related to the results on the envelope of holomorphy of a neighbourhood of a real surface in a complex surface, [7, 8, 9].

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Let $K \subseteq \overline{\Delta} \times \overline{\Delta}$ be a compact set. An obvious necessary condition that every function holomorphic in a neighbourhood of K extends holomorphically to a neighbourhood of $\overline{\Delta} \times \overline{\Delta}$ is that the rational hull of K equals $\overline{\Delta} \times \overline{\Delta}$. In this context Chirka [4] asked the following.

Question (Chirka). Let $K \subset \overline{\Delta} \times \overline{\Delta}$ be a connected compact set containing $\partial\Delta \times \overline{\Delta}$ and such that its rationally convex hull coincides with $\overline{\Delta} \times \overline{\Delta}$. Is it true that every function holomorphic in a connected neighbourhood of K extends holomorphically to $\overline{\Delta} \times \overline{\Delta}$?

Motivated by this question, we consider the following generalizations of Hartogs' extension lemma.

Theorem 1. For $\varepsilon > 0$, let Σ_ε be given by

$$\Sigma_\varepsilon = \{(z, \varepsilon) \in \overline{\Delta} \times \mathbb{C}; \operatorname{Re} z \geq -1/2\} \cup \{(z, -\varepsilon) \in \Delta \times \mathbb{C}; \operatorname{Re} z \leq 1/2\}.$$

Then there exist constants $\gamma > 1$ and $0 < \varepsilon_0 < 1$ such that for any $0 < \varepsilon < \varepsilon_0$, any holomorphic function in a connected ε^γ -neighbourhood (in the w -direction) of $K_\varepsilon = \Sigma_\varepsilon \cup (\partial\Delta \times \overline{\Delta})$ extends holomorphically to a neighbourhood of $\overline{\Delta} \times \overline{\Delta}$.

Observe that the Σ_ε -part of the set K_ε is not connected and that for each connected component of Σ_ε one cannot directly use any of the previous theorems.

An ε^γ -neighbourhood of K_ε in the w -direction is the set

$$\bigcup_{(z,w) \in K_\varepsilon} (\Delta(z, \delta) \times \Delta(w, \varepsilon^\gamma))$$

for some $\delta > 0$.

As a first example to look at, Chirka considered the set K_ε , which satisfies the assumption on its rational convex hull in the question for $\varepsilon > 0$ small enough. Nevertheless, in a private communication, J. P. Rosay has pointed out to us that the answer to Chirka's question is negative. An example is K_ε itself for some *small enough* neighbourhood. Note that in Theorem 1 we tacitly assume that the neighborhood from which *we have the extension* must be 'fat' enough near the 'faces' of Σ_ε as specified in the hypothesis.

While the case where Σ is not connected might be a 'natural' place to look for a counterexample to Chirka's question, this does not seem to be the case for 'continuous' varieties over $\overline{\Delta}$.

Problem. Let $a_0, \dots, a_{n-1}$ be continuous functions on $\overline{\Delta}$ such that the set

$$\Sigma = \{(z, w) \in \overline{\Delta} \times \mathbb{C}; w^n + a_{n-1}(z)w^{n-1} + \dots + a_0(z) = 0\}$$

is a subset of $\overline{\Delta} \times \overline{\Delta}$. Does every function holomorphic in a connected neighbourhood Ω of the set $K = \Sigma \cup (\partial\Delta \times \overline{\Delta})$ such that $\overline{\Omega} \cap (\overline{\Delta} \times \overline{\Delta})$ is connected extend holomorphically to a neighbourhood of the bidisc $\overline{\Delta} \times \overline{\Delta}$?

In this direction we have the following result.

Theorem 2. Let $a_0, \dots, a_{n-1}$ be continuous functions on $\overline{\Delta}$ and let

$$\Sigma = \{(z, w) \in \overline{\Delta} \times \mathbb{C}; w^n + a_{n-1}(z)w^{n-1} + \dots + a_0(z) = 0\}$$

be a continuous variety over $\overline{\Delta}$. Then every function holomorphic in a connected neighbourhood of the set $K = \Sigma \cup (\partial\Delta \times \mathbb{C})$ extends holomorphically to a neighbourhood of $\overline{\Delta} \times \mathbb{C}$.

The assumptions in Theorem 2 are much stronger than those in the Problem; that is, we assume holomorphicity on a neighbourhood of a continuous variety Σ and the *infinite* cylinder $\partial\Delta \times \mathbb{C}$ to conclude holomorphic extendability into $\overline{\Delta} \times \mathbb{C}$.

2. PROOF OF THEOREM 1

Let Ω be a connected neighborhood of K_ε and f a holomorphic function in Ω . Since K_ε is invariant under the transformation $(z, w) \mapsto (-z, -w)$, shrinking Ω if necessary, we may assume that Ω is invariant under such a transformation as well and hence f can be decomposed into its even and odd parts. Therefore, to extend f , we can further assume that f is even (for the odd part the reasoning being the same).

Let $\rho^\pm$, defined in a smooth neighborhood $D^\pm$ of $\pm \operatorname{Re} z \geq -1/2$ and $|z| \leq 1$, be the radius of convergence of the Taylor expansion of f ,

$$f(z, \pm\varepsilon + w) = \sum_{n=0}^{\infty} a_n^\pm(z) w^n$$

respectively, that is,

$$\log \frac{1}{\rho^\pm(z)} = \limsup_{n \rightarrow \infty} \frac{1}{n} \log |a_n^\pm(z)|.$$

Since f is holomorphic and even in Ω , the functions $u^\pm := \log \frac{1}{\rho^\pm}$ are subharmonic where defined, and moreover $u^+(-z) = u^-(z)$ if $\operatorname{Re} z \leq 1/2$ and $|z| \leq 1$. Denote by $\omega^+(z)$ the harmonic measure of the segment $\{\operatorname{Re} z = -1/2\} \cap D^+$ based at z with respect to the domain $D^+ \cap \{\operatorname{Re} z > -1/2\}$ and let $\gamma > 1$ be chosen so that $\eta := \sup_{|t| \leq 1} \omega(it) < 1/\gamma < 1$. By our hypothesis, $\rho^+(z) \geq 1 - \varepsilon$ for $|z|$ close to 1 and $\geq \varepsilon^\gamma$ for $\operatorname{Re} z \geq -1/2$ and $|z| < 1$. By the Maximum Principle we conclude that

$$u^+(it) \leq \log \frac{1}{1 - \varepsilon} + \alpha \log \frac{1}{\varepsilon},$$

where $\alpha = \gamma\eta < 1$. Thus, there is $0 < \varepsilon_0 < 1$ so that $u^+(it) < \log \frac{1}{\varepsilon}$ for all $0 < \varepsilon < \varepsilon_0$ and $|t| \leq 1$. But, by the symmetry mentioned above that $u^\pm$ satisfy, this clearly means that f has a holomorphic extension to a domain $\tilde{\Omega}$ that contains a graph of a continuous function to which Chirka's Theorem can be applied. This finishes the proof.

3. PROOF OF THEOREM 2

For the proof of Theorem 2 we will use Bharali's approach [2] to the generalization of the Chirka extension theorem. Let $a_0, \dots, a_{n-1}$ be continuous functions on $\overline{\Delta}$ which define the continuous variety

$$\Sigma = \{(z, w) \in \overline{\Delta} \times \mathbb{C}; w^n + a_{n-1}(z)w^{n-1} + \dots + a_0(z) = 0\}$$

over $\overline{\Delta}$. The mapping $\varphi : \overline{\Delta} \rightarrow \mathbb{C}^{n+1}$,

$$\varphi : z \mapsto (z, a_0(z), \dots, a_{n-1}(z)),$$

defines a continuous disc in $\mathbb{C}^{n+1}$. Any perturbation

$$\tilde{\varphi} : z \mapsto (z, \tilde{a}_0(z), \dots, \tilde{a}_{n-1}(z))$$

of this disc within the class of continuous functions on $\overline{\Delta}$ defines a new continuous variety over $\overline{\Delta}$,

$$\tilde{\Sigma} = \{(z, w) \in \overline{\Delta} \times \mathbb{C}; w^n + \tilde{a}_{n-1}(z)w^{n-1} + \cdots + \tilde{a}_0(z) = 0\},$$

which is a perturbation of Σ .

For $\varepsilon > 0$ let

$$U(\varepsilon) = \bigcup_{(z,w) \in \Sigma} (\{z\} \times \overline{\Delta(w, \varepsilon)})$$

be a closed neighbourhood of Σ in $\overline{\Delta} \times \mathbb{C}$ and similarly let

$$V(\varepsilon) = \bigcup_{z \in \overline{\Delta}} (\{z\} \times \overline{\Delta(a_0(z), \varepsilon)} \times \cdots \times \overline{\Delta(a_{n-1}(z), \varepsilon)})$$

be a closed neighbourhood of the disc $\varphi(\overline{\Delta})$ in $\overline{\Delta} \times \mathbb{C}^n$. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every continuous disc

$$\tilde{\varphi}: z \mapsto (z, \tilde{a}_0(z), \dots, \tilde{a}_{n-1}(z)),$$

the condition

$$(z, \tilde{a}_0(z), \dots, \tilde{a}_{n-1}(z)) \in V(\delta) \quad \text{for some } z \in \overline{\Delta}$$

implies that every point $(z, w) \in \tilde{\Sigma}$ lies in $U(\varepsilon)$; that is, whenever a part of the image of the disc $\tilde{\varphi}(\overline{\Delta})$ lies inside $V(\delta)$, the corresponding part of the continuous variety $\tilde{\Sigma}$ lies within $U(\varepsilon)$.

Hence, if we can perturb the disc φ within a continuous family of continuous discs φ_t , $t \in [0, 1]$, on $\overline{\Delta}$ in such a way that $\varphi_0 = \varphi$, φ_1 is a holomorphic disc and each φ_t is holomorphic on the set of points which are mapped outside a closed neighbourhood of the disc $\varphi(\Delta)$, then we can perturb the continuous variety Σ through a continuous family Σ_t , $t \in [0, 1]$, of continuous varieties over $\overline{\Delta}$ in such a way that $\Sigma_0 = \Sigma$, Σ_1 is an analytic variety and each continuous variety Σ_t is such that it is holomorphic outside a given closed neighbourhood of Σ . Then, using analytic continuation [6], one can extend a given holomorphic function f in a connected neighbourhood of the set $K = \Sigma \cup (\partial\Delta \times \mathbb{C})$ to a neighbourhood of $K_1 = \Sigma_1 \cup (\partial\Delta \times \mathbb{C})$. By Hartogs' extension theorem we conclude that f extends holomorphically to a neighbourhood of $\overline{\Delta} \times \mathbb{C}$.

Remark 3. The continuation of f given by [6] does not *a priori* rule out multi-valuedness. However, in a tubular neighbourhood $\mathcal{V}$ of $\partial\Delta_z \times \mathbb{C}_w$, the family $\{\Sigma_t|_{\mathcal{V}}\}$ evolves in the direction of $\mathbb{C}_w$, which is simply connected. Thus, this extension agrees with f on $\mathcal{V}$. Hence they agree everywhere where both are defined.

We then seek for a perturbation of the disc $\varphi(\overline{\Delta})$ within a continuous family of continuous discs φ_t , $t \in [0, 1]$, on $\overline{\Delta}$ in such a way that $\varphi_0 = \varphi$, φ_1 is a holomorphic disc and each φ_t is holomorphic on the set of points which are mapped outside a closed neighbourhood of $\varphi(\overline{\Delta})$. This has basically already been done in Bharali's paper [2], and we include the necessary argument for the sake of completeness.

First let us observe that without loss of generality we may assume that $a_j(z) = P_j(z, \bar{z})$, $j = 0, \dots, n-1$, for some holomorphic polynomials $P_j(z, \xi)$ in two complex variables. The mapping

$$(z, \xi) \in \overline{\Delta} \times \mathbb{C} \longrightarrow (z, P_0(z, \xi), \dots, P_{n-1}(z, \xi)) \in \overline{\Delta} \times \mathbb{C}^n$$

is holomorphic and maps the maximal real disc $\mathcal{R} = \{(z, \bar{z}) \in \bar{\Delta} \times \mathbb{C}\}$ into $\varphi(\bar{\Delta})$ and a neighbourhood of $\mathcal{R}$ in $\bar{\Delta} \times \mathbb{C}$ into a neighbourhood of $\varphi(\bar{\Delta})$ in $\bar{\Delta} \times \mathbb{C}^n$. In the proof of Lemma 1.4 in [2], Bharali constructed an explicit family of holomorphic discs in $\bar{\Delta} \times \mathbb{C}$ with boundaries in $\mathcal{R}$ and $\partial\Delta \times \mathbb{C}$. For $t \in [-1, 1]$, let

$$\mathcal{A}_t(z) = (z, 2t - z).$$

For $t \in [0, 1]$ the disc $\mathcal{A}_t$ is defined on

$$\mathcal{D}_t^+ = \{x + iy; t \leq x \leq 1, x^2 + y^2 \leq 1\},$$

while for $t \in [-1, 0]$ it is defined on

$$\mathcal{D}_t^- = \{x + iy; -1 \leq x \leq t, x^2 + y^2 \leq 1\}.$$

For each $t \in [0, 1]$ we define a continuous disc Φ_t on $\bar{\Delta}$ as

$$\Phi_t(z) = \begin{cases} \mathcal{A}_{t-1}(z) & \text{for } z \in \mathcal{D}_{t-1}^-, \\ \mathcal{A}_{1-t}(z) & \text{for } z \in \mathcal{D}_{1-t}^+, \\ (z, \bar{z}) & \text{elsewhere on } \bar{\Delta}. \end{cases}$$

Finally we define for $t \in [0, 1]$,

$$\varphi_t(z) = (z, P_0(\Phi_t(z)), \dots, P_{n-1}(\Phi_t(z))).$$

The discs $\{\varphi_t\}_{t \in [0, 1]}$ form a continuous family of continuous discs on $\bar{\Delta}$ such that $\varphi_0 = \varphi$, φ_1 is a holomorphic disc and each φ_t is holomorphic on the set of points which are mapped outside a closed neighbourhood of the mapping φ . This finishes the proof of Theorem 2.

Remark 4. Another approach to the problem of extendability from a neighbourhood of the continuous variety $K = \Sigma \cup (\partial\Delta \times \mathbb{C})$ is also the following. Namely, one can prove that within a neighbourhood of Σ there is a smooth variety $\tilde{\Sigma}$, a bordered Riemann surface S , a proper holomorphic function $a : S \rightarrow \Delta$ and a smooth function $\phi : \bar{S} \rightarrow \mathbb{C}$ such that

$$(a, \phi) : \bar{S} \longrightarrow \bar{\Delta} \times \mathbb{C}$$

is a smooth embedding whose image is $\tilde{\Sigma}$. Using this fact, one is left to consider a similar problem as Chirka above the bordered Riemann surface S : Let S be a bordered Riemann surface and let $\phi : S \rightarrow \bar{\Delta}$ be a continuous function on S . Does every function holomorphic in a connected neighbourhood of $\text{graph}(\phi) \cup (\partial S \times \bar{\Delta})$ holomorphically extend to a neighbourhood of $S \times \bar{\Delta}$? If S is a planar domain, that is, the genus of S is 0, then the same proof as Chirka's [4] works. The case of higher genus $g > 0$ seems to be much harder and is generally open. In cases where the degree of the 'Weierstrass polynomial' is greater than or equal to the genus of (a uniformly small smooth perturbation of) S , the extension result follows from Theorem 12 in [9].

For Chirka's technique to work, it is necessary to have *sup* norm control on the solution to a certain quasilinear $\bar{\partial}$ -equation on S , something we do not have when $g > 0$. Also, this is essentially the *raison d'être* for Bharali's class of functions in [2] to be that restrictive. An attempt to get such control via Riemann-Hilbert boundary value problems for quasilinear $\bar{\partial}$ -equations on bordered Riemann surfaces yielded [3]. However the *sup* norm control one needs to conclude the proof of the generalization of Chirka's result over bordered Riemann surfaces with genus $g > 0$ seems to be still unknown.

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