ON SPECTRAL GAP RIGIDITY AND CONNES INVARIANT $\chi(M)$

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ABSTRACT. We calculate Connes' invariant $\chi(M)$ for certain II_1 factors M that can be obtained as inductive limits of subfactors with spectral gap. Then we use this to answer a question he posed in 1975 on the structure of McDuff factors M with $\chi(M) = 1$.

1. INTRODUCTION

Given a II_1 factor M , one denotes by $\chi(M)$ the image in the outer automorphism group $\text{Out}(M) = \text{Aut}(M)/\text{Int}(M)$ of the group of automorphisms of M that are both approximately inner and centrally free. This invariant for II_1 factors was introduced by Connes in [2], who used it to solve several famous problems in von Neumann algebras. On this occasion, he raised some questions on $\chi(M)$, one of them being whether a *McDuff factor* (i.e. a II_1 factor that splits off the hyperfinite II_1 factor R) with $\chi(M) = 1$ is necessarily of the form $M = Q \overline{\otimes} R$ with Q a non-Gamma II_1 factor; i.e., in the terminology used hereafter, M is *s-McDuff* (*strong McDuff*).¹

We answer this question here by providing several classes of examples of McDuff factors who have trivial $\chi(M)$ but are not s-McDuff. We do this by using the deformation-rigidity methods in [17, 18, 19]. Thus, we first show that any infinite tensor product of non-Gamma II_1 factors, $M = \overline{\otimes}_n Q_n$, satisfies these properties. The second class of examples comes from subfactor theory. Thus, starting from a non-trivial irreducible inclusion of non-Gamma factors $P \subset N$ with finite depth, we consider the enveloping factor N_∞ , obtained as the inductive limit of the associated Jones tower of factors, $P \subset N \subset N_1 \subset N_2 \subset \dots$. We then use [14] to show that if the graph $\Gamma = \Gamma_{P,N}$ of $P \subset N$ has the property that the Perron-Frobenius eigenvector $\vec{s}$ (resp. $\vec{t}$) of $\Gamma^t \Gamma$ (resp. $\Gamma \Gamma^t$) has distinct entries (e.g. if $\Gamma_{P,N} = A_{2n}$), then $\chi(N_\infty) = 1$. On the other hand, by using deformation-rigidity we prove that N_∞ is not s-McDuff.

We mention that the deformation of the ambient factors that we use in our arguments is by *inductive limits*, while the rigidity part is played by the *spectral*

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gap property, considered in [19]. More applications of these ideas and techniques can be found in [20].

2. SPECTRAL GAP AND AUTOMORPHISMS IN $\text{Ctr}(M), \overline{\text{Int}(M)}$

2.1. Definition. Let M be a II_1 factor and let $Q \subset M$ be a subfactor. We say that Q has spectral gap in M if $\forall \varepsilon > 0$, $\exists u_1, \dots, u_n \in \mathcal{U}(Q)$ and $\delta > 0$ such that if $x \in M$ satisfies $\|[x, u_i]\|_2 \leq \delta \|x\|_2$, $\forall i$, then $\|x - E_{Q' \cap M}(x)\|_2 \leq \varepsilon \|x\|_2$.

The following lemma relates this property with a somewhat weaker version of spectral gap, useful in applications. As usual, we denote by ω a free ultrafilter on $\mathbb{N}$ and by M^ω the ultrapower of M (itself a II_1 factor), as defined in [8].

2.2. Lemma. *If $Q \subset M$ is an inclusion of factors and Q has spectral gap in M , then $Q' \cap M^\omega = (Q' \cap M)^\omega$.*

Proof. Trivial by the definitions. $\square$

2.3. Lemma. *Assume the II_1 factor M is an inductive limit of subfactors $N_n \nearrow M$ which have spectral gap in M . Then $M' \cap M^\omega = \bigcap_n (N'_n \cap M)^\omega = \{(x_n) \mid \exists k_n \text{ such that } x_n \in N'_{k_n} \cap M \text{ and } \lim_\omega k_n = \infty\}$.*

Proof. To see the first equality, note that we have $M' \cap M^\omega = (\bigcup_n N_n)' \cap M^\omega = \bigcap_n (N'_n \cap M^\omega)$, which by Lemma 2.2 is equal to $\bigcap_n (N'_n \cap M)^\omega$.

To show the second equality, denote by $\mathcal{Y}$ the given set. We clearly have $\mathcal{Y} \subset \bigcap_n (N'_n \cap M)^\omega$. Conversely, if $(x_n)_n \in \bigcap_n (N'_n \cap M)^\omega$, then for any m there exists a neighborhood V_m of ω such that $\|x_k - E_{N'_m \cap M}(x_k)\|_2 \leq 2^{-m}$, $\forall k \in V_m$. Moreover, we can take $V_m \subset V_{m-1}$. Thus, if for each $n \in V_m \setminus V_{m-1}$ we denote $y_n = E_{N'_m \cap M}(x_n)$ and put $k_n = m$, then $(y_n)_n$ coincides with $(x_n)_n$ in M^ω and clearly $y \in \mathcal{Y}$ and k_n satisfy the required conditions. $\square$

2.4. Lemma. *Assume that $Q \subset P$ is an inclusion of factors. Let $0 < \varepsilon < 1/2$ and set $\delta = (\varepsilon/6)^8$. If $\theta \in \text{Aut}(P)$ satisfies $\|\theta(v) - v\|_2 \leq \delta$, $\forall v \in \mathcal{U}(Q)$, then there exists a partial isometry $w \in \mathcal{U}(P)$ such that $\theta(x)w = wx$, $\forall x \in Q$, and $\|w - 1\|_2 \leq \varepsilon$. If in addition $Q' \cap P$ is a factor, then there exists $u \in \mathcal{U}(P)$ such that u still satisfies $\theta(x)u = ux$, $\forall x \in Q$, and $\|u - 1\|_2 \leq 2\varepsilon$.*

Proof. Let $K = \overline{\text{co}}^w \{\theta(v)v^* \mid v \in \mathcal{U}(Q)\}$. Note that $\|y\| \leq 1$ and $\|y - 1\|_2 \leq \delta$, for all $y \in K$. Since K is convex and weakly compact in $P \subset L^2(P)$, there exists a unique element ξ of minimal norm $\|\cdot\|_2$ in K . Since $\theta(v)\xi v^* \in K$ and $\|\theta(v)\xi v^*\|_2 = \|\xi\|_2$, $\forall v \in \mathcal{U}(Q)$, by the uniqueness of ξ it follows that $\theta(x)\xi = \xi x$, $\forall x \in Q$. Thus, the partial isometry $w \in P$ in the polar decomposition of ξ still satisfies $\theta(x)w = wx$, $\forall x \in Q$. Moreover, by [3] we have $\|w - 1\|_2 \leq 6\delta^{1/8} = \varepsilon$.

If in addition $Q' \cap P$ is a factor and we assume $\varepsilon < 1/2$, then $\tau(ww^*) \geq 1/2$ and we can take partial isometries $e_{12} \in Q' \cap P$, $f_{21} \in \theta(Q)' \cap P$ such that $e_{12}e_{12}^* \leq w^*w$, $e_{12}^*e_{12} + w^*w = 1$, $f_{21}^*f_{21} = we_{12}e_{12}^*w^*$, $f_{21}f_{21}^* + ww^* = 1$. Thus, if we define $u = w + f_{21}we_{12}$, then u is a unitary satisfying $\theta(x)u = ux$, $\forall x \in Q$, and $\|u - 1\|_2 \leq 2\varepsilon$, proving the statement. $\square$

2.5. Lemma. *Let $\theta \in \text{Aut}(M)$, $\{u_n\}_n \subset \mathcal{U}(M)$. Then we have:*

(a) *If $N_0 \subset M$ has spectral gap and $\theta(x) = \lim_n u_n x u_n^*$, $\forall x \in M$, then*

$$\lim_{n,m \rightarrow \infty} \|u_n^* u_m - E_{N_0' \cap M}(u_n^* u_m)\|_2 = 0.$$

If in addition $\theta|_{N_0} = id$, then there exist $v_n \in \mathcal{U}(N'_0 \cap M)$ such that $\theta(x) = \lim_n v_n x v_n^*$, $\forall x \in M$.

(b) Assume M is an inductive limit of subfactors $N_n \nearrow M$ with spectral gap such that $N'_k \cap M$ is a factor, $\forall k \geq 0$. Then $\theta \in \overline{\text{Int}(M)}$ iff $\exists v_k \in \mathcal{U}(N)$ such that $\theta|_{N_k} = \text{Ad}(v_k)|_{N_k}$, $\forall k \geq 0$. Moreover, if this is the case, then $U_n = v_0^* v_n \in N'_0 \cap M$ and we have $\theta(x) = \text{Adv}_0 \lim_n \text{Ad} U_n(x)$, $\forall x \in M$.

Proof. The first part of (a) is trivial. To prove the second part, note that the spectral gap of N_0 in M and the condition $\lim_n u_n x u_n^* = x$, $\forall x \in N_0$, imply $\lim_n \|u_n - E_{N'_0 \cap M}(u_n)\|_2 = 0$. By [3], there exist unitary elements $v_n \in \mathcal{U}(N'_0 \cap M)$ such that $\lim_n \|v_n - E_{N'_0 \cap M}(u_n)\|_2 = 0$, and thus $\lim_n \|u_n - v_n\|_2 = 0$ as well, showing that $\theta = \lim \text{Ad}(v_n)$. To prove part (b), assume $\theta = \lim_n \text{Ad}(u_n)$, fix $k \geq 1$ and let n_0 be such that $\|u_n^* u_m - E_{N'_k \cap M}(u_n^* u_m)\|_2 \leq 12^{-8}$, $\forall n, m \geq n_0$ (by (a)). Then, for $x \in \mathcal{U}(N_k)$ we have $u_{n_0}^* \theta(x) u_{n_0} = \lim_{m \rightarrow \infty} u_{n_0}^* u_m x u_m^* u_{n_0}$ and thus

$$\|u_{n_0}^* \theta(x) u_{n_0} - x\|_2 \leq 2 \limsup_m \|u_{n_0}^* u_m - E_{N'_k \cap M}(u_{n_0}^* u_m)\|_2 \leq 2 \cdot 12^{-8}.$$

But this implies $\text{Ad}(u_{n_0}^*) \circ \theta$ is inner on N_k , by Lemma 2.4. Thus, there exists $V_k \in \mathcal{U}(M)$ such that $v_k = u_{n_0} V_k$ satisfies $\theta(x) = \text{Adv}_k(x)$, $\forall x \in N_k$. $\square$

2.6. Lemma. Assume $N_n \nearrow M$, N_n has spectral gap in M and $N'_n \cap M$ is a factor, $\forall n$. Then $\theta \in \text{Ctr}(M)$ iff there exist n and $u \in \mathcal{U}(M)$ such that $\theta|_{N'_n \cap M} = \text{Ad}(u)$.

Proof. By Lemma 2.3, if there exist n and $u \in \mathcal{U}(M)$ such that $\theta(x) = u x u^*$, $\forall x \in N'_n \cap M$, then $\theta \in \text{Ctr}(M)$. To prove the converse, by Lemma 2.4 it is sufficient to show that there exists n such that $\|\theta(v) - v\|_2 \leq 12^{-8}$, $\forall v \in \mathcal{U}(N'_n \cap M)$. Assume on the contrary that $\forall n$, $\exists v_n \in \mathcal{U}(N'_n \cap M)$ such that $\|\theta(v_n) - v_n\|_2 > 12^{-8}$. But this implies that $v = (v_n) \in M' \cap M^\omega$ satisfies $\theta(v) \neq v$, contradicting the central triviality of θ . $\square$

2.7. Corollary. Assume the II_1 factor M is an inductive limit of subfactors $N_n \nearrow M$ such that N_n has spectral gap in M and $N'_n \cap M$ is a factor, $\forall n$.

1. If $\theta \in \text{Ctr}(M) \cap \overline{\text{Int}(M)}$, then for any large enough n , there exist $u, v \in \mathcal{U}(M)$ such that $\theta = \text{Adu}$ on N_n and $\theta = \text{Adv}$ on $N'_n \cap M$. Moreover, if for such an n we denote $\theta' = \text{Adu}^* \theta$, then for any $\varepsilon > 0$, there exist $m \geq n$ and a non-zero partial isometry $w \in N'_m \cap M$ such that $\theta(x)w = wx$, $\forall x \in N'_m \cap M$, and $\|w - 1\|_2 \leq \varepsilon$.

2. If in addition $\mathcal{N}_M(N'_n \cap M)$ acts innerly on $N'_n \cap M$, then $\text{Ctr}(M) \cap \overline{\text{Int}(M)} / \text{Int}(M)$ naturally embeds into the group of automorphisms $\theta \in \text{Aut}(M)$ which act trivially on $N_n \vee N'_n \cap M$.

Proof. By Lemmas 2.5 and 2.6, for any large enough n there exist $u, v \in \mathcal{U}(M)$ such that $\theta' = \text{Ad}(u^*)\theta$ is trivial on N_n and equal to $\text{Ad}(v)$ on $N'_n \cap M$. The rest of (1) follows from Lemma 2.4. Under the additional assumption in (2), it follows that θ' can be perturbed by an element in $\text{Int}(N'_n \cap M)$ so that it can act trivially on $N_n \vee N'_n \cap M$. $\square$

3. DEFORMATION-RIGIDITY LEMMA AND EXAMPLES

3.1. Lemma. Let $Q \subset M$ be an inclusion of factors and assume that Q has spectral gap in M . If $N_n \subset M$ are von Neumann subalgebras such that $\lim_n \|E_{N_n}(x) - x\|_2 = 0$, $\forall x \in M$, then for any $\varepsilon > 0$, there exists n such that $N'_n \cap M \subset_\varepsilon Q' \cap M$.

Proof. Since Q has spectral gap in M , by definition there exist $\delta > 0$ and $u_1, \dots, u_m \in U(Q)$ such that if $x \in (M)_1$ satisfies $\| [u_i, x] \|_2 < \delta, \forall i$, then $x \in_\varepsilon Q' \cap M$. By the hypothesis, there exists n such that $\| E_{N_n}(u_i) - u_i \|_2 \leq \delta/2, \forall i$. Thus, if $x \in N'_n \cap M$, then

$$\| [x, u_i] \|_2 \leq 2\|x\| \|E_{N_n}(u_i) - u_i\|_2 \leq \delta, \forall i,$$

implying that $x \in_\varepsilon Q' \cap M$. $\square$

We next look for conditions which are sufficient for the assumptions in Corollary 2.7 to be satisfied and for which Lemma 3.1 thus applies. As in [15], we denote by $\mathcal{G}_{P,N}$ the standard invariant of an inclusion of factors with finite index $P \subset N$ and we denote by $\Gamma_{P,N}$ its standard graph.

3.2. Proposition. 1. *If N is a non-Gamma II_1 factor and S is an arbitrary finite factor, then N has spectral gap in $M = N \overline{\otimes} S$. Moreover, any subfactor of finite index $P \subset N$ has spectral gap in M .*

2. *If N has the property (T) and M is a II_1 factor containing M , then N has spectral gap in M . In particular, if P is a subfactor of finite index of a property (T) factor N , $P \subset N \subset N_1 \subset \dots$ is the Jones tower and N_∞ is the associated enveloping algebra, then N_n has spectral gap in $N_\infty, \forall n$.*

3. *If N is non-Gamma, $P \subset N$ is a subfactor with finite depth, and $N_n \nearrow N_\infty$ is the associated tower and enveloping algebra, then N_n has spectral gap in $N_\infty, \forall n$.*

4. *Let $\mathcal{G}$ be a standard λ -lattice, let Q be a II_1 factor and denote by $P = M_{-1}^{\mathcal{G}}(Q) \subset M_0^{\mathcal{G}}(Q) = N$ the inclusion of II_1 factors with standard invariant $\mathcal{G}$, as constructed in [15], [16], with $N_n \nearrow N_\infty$ the associated tower and enveloping algebra. Then N_n has spectral gap in $N_\infty, \forall n$. Also, if $Q = L(F_\infty)$, then $P \simeq N_n \simeq L(\mathbb{F}_\infty), \forall n$.*

Moreover, in part 3, $N'_n \cap N_\infty$ is a factor $\forall n$. In turn, in parts 2 and 4, $N' \cap N_\infty$ is a factor if and only if the standard graph of $P \subset N$ is ergodic. This is the case if for instance $\mathcal{G}_{N,M}$ is strongly amenable or if $P \subset N$ has graph A_∞ , i.e. when $[N : P] \geq 4$ and the relative commutants $N' \cap N_n$ are generated by the Jones projections (i.e. $\mathcal{G}_{P,N}$ is the so-called Temperley-Lieb standard lattice).

Proof. Part 1 is essentially due to Connes (see [3]), and part 2 is trivial. Part 4 is immediate by the proofs of 7.1 and 7.3 in [13]. To prove part 3, note first that if $N \supset P \supset P_1 \supset P_2 \supset \dots$ is a tunnel, then P_k has spectral gap in N and thus in $N \vee N' \cap N_\infty$, for any k . Since $P \subset N$ has finite depth, so does $N \vee N' \cap N_\infty \subset N_\infty$ (see e.g. [9] or [4]) and there exists k such that $P'_k \cap N_\infty$ contains an orthonormal basis $\{m_j\}_j$ of N_∞ over $N \vee N' \cap N_\infty$. Thus, if $\xi \in N' \cap L^2(N_\infty)^\omega$, then $\xi = \sum_j m_j \xi_j$, for some unique “coefficients” ξ_j lying in $L^2(N \vee N' \cap N_\infty)^\omega$. Since ξ and m_j commute with P_k , it follows that the ξ_j commute with P_k as well, so in fact all ξ_j lie in $(P'_k \cap N) \vee L^2(N' \cap N_\infty)^\omega$. Altogether, $\xi \in N' \cap L^2(P'_k \cap N_\infty)^\omega = L^2(N' \cap N_\infty)^\omega$. $\square$

4. CALCULATIONS OF $\chi(M)$ AND AN ANSWER TO CONNES’ QUESTION

4.1. Theorem. *If $M = \overline{\otimes}_k Q_k$, with Q_k a sequence of non-Gamma II_1 factors, then $\text{Ctr}(M) \cap \overline{\text{Int}}(M) = \text{Int}(M)$ (equivalently $\chi(M) = 1$) and M is McDuff but not s -McDuff.*

Proof. By Corollary 2.7.2 we have $\chi(M) = 1$ (this calculation was in fact already done in [2]). Assume $M = Q \overline{\otimes} R$ for some non-Gamma factor Q . Thus, Q has

spectral gap in M . Denote $N_n = \overline{\otimes}_{k \leq n} Q_k$. Then $N_n \nearrow M$ and, by Lemma 3.2 there exists n such that $\overline{\otimes}_{k > n} Q_k = N'_n \cap M \subset_\varepsilon Q' \cap M = R$. By [18], this implies there exists an isomorphism of $\overline{\otimes}_{k > n} Q_k$, which is a non-amenable factor, into an amplification of R . Since R is amenable, this is a contradiction. $\square$

From here on, we consider the following special case of inductive limits of factors: Let $P \subset N$ be a subfactor of finite Jones index, let $P \subset N \subset N_1 \subset N_2 \subset \dots$ be its Jones tower and let $M = N_\infty = \overline{\bigcup}_n N_n$ be the associated *enveloping Π_1 factor*. Under the assumption that N has spectral gap in N_∞ , we relate Connes' χ -invariant of N_∞ with Kawahigashi's χ -invariant of the inclusion $N' \cap N_\infty \subset P' \cap N_\infty$ and use this to calculate $\chi(N_\infty)$ for many enveloping factors. In particular, this will provide more examples of factors with trivial χ -invariant which are McDuff but not s-McDuff.

Thus, recall from [7] that if $S \subset R$ is an inclusion of finite von Neumann algebras, then

$$\chi(R, S) \stackrel{\text{def}}{=} \overline{\text{Int}(R, S)} \cap \text{Ctr}(R, S) / \text{Int}(R, S),$$

where $\text{Int}(R, S)$ is the group of inner automorphisms of R implemented by unitaries in S , $\overline{\text{Int}(R, S)}$ is its closure and $\text{Ctr}(R, S)$ is the group of automorphisms of R leaving S invariant and acting trivially on $R' \cap S^\omega$. We consider this invariant for inclusions of the form $S = N' \cap N_\infty \subset P' \cap N_\infty = R$, where $P \subset N$ is a (proper) subfactor of finite index and N_∞ is its enveloping algebra, as above.

4.2. Theorem. *If N has spectral gap in N_∞ , then there is a natural isomorphism $\chi(N_\infty) \simeq \chi(P' \cap N_\infty, N' \cap N_\infty)$.*

Proof. Note first that by [12], if σ is an automorphism of $R = P' \cap N_\infty$ leaving $S = N' \cap N_\infty$ invariant, then there exists $u \in \mathcal{U}(N' \cap N_\infty)$ such that $u\sigma(e_1)u^* = e_1$, where $e_1 = e_P$. Thus, any element in $\overline{\text{Int}(R, S)} / \text{Int}(R, S)$ can be represented by an automorphism of the form $\sigma = \lim_n \text{Ad}(u_n)$, with $u_n \in \mathcal{U}(N'_1 \cap N_\infty)$. Denote by $\mathcal{G} \subset \text{Aut}(R, S)$ the group of automorphisms of this form. Denote also $\mathcal{G}_0 = \mathcal{G} \cap \text{Ctr}(R, S)$. Let also $\mathcal{H} \subset \text{Aut}(N_\infty)$ be the group of automorphisms in $\overline{\text{Int}(N_\infty)}$ which act trivially on N_1 and $\mathcal{H}_0 = \mathcal{H} \cap \text{Ctr}(N_\infty)$.

If $\sigma \in \mathcal{G}$, then let $\psi = \Psi(\sigma)$ be the Banach limit $\psi(x) \stackrel{\text{def}}{=} \text{Lim}_\omega u_n x u_n^*$, $x \in N_\infty$. Thus ψ is a unital trace preserving a completely positive (c.p.) map on N_∞ , and by the choice of u_n , we have $\psi(x) = x$ for $x \in N$, $\psi(e_1) = e_1$ and $\psi(x) = \sigma(x)$, $x \in P' \cap N_\infty$. Note now that if ψ is a unital trace preserving c.p. map on a finite von Neumann algebra N_∞ , then the set $\mathcal{X} \subset N_\infty$ on which it is $\|\cdot\|_2$ -isometric is a von Neumann subalgebra and ψ is multiplicative on $\mathcal{X}$. Indeed, $\mathcal{X}$ coincides with the space of fixed points of the unital trace preserving c.p. map $\psi^* \psi$, which is well known to be a von Neumann algebra. Thus, $\psi|_{\mathcal{X}} : \mathcal{X} \rightarrow N_\infty$ is a unital trace preserving, $\|\cdot\|_2$ -isometric c.p. map, implying that it is a von Neumann algebra *-morphism. Since in our case ψ is isometric on $N, e_1, N' \cap N_\infty$ and the algebra generated by N, e_1 and $N' \cap N_\infty$ is $\|\cdot\|_2$ -dense in N_∞ , it follows that $\mathcal{X} = N_\infty$. Thus, $\psi \in \text{Aut}(N_\infty)$ and so, by the way it was defined, $\psi \in \overline{\text{Int}(N_\infty)}$. Moreover, since N has spectral gap in N_∞ , $N'_\infty \cap N_\infty^\omega = N'_\infty \cap (N' \cap N_\infty)^\omega = R' \cap S^\omega$. Thus, if $\sigma \in \mathcal{G}_0$, then $\psi \in \text{Ctr}(N_\infty)$.

Conversely, if $\theta \in \mathcal{H}$, then define $\Phi(\theta)$ to be the restriction of θ to $R = P' \cap N_\infty$. Note that by Lemma 2.5 (a), we have $\Phi(\theta) \in \mathcal{G}$. Also, if $\theta \in \mathcal{H}_0$, then $\Phi(\theta) \in \mathcal{G}_0$. Since we clearly have $\Phi \circ \Psi = \text{id}$ and $\Psi \circ \Phi = \text{id}$, the statement follows. $\square$

Note that, for a special class of inclusions of factors $P \subset N$, the equality $\chi(N, P) = \chi(N_\infty)$ was already established in [23].

4.3. Theorem. *Let $P \subset N$ be a subfactor of finite index. Assume N has spectral gap in N_∞ and $N' \cap N_\infty$ is a factor (e.g. N is non-Gamma and $P \subset N$ has finite depth).*

(i) *N_∞ is s-McDuff if and only if $P \subset N$ is a “matricial” inclusion; i.e., it is of the form $P \subset M_n(P)$, for some n , or equivalently $N = P \vee (P' \cap N)$.*

(ii) *If $P' \cap N = \mathbb{C}$ and $\Gamma_{P,N}$ is strongly amenable and its canonical weight vectors $\vec{s} = (s_k)_k$, resp. $\vec{t} = (t_l)_l$, have distinct entries (i.e. $s_k \neq s_{k'}$ for $k \neq k'$ and $t_l \neq t_{l'}$ if $l \neq l'$), then $\chi(N_\infty) = 1$.*

Proof. To prove (i), assume $Q \overline{\otimes} R = N_\infty$. Let $R_n \nearrow R$ be an increasing sequence of finite dimensional subfactors exhausting R . By Lemma 3.1 there exists n such that $N \subset_{1/4} Q \overline{\otimes} R_n$. Since $N' \cap N_\infty$ is a factor, by Proposition 12 in [10], it follows that there exists a unitary element $u \in N_\infty$ such that $uNu^* \subset Q^t$, for some $t > 0$. Thus, by replacing Q by $u^*Q^t u$ and R by $u^*R^{1/t}u$, we may assume the decomposition $N_\infty = Q \overline{\otimes} R$ is such that $N \subset Q$. Hence, $R = Q' \cap N_\infty \subset N' \cap N_\infty$.

Now note that since R splits off N_∞ , it also splits off $N' \cap N_\infty$; i.e., if we denote $B = R' \cap (N' \cap N_\infty)$, then $N' \cap N_\infty = R \overline{\otimes} B$. In particular, B is a factor.

On the other hand, by applying Lemma 3.1 again, for any $\varepsilon > 0$ there exists n such that $Q \subset_{\varepsilon/2} N_n$. Taking relative commutants, it follows that $N'_n \cap N_\infty \subset_\varepsilon R$. Thus, with the notation in [18], if we take $\varepsilon < 1$, then we get $N'_n \cap N_\infty \prec_{N_\infty} R$, and thus $R \overline{\otimes} B = N' \cap N_\infty \prec_{N_\infty} R$ as well (because $N'_n \cap N_\infty \subset N' \cap N_\infty$ has finite index). In particular, this shows that B must be a finite dimensional factor. Thus, by replacing if necessary R by $R \overline{\otimes} B$, we may actually assume $N' \cap N_\infty = R$. So $N' \cap N_\infty$ splits off N_∞ . Similarly $P' \cap N_\infty$ splits off N_∞ as well, implying that $N' \cap N_\infty$ splits off $P' \cap N_\infty$. Thus, $N' \cap N_\infty \subset P' \cap N_\infty$ is matricial, in particular extremal, which in turn implies $(P' \cap N_\infty)' \cap N_\infty = P \subset N = (N' \cap N_\infty)' \cap N_\infty$ is matricial as well.

To prove (ii), note that by Theorem 4.1 we have $\chi(N_\infty) = \chi(P' \cap N_\infty, N' \cap N_\infty)$. Note also that by part 4 of Theorem 1.6 in [14], any automorphism θ of $N' \cap N_\infty \subset P' \cap N_\infty$ either is implemented by a unitary $u \in P' \cap N_\infty$ normalizing $N' \cap N_\infty$ or is properly outer. If u is not in $N' \cap N_\infty$, then the canonical vector of $\Gamma_{N' \cap N_\infty, P' \cap N_\infty}$ would have an even vertex other than $*$ equal to 1. By strong amenability, $P = (P' \cap N_\infty)' \cap N_\infty \subset (N' \cap N_\infty)' \cap N_\infty = N$, which implies the same is true for $\Gamma_{P,N}$ and its standard vector, contradicting the hypothesis.

Thus, either $\theta = \text{Ad}(u)$ for some $u \in N' \cap N_\infty$ or θ is properly outer on the inclusions $N' \cap N_\infty \subset P' \cap N_\infty$. But by part 6 of Theorem 1.6 in [14], any properly outer automorphism of $N' \cap N_\infty \subset P' \cap N_\infty$ is centrally free; i.e., it acts freely on $N' \cap (P' \cap N_\infty)^\omega = N'_\infty \cap N_\infty^\omega$, a contradiction. Thus, $\chi(P' \cap N_\infty, N' \cap N_\infty)$ must be trivial, and so $\chi(N_\infty) = 1$ as well. $\square$

4.4. Corollary. *For each $2 \leq n \leq \infty$, the factor $L(\mathbb{F}_n) \overline{\otimes} R$ has a sequence of irreducible subfactors M_m with distinct indices satisfying $\chi(M_m) = 1$ and which are McDuff but not s-McDuff.*

Proof. By [22], for each odd number $m \geq 5$ there exists a subfactor $P \subset N = L(\mathbb{F}_n)$ of index $4 \cos^2 \pi/m$ and standard graph equal to A_{m-1} . Moreover, subfactors with such a graph have weight vectors with distinct entries, $\vec{s}_m = (s_{m,k})_k$ and the numbers $r_m = \sum_k s_{m,k}^2$ are distinct, for $m \geq 5$ (see e.g. [5] or [4]). Also, if we fix

m and denote by N_∞ the enveloping factor of the inclusion $P \subset N$ as before and let $R = N' \cap N_\infty$, then the inclusion of factors $L(\mathbb{F}_n) \vee R \subset N_\infty$ has index equal to r_m (see e.g. [4]). By [6], there exists a subfactor $M_m \subset L(\mathbb{F}_n) \vee R$ such that $L(\mathbb{F}_n) \vee R \subset N_\infty$ is the basic construction of $M_m \subset L(F_n) \vee R$, and by [12], M_m is an amplification of N_∞ . Thus, since by Theorem 4.3 N_∞ is not s-McDuff and $\chi(N_\infty) = 1$, it follows that M_m satisfies the same properties as well. $\square$

4.5. Theorem. *Let $P \subset N$ be an inclusion of non-Gamma II_1 factors with Temperley-Lieb standard invariant and either $[N : P] < 4$, $\Gamma_{P,N} = A_{2n}$ for some $n \geq 2$ or $[N : P] \geq 4$, $\Gamma_{P,N} = A_\infty$. Assume N has spectral gap in N_∞ (note that this is automatic in the case where $[N : P] < 4$). If $[N : P] \leq 4$, then $\chi(N_\infty) = 1$; while if $[N : P] > 4$, then $\chi(N_\infty) = \mathbb{T}$.*

Proof. The case $[N : P] \leq 4$ is a consequence of 4.3.(ii). To prove the case $[N : P] > 4$, note that by [13], the inclusion of factors $(P' \cap N_\infty)' \cap N_\infty = \tilde{P} \subset \tilde{N} = (N' \cap N_\infty)' \cap N_\infty$ is locally trivial. More precisely, there is an isomorphism $\sigma : p\tilde{N}p \simeq (1-p)\tilde{N}(1-p)$, where $p \in \tilde{P} \cap \tilde{N}$ is a projection satisfying $\tau(p)\tau(1-p) = [N : P]^{-1}$ such that $\tilde{P} = \{x + \sigma(x) \mid x \in p\tilde{N}p\}$. Also, the enveloping factor of $\tilde{P} \subset \tilde{N}$ is equal to N_∞ and $\tilde{N}$ has spectral gap in N_∞ (because N does). But then Corollary 2.7.1 applies, showing that any element in $\overline{\text{Int}(N_\infty)} \cap \text{Ctr}(N_\infty)/\text{Int}(N_\infty)$ is represented by an automorphism θ of N_∞ which acts trivially on $\tilde{N}$ and for which there exist a non-zero partial isometry $w \in \tilde{N}' \cap N_\infty$ and some m such that $\theta(x)w = wx$, $\forall x \in \tilde{N}'_m \cap N_\infty$. Note that w^*w lies in $(\tilde{N}'_m \cap N_\infty)' \cap (\tilde{N}' \cap N_\infty) = \tilde{N}' \cap \tilde{N}_m$.

By multiplying from the right on both sides with a minimal projection $q \leq w^*w$ in $\tilde{N}' \cap \tilde{N}_m$, we may in fact assume $w^*w = q$. Since the minimal projections in distinct direct summands of $\tilde{N}' \cap \tilde{N}_m$ have distinct traces, it follows that both ww^* and $\theta(q)$ are minimal projections in the same direct summand of $\theta(\tilde{N}' \cap \tilde{N}_m)$. Thus, by multiplying w from the left with an appropriate partial isometry in $\theta(\tilde{N}' \cap \tilde{N}_m)$, we may assume $ww^* = \theta(q)$. So, now we have $\theta(x)w = wx$, $\forall x \in (\tilde{N}'_m \cap N_\infty)q = q(\tilde{N}' \cap N_\infty)q$. But this implies there exists a unitary element $v \in \tilde{N}' \cap N_\infty$ such that $\theta = \text{Adv}$ on $\tilde{N}' \cap N_\infty$. Altogether, this shows that any element in $\chi(N_\infty)$ can be represented by an automorphism acting trivially on $Q = \tilde{N} \vee \tilde{N}' \cap N_\infty$; i.e., $\chi(N_\infty)$ naturally embeds into the group G of automorphisms of N_∞ acting trivially on Q .

But N_∞ is isomorphic to the crossed product of Q by an aperiodic automorphism, whose amplification to $Q \overline{\otimes} \mathcal{B}(\mathcal{H}) \overline{\otimes} \mathcal{B}(\mathcal{H})$ acts by $\sigma \otimes \sigma_0$, where σ is an automorphism of $\tilde{N} \overline{\otimes} \mathcal{B}(\mathcal{H})$ scaling the trace by $t/(1-t)$ and σ_0 is an automorphism of $\tilde{N}' \cap N_\infty \overline{\otimes} \mathcal{B}(\mathcal{H}) \simeq R \overline{\otimes} \mathcal{B}(\mathcal{H})$ scaling the trace by $(1-t)/t$ (given by the inclusion $\tilde{N}' \cap N_\infty \subset \tilde{P}' \cap N_\infty$). Thus $G = \mathbb{T}$, with each $\lambda \in \mathbb{T}$ corresponding to the automorphism θ_λ of $N_\infty = Q \rtimes \mathbb{Z}$ which leaves Q pointwise fixed and satisfies $\theta_\lambda(u) = \lambda u$, where $u \in N_\infty$ is the canonical unitary implementing the action of $\mathbb{Z}$ on Q . To see that each θ_λ lies in $\overline{\text{Int}(N_\infty)} \cap \text{Ctr}(N_\infty)$, note that one can take a sequence of unitary elements u_n in the hyperfinite II_1 factor $R = \tilde{N}' \cap N_\infty$ such that $\sigma_0(u_n) = \lambda u_n$ and $(u_n)_n \in R' \cap R^\omega$. To see this, note that by Connes' classification of trace scaling automorphisms of the hyperfinite II_∞ factor $R^\infty \simeq \tilde{N}' \cap N_\infty \overline{\otimes} \mathcal{B}(\mathcal{H})$ ([1]), we may assume $\tilde{N}' \cap N_\infty = R_1 \overline{\otimes} R_2$ and $\sigma_0 = \sigma_1 \otimes \sigma_2$, with σ_1 an automorphism of $R_1 \overline{\otimes} \mathcal{B}(\mathcal{H})$ scaling the trace by $(1-t)/t$ and σ_2 an automorphism of $R_2 = R^{\otimes \infty}$ which is an infinite tensor product $\sigma_2 = \alpha^{\otimes \infty}$ of copies of an automorphism α of R such that $\alpha(v) = \lambda v$, for some $v \in \mathcal{U}(R)$. Thus, if we take $u_n = 1 \otimes 1 \otimes \dots \otimes v \otimes 1 \dots$,

with v on the n th position of the infinite tensor product, then $\theta = \lim_n \text{Ad} u_n$ acts trivially on Q while $\theta(u) = \lambda u$. Moreover, θ clearly belongs to $\overline{\text{Int}}(N_\infty) \cap \text{Ctr}(N_\infty)$ by the way it was defined. $\square$

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