TRACE-POSITIVE COMPLEX POLYNOMIALS IN THREE UNITARIES

STANISLAV POPOVYCH

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ABSTRACT. We consider the quadratic polynomials in three unitary generators, i.e. the elements of the group $*$ -algebra of the free group with generators u_1, u_2, u_3 of the form $f = \sum_{j,k=1}^3 \alpha_{jk} u_j^* u_k$, $\alpha_{jk} \in \mathbb{C}$. We prove that if f is self-adjoint and $\text{Tr}(f(U_1, U_2, U_3)) \geq 0$ for arbitrary unitary matrices U_1, U_2, U_3 , then f is a sum of hermitian squares. To prove this statement we reduce it to the question whether a certain Tarski sentence is true. Tarski's decidability theorem thus provides an algorithm to answer this question. We use an algorithm due to Lazard and Rouillier for computing the number of real roots of a parametric system of polynomial equations and inequalities implemented in Maple to check that the Tarski sentence is true.

As an application, we describe the set of parameters a_1, a_2, a_3, a_4 such that there are unitary operators $U_1, \dots, U_4$ connected by the linear relation $a_1 U_1 + a_2 U_2 + a_3 U_3 + a_4 U_4 = 0$.

1. INTRODUCTION

Let R be a hyperfinite II_1 factor and $\omega \in \beta(\mathbb{N}) \setminus \mathbb{N}$ be a free ultrafilter on $\mathbb{N}$. Connes' embedding conjecture states that every von Neumann II_1 factor $\mathcal{M}$ with a separable predual and a fixed normal faithful tracial state can be embedded into the ultrapower R^ω in a trace-preserving way.

The following algebraic reformulation of the conjecture was obtained in [6].

Theorem. *Let $\mathbb{F}$ denote the free group with a countable family of free generators $\{u_1, u_2, \dots\}$. Denote by $\mathcal{F}$ its group $*$ -algebra. Connes' embedding conjecture is true iff for any selfadjoint $f \in \mathcal{F}$ of the form*

$$f(u_1, \dots, u_n) = \sum_{i,j=1}^n \alpha_{ij} u_i^* u_j, \quad \alpha_{ij} \in \mathbb{C}, \quad i, j = 1, \dots, n,$$

the condition

$$(1.1) \quad \text{Tr}(f(V_1, \dots, V_n)) \geq 0$$

for every $m \geq 1$ and every n -tuple of unitary $m \times m$ matrices $V_1, \dots, V_n$ implies that

$$(1.2) \quad \text{for every } \varepsilon > 0, \quad \varepsilon e + f = g + c,$$

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where e is the unit of $\mathcal{F}$, c is a sum of commutators in $\mathcal{F}$ and g is a sum of hermitian squares.

Here, a *commutator* means a ring commutator, i.e. an element of the form $gh - hg$ with $h, g \in \mathcal{F}$. A *hermitian square* in $\mathcal{F}$ is an element of the form h^*h with $h \in \mathcal{F}$.

We will call the elements of the form $f = \sum_{i,j=1}^n \alpha_{ij} u_i^* u_j$ *quadratic polynomials in unitary variables*. If condition (1.1) holds, we will call f *trace-positive*. An element f such that $f(U_1, \dots, U_n)$ is a positive operator for every unitary operator $U_1, \dots, U_n$ will be called *positive*.

It was proved in [12] that every self-adjoint trace-positive quadratic polynomial with real coefficients is a sum of hermitian squares. In particular, every such polynomial is positive semidefinite. However, little is known about trace-positive self-adjoint quadratic polynomials with complex coefficients.

The aim of the present paper is to show that every trace-positive complex quadratic polynomial in three unitaries is a sum of hermitian squares. Firstly, we will reduce this statement to the validity of a Tarski sentence. It is well known that there exists an algorithm to check that a given Tarski sentence is true or false (see [14]). However, this algorithm is quite slow to be of practical help. Fortunately, there is an alternative approach based on computational methods of real algebraic geometry. In Section 2, we use the algorithm due to D. Lazard and F. Rouillier for counting the number of real roots of a multivariate system of polynomial equations and inequalities to prove the above result. It is based on the notion of a discriminant variety (see [9]) and has been implemented in Maple.

Note that a quadratic self-adjoint polynomial f in 3 generators, by the change of variables $u = u_1^* u_2, v = u_1^* u_3$, can be presented in the form $f(u, v) = a_0 + au + \bar{a}u^* + bv + \bar{b}v^* + cu^*v + \bar{c}v^*u$ with $a_0 \in \mathbb{R}, a, b, c \in \mathbb{C}$. Thus, f is trace-positive if and only if

$$a_0 + 2\operatorname{Re}(a \operatorname{tr}(U) + b \operatorname{tr}(V) + c \operatorname{tr}(U^*V)) \geq 0$$

for all unitary matrices $U, V \in M_m(\mathbb{C})$ with arbitrary $m \geq 1$. Here, $\operatorname{tr}(\cdot) = \frac{1}{m} \operatorname{Tr}(\cdot)$ denotes the normalized trace. Thus, a description of the trace-positive of the trace positive polynomials in 3 variables amounts to a description of the closure in $\mathbb{C}^3$ of the set

$$\begin{aligned} T &= \{(\operatorname{tr}(U), \operatorname{tr}(V), \operatorname{tr}(U^*V)) \mid m \geq 1, U, V \in \mathcal{U}(m)\} \\ &= \{(\operatorname{tr}(U), \operatorname{tr}(V), \operatorname{tr}(UV)) \mid m \geq 1, U, V \in \mathcal{U}(m)\}, \end{aligned}$$

where $\mathcal{U}(m)$ denotes the group of the unitary $m \times m$ matrices.

Note that the lists of the possible eigenvalues of U , V and UV can be completely described by a generalization of Horn's inequalities (see [1, 5]). However, only a simple connection between the normalized traces of two unitaries and their product has been known until recently. It is the inequality

$$\sqrt{1 - |\operatorname{tr}(UV)|^2} \leq \sqrt{1 - |\operatorname{tr}(U)|^2} + \sqrt{1 - |\operatorname{tr}(V)|^2},$$

established in [15].

Let Θ_n denote the set of all positive semidefinite matrices with 1's on the main diagonal. Let $\mathcal{F}_n$ denote the closure of the set of matrices

$$\{(\operatorname{tr}(U_k^* U_j))_{k,j} \mid m \in \mathbb{N}, U_1, \dots, U_n \in \mathcal{U}(m)\},$$

where $\mathcal{U}(m)$ is the set of the unitary $m \times m$ matrices. Clearly, $\mathcal{F}_n \subseteq \Theta_n$. It can be easily inferred from the results of [4] that the set T is closed, and thus the map

$$(a, b, c) \mapsto \begin{pmatrix} 1 & a & b \\ \bar{a} & 1 & c \\ \bar{b} & \bar{c} & 1 \end{pmatrix}$$

is one-to-one from T onto $\mathcal{F}_3$. A complete description of $\mathcal{F}_3$ was obtained in [4]. However, no description of the set $\mathcal{F}_n$ for $n > 3$ is known at present. Let $\mathcal{F}_n^c$ denote the closure of the set of matrices $(\operatorname{tr}(U_k^* U_m))_{k,m=1}^n$, where unitary matrices $U_1, \dots, U_n \in \mathcal{U}(d)$ (of arbitrary size $d \geq 1$) commute:

$$U_k U_m = U_m U_k \text{ for all } k, m = 1, \dots, n.$$

The set $\mathcal{F}_n^c$ can be alternatively described as the convex hull of the rank one matrices from Θ_n (see [4]).

The main result of the paper is the following:

Theorem 1.1. *For a self-adjoint quadratic polynomial*

$$(1.3) \quad f(u_1, u_2, u_3) = a_0 + au_1^* u_2 + \bar{a}u_2^* u_1 + bu_1^* u_3 + \bar{b}u_3^* u_1 + cu_2^* u_3 + \bar{c}u_3^* u_2$$

the following statements are equivalent:

- (1) $f(z_1, z_2, z_3) \geq 0$ for $z_1, z_2, z_3 \in \mathbb{C}$ and $|z_1| = |z_2| = |z_3| = 1$.
- (2) f is trace-positive.
- (3) f is positive.
- (4) f is a sum of hermitian squares.

As a byproduct of Lemma 2.1 we also show the equality $\Theta_3 = \mathcal{F}_3 = \mathcal{F}_3^c$ proved earlier in [4] by different methods.

Remark 1.2. It was also shown in [4] that $\mathcal{F}_6 \neq \mathcal{F}_6^c$. Thus, for self-adjoint quadratic polynomials f in 6 unitary variables the condition that $f(z_1, \dots, z_4) \geq 0$ for every $z_1, \dots, z_6 \in \mathbb{C}$ with $|z_1| = \dots = |z_6| = 1$ does not imply that f is trace-positive. It is still unknown, however, whether $\mathcal{F}_4 = \mathcal{F}_4^c$.

A polynomial f of the form (1.3) will be called Θ_3 -positive if $a_0 + 2 \operatorname{Re}(a\alpha + b\beta + c\gamma) \geq 0$ for every matrix

$$A = \begin{pmatrix} 1 & \alpha & \beta \\ \bar{\alpha} & 1 & \gamma \\ \bar{\beta} & \bar{\gamma} & 1 \end{pmatrix} \in \Theta_3.$$

Clearly, every Θ_3 -positive polynomial is trace-positive. Put $Q(a, b, c, A) = \operatorname{Re}(a\alpha + b\beta + c\gamma)$ and $P(a, b, c, \phi, \psi) = \operatorname{Re}(ae^{i\phi} + be^{i\psi} + ce^{i(\psi-\phi)})$.

Since for every Θ_3 -positive f we have

$$a_0 + 2 \operatorname{Re}(a \operatorname{tr}(V_1) + b \operatorname{tr}(V_2) + c \operatorname{tr}(V_1^* V_2)) \geq 0$$

with $V_1 = e^{i\phi}$, $V_2 = e^{i\psi}$, we get the following inequality:

$$\min_{A \in \Theta_3} Q(a, b, c, A) \leq \min_{\phi, \psi \in [0, 2\pi)} P(a, b, c, \phi, \psi).$$

Meanwhile, if $a_0 \geq x_1 + x_2 + x_3$ for some x_1, x_2, x_3 such that the matrix

$$B = \begin{pmatrix} x_1 & a & b \\ \bar{a} & x_2 & c \\ \bar{b} & \bar{c} & x_3 \end{pmatrix} \geq 0,$$

then the Schur product $B \circ A$ is positive semidefinite for every $A \in \Theta_3$ and, consequently,

$$x_1 + x_2 + x_3 + 2\operatorname{Re}(a\alpha + b\beta + c\gamma) = (1, 1, 1)(B \circ A)(1, 1, 1)^T \geq 0.$$

Hence,

(1.4)

$$\min \left\{ x_1 + x_2 + x_3 \mid \begin{pmatrix} x_1 & a & b \\ \bar{a} & x_2 & c \\ \bar{b} & \bar{c} & x_3 \end{pmatrix} \geq 0 \right\} \geq \max_{A \in \Theta_3} (-2Q(a, b, c, A))$$

(1.5) $\geq \max_{\phi, \psi \in [0, 2\pi)} (-2P(a, b, c, \phi, \psi)).$

2. PROOFS

Lemma 2.1. *For every $a, b, c \in \mathbb{C}$ the following formula holds:*

(2.1)

$$\min \left\{ x_1 + x_2 + x_3 \mid \begin{pmatrix} x_1 & a & b \\ \bar{a} & x_2 & c \\ \bar{b} & \bar{c} & x_3 \end{pmatrix} \geq 0 \right\} = \max_{\phi, \psi \in [0, 2\pi)} (-2P(a, b, c, \phi, \psi)).$$

Remark 2.2. The analog of formula (2.1) for the 4×4 matrices does not hold (see Remark 2.5).

Proof. By inequalities (1.4)-(1.5), the above equality holds if and only if there exists $x_1, x_2, x_3, \phi, \psi$ such that

$$(2.2) \quad x_1 + x_2 + x_3 = -2\operatorname{Re}(ae^{i\phi} + be^{i\psi} + ce^{i(\psi-\phi)}),$$

$$(2.3) \quad \begin{pmatrix} x_1 & a & b \\ \bar{a} & x_2 & c \\ \bar{b} & \bar{c} & x_3 \end{pmatrix} \geq 0.$$

Clearly, it is enough to prove the above formula for $a \geq 0$ and $b \geq 0$ and $c = c_r + ic_i$ with $c_r, c_i \in \mathbb{R}$. This restriction will be assumed in the sequel.

If $x_1, x_2, x_3, \phi, \psi$ satisfy (2.2)-(2.3), then

$$A = \begin{pmatrix} x_1 & a & b \\ \bar{a} & x_2 & c \\ \bar{b} & \bar{c} & x_3 \end{pmatrix} \geq 0.$$

Hence, the matrix $XAX^* \geq 0$ for every matrix X . With $X = \operatorname{diag}(1, e^{i\phi}, e^{i\psi})$, we have

$$B = XAX^* = \begin{pmatrix} x_1 & ae^{i\phi} & be^{i\psi} \\ \bar{a}e^{-i\phi} & x_2 & ce^{i(\psi-\phi)} \\ \bar{b}e^{-i\psi} & \bar{c}e^{-i(\psi-\phi)} & x_3 \end{pmatrix} \geq 0.$$

It is easy to check that equation (2.2) is equivalent to $(1, 1, 1)B(1, 1, 1)^T = 0$. The positivity of B implies that $B(1, 1, 1)^T = 0$. Hence,

$$(2.4) \quad x_1 = -ae^{i\phi} - be^{i\psi},$$

$$(2.5) \quad x_2 = -\bar{a}e^{-i\phi} - ce^{i(\psi-\phi)},$$

$$(2.6) \quad x_3 = -\bar{b}e^{-i\psi} - \bar{c}e^{-i(\psi-\phi)}.$$

Write $e^{i\phi} = u_1 + iv_1$ and $e^{i\psi} = u_2 + iv_2$. The conditions

$$(2.7) \quad u_1^2 + v_1^2 = 1, u_2^2 + v_2^2 = 1,$$

$$(2.8) \quad \operatorname{Im}(x_1) = \operatorname{Im}(x_2) = \operatorname{Im}(x_3) = 0$$

are equivalent to the following formulas:

$$(2.9) \quad u_1 = -\frac{c_r x b \left((b^2(a^2 - |c|^2) + a^2 |c|^2)x^2 - 2xa^3c_i + a^2c_i^2 \right)}{2b^2a^2|c|^2x^3 - \left(b^2a^2 + (b^2 + a^2)|c|^2 \right)ac_ix^2 + a^3c_i^3},$$

$$(2.10) \quad v_1 = \frac{-bx}{a},$$

$$(2.11) \quad u_2 = \frac{1}{2} \frac{2b^2ac_ix^3 + \left(-a^2b^2 + a^2(c_r^2 - c_i^2) - b^2|c|^2 \right)x^2 + a^2c_i^2}{xac_r(xb^2 - ac_i)},$$

$$(2.12) \quad v_2 = x,$$

where x is any real root of the polynomial

$$\begin{aligned} q(x) = & 4b^4a^2|c|^2x^6 - 4ab^2c_i(b^2|c|^2 + a^2|c|^2 + b^2a^2)x^5 \\ & + (4b^2a^3(c^2 + c_r^2)c_i + 2b^4a^2|c|^2 + (a^2 + b^2)^2|c|^4 \\ & - b^2a^2(4|c|^2c_r^2 + 2a^2|c|^2 - 4a^2c_r^2 - 4c_r^2b^2 + a^2b^2))x^4 \\ & + 4b^2a^3c_i(c_i^2 + 2c_r^2)x^3 \\ & - 2a^2(|c|^2 - c_r^2)(b^2|c|^2 + a^2c^2 + b^2a^2)x^2 + a^4c_i^4. \end{aligned}$$

The substitution of the above expressions for u_1, v_1, u_2, v_2 into the formulas for x_1, x_2, x_3 and then into the matrix B gives us an expression for the coefficients of B as rational functions depending on a, b, c_r, c_i, x . Thus,

$$(2.13) \quad b_{11} = \frac{p_1(a, b, c_r, c_i, x)}{q_1(a, b, c_r, c_i, x)},$$

$$(2.14) \quad \det \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \frac{p_2(a, b, c_r, c_i, x)}{q_2(a, b, c_r, c_i, x)},$$

$$(2.15) \quad \det B = 0,$$

where p_1, p_2, q_1, q_2 are some polynomials in a, b, c_r, c_i, x which can be easily computed explicitly. Thus, formulas (2.2)-(2.3) are valid if the following Tarski sentence is true:

$$(2.16) \quad \forall a \forall b \forall c_r \forall c_i \exists x \ q(x) = 0 \ \& \ p_1(a, b, c_r, c_i, x)q_1(a, b, c_r, c_i, x) > 0$$

$$(2.17) \quad \& p_2(a, b, c_r, c_i, x)q_2(a, b, c_r, c_i, x) > 0.$$

We need to prove that $q(x)$ has at least one *real* root for arbitrary parameters a, b, c .

However, there exists an algorithm (see [14]) to check the validity of any Tarski sentence, but it is quite slow in practice. We find it more efficient to use an algorithm for counting the number of real roots of a multivariate system of polynomial equations and inequalities based on the notion of a discriminant variety (see [9]).

For a basic semi-algebraic set

$$\mathcal{S} = \{x \in \mathbb{R}^n \mid p_1(x) = 0, \dots, p_s(x) = 0, f_1(x) > 0, \dots, f_l(x) > 0\},$$

and a basic constructable set

$$\mathcal{C} = \{x \in \mathbb{R}^n \mid p_1(x) = 0, \dots, p_s(x) = 0, f_1(x) \neq 0, \dots, f_l(x) \neq 0\},$$

where p_j, f_t are polynomials with rational coefficients, the notion of *discriminant variety* has been introduced in [9].

The indeterminates $x = (U_1, \dots, U_d, X_{d+1}, \dots, X_n)$ are partitioned into two sets: *the parameters* $(U_1, \dots, U_d)$ and *the unknowns* $(X_{d+1}, \dots, X_n)$. Let $\Pi_U: \mathbb{C}^n \rightarrow \mathbb{C}^d$,

$$\Pi_U(U_1, \dots, U_d, X_{d+1}, \dots, X_n) = (U_1, \dots, U_d)$$

be the canonical projection on the parameter space. For a given ideal I in the ring of (commutative) polynomials, we denote by $\mathbf{V}(I)$ the associated algebraic variety. Let $\overline{\Pi_U(\mathcal{C})}$ be the closure of $\Pi_U(\mathcal{C})$ in the topology of $\mathbb{C}^n$ or Zariski closure (which are the same [11]).

Definition 2.3 (Lazard, Rouillier). An algebraic variety W is a *discriminant variety* of $\mathcal{C}$ with respect to Π_U if

- (1) $W \subseteq \overline{\Pi_U(\mathcal{C})}$.
- (2) $W = \overline{\Pi_U(\mathcal{C})}$ if and only if $\Pi_U^{-1}(u) \cap \mathcal{C}$ is infinite for almost all $u \in \overline{\Pi_U(\mathcal{C})}$.
- (3) The connected components $\mathcal{U}_1, \dots, \mathcal{U}_k$ of $\overline{\Pi_U(\mathcal{C})} \setminus W$ are analytic submanifolds (i.e., nonsingular) of dimension δ .
- (4) $(\Pi_U^{-1}(\mathcal{U}_i) \cap \mathcal{C}, \Pi_U)$ is an analytic covering for $\mathcal{U}_i$ for all i .

There is an algorithm for computing the minimal discriminant variety described in [9]. It has been implemented as Maple package `RootFinding[Parametric]`. In particular, for the constructable set

$$\{q(x)=0, p_1(a, b, c_r, c_i, x)q_1(a, b, c_r, c_i, x)>0, p_2(a, b, c_r, c_i, x)q_2(a, b, c_r, c_i, x)>0\},$$

the minimal discriminant variety is the union

$$\begin{aligned} & \mathbf{V}(a) \cup \mathbf{V}(b) \cup \mathbf{V}(c_i) \cup \mathbf{V}(c_r) \cup \mathbf{V}(a-b) \cup \mathbf{V}(a+b) \\ & \cup \mathbf{V}(c_r^2 + c_i^2) \cup \mathbf{V}(b^2 - c_r^2 - c_i^2) \cup \mathbf{V}(a^2 - c_r^2 - c_i^2) \cup \mathbf{V}(f), \end{aligned}$$

where

$$\begin{aligned}
f = & (6a^{10}b^2 + 20a^6b^6 + b^{12} + 6a^2b^{10} + a^{12} + 15a^8b^4 + 15a^4b^8) |c|^{12} \\
& + (-12a^{10}b^2c_r^2 + 6a^{12}b^2 - 48a^4b^8c_r^2 - 12a^2b^{10}c_r^2 - 48a^8c_r^2b^4 \\
& - 102a^6b^8 - 24a^{10}b^4 + 6a^2b^{12} - 102a^8b^6 - 24a^4b^{10} - 72a^6c_r^2b^6) |c|^{10} \\
& + (495a^8b^8 + 15a^4b^{12} + 60a^6c_r^2b^8 - 102a^6b^{10} + 15a^{12}b^4 \\
& + 48a^4b^8c_r^4 - 12a^{12}c_r^2b^2 - 12a^2c_r^2b^{12} + 96a^6c_r^4b^6 + 48a^8c_r^4b^4 + 60a^8c_r^2b^6 \\
& - 102a^{10}b^6 + 216a^4b^{10}c_r^2 + 216a^{10}b^4c_r^2) |c|^8 \\
& + (-174a^{10}b^4c_r^4 - 48a^{12}b^4c_r^2 + 6a^6c_r^4b^8 - 102a^{10}b^8 - 174a^4b^{10}c_r^4 \\
& - 432a^8c_r^2b^8 - 64a^6c_r^6b^6 + 20a^{12}b^6 - 48a^4b^{12}c_r^2 - 102a^8b^{10} + 20a^6b^{12} \\
& + 6a^8c_r^4b^6 + 60a^{10}b^6c_r^2 + 60a^6c_r^2b^{10}) |c|^6 \\
& + (-72a^6c_r^2b^{12} + 24a^6c_r^6b^8 + 15a^{12}b^8 + 60a^8c_r^2b^{10} \\
& + 24a^8c_r^6b^6 + 6a^{10}b^6c_r^4 - 24a^{10}b^{10} - 90a^8c_r^4b^8 \\
& + 60a^{10}b^8c_r^2 + 15a^8b^{12} + 48a^{12}b^4c_r^4 - 72a^{12}b^6c_r^2 + 48a^4b^{12}c_r^4 + 6a^6c_r^4b^{10}) |c|^4 \\
& + (48a^8c_r^6b^8 + 6a^8c_r^4b^{10} + 96a^{12}c_r^4b^6 + 6a^{10}b^8c_r^4 - 48a^{12}c_r^2b^8 - 48a^8c_r^2b^{12} \\
& + 24a^6c_r^6b^{10} + 24a^{10}b^6c_r^6 + 216a^{10}b^{10}c_r^2 + 96a^6c_r^4b^{12} + 6a^{10}b^{12} + 6a^{12}b^{10}) |c|^2 \\
& + 24a^8c_r^6b^{10} - 64a^6c_r^6b^{12} + 48a^{12}c_r^4b^8 - 12a^{12}c_r^2b^{10} \\
& - 64a^{12}c_r^6b^6 - 174a^{10}b^{10}c_r^4 + 24a^{10}b^8c_r^6 - 12a^{10}b^{12}c_r^2 \\
& + 48a^8c_r^4b^{12} + a^{12}b^{12} - 27a^8c_r^8b^8.
\end{aligned}$$

The algorithm also provides us with an open cylindrical algebraic decomposition (see [3]) of the set $T = (\overline{\Pi_U(\mathcal{C})} \setminus W) \cap \mathbb{R}^4$ compatible with the set of polynomials F defining W . This gives a representation of T as a disjoint finite union of cells (semi-algebraic sets isomorphic to $\mathbb{R}^4$). Each cell is contained in a connected component of T , and each connected component of T has the same closure as the union of the cells it contains. Moreover, each polynomial in F has a constant sign on each cell of the decomposition. The number of real solutions is constant for parameters belonging to the same cell.

In our case there are 2688 cells, the number of real solutions ranging between 1 and 3. In particular, this calculation proves that system (2.2)-(2.3) has a solution $x_1, x_2, x_3, \phi, \psi$ for every vector of parameters in T . Hence, equation (2.1) holds for every $(a, b, c) \in T$.

Since both the left-hand side and the right-hand side of (2.1) are, clearly, continuous functions in a, b, c_r, c_i , it is therefore sufficient to check formula (2.1) for (a, b, c_r, c_i) belonging to some Zariski open subset. In particular, we can take the subset T . This proves that (2.1) holds for all complex a, b, c . $\square$

Proof of Theorem 1.1. If $f(z_1, z_2, z_3) \geq 0$ for complex z_1, z_2, z_3 of modulus one, then $a_0 + 2\operatorname{Re}(ae^{i\phi} + be^{i\psi} + ce^{i(\psi-\phi)}) \geq 0$ for all $\phi, \psi \in \mathbb{R}$. By Lemma 2.1, there are $x_1, x_2, x_3 \in \mathbb{R}$ such that $a_0 \geq x_1 + x_2 + x_3$ and the matrix

$$\begin{pmatrix} x_1 & a & b \\ \bar{a} & x_2 & c \\ \bar{b} & \bar{c} & x_3 \end{pmatrix}$$

is positive semidefinite. From [12, Lemma 4] it follows that f is a sum of hermitian squares. This proves the implication (1) $\Rightarrow$ (4). The rest of the equivalences follow easily. $\square$

Corollary 2.4. $\Theta_3 = \mathcal{F}_3 = \mathcal{F}_3^c$.

Proof. Clearly $\mathcal{F}_3^c \subseteq \mathcal{F}_3 \subseteq \Theta_3$. The sets Θ_3 and $\mathcal{F}_3^c$ are closed convex sets in $\mathbb{C}^3$. If we assume that $\mathcal{F}_3^c \subsetneq \Theta_3$, then we can find a linear functional $F : \mathbb{C}^3 \rightarrow \mathbb{R}$ and a real number t such that $F(v) \geq t$ for $v \in \mathcal{F}_3^c$ but $F(w) < t$ for some $w \in \Theta_3$. Clearly, the functional F can be represented as $F(\alpha, \beta, \gamma) = \operatorname{Re}(\alpha a + \beta b + \gamma c)$ for some $a, b, c \in \mathbb{C}$. From (1.5) and Lemma 2.1, we have the equality

$$\max_{A \in \Theta_3} (-Q(a, b, c, A)) = \max_{\phi, \psi \in [0, 2\pi)} (-P(a, b, c, \phi, \psi)).$$

This gives a contradiction since the left-hand side of this equality is greater than $-t$ and the right-hand side is less than or equal to $-t$. $\square$

Remark 2.5. If the natural analog of Lemma 2.1 for the 4×4 matrices were valid, then, as in the proof above, we would have $\Theta_4 = \mathcal{F}_4^c$. However, it was shown in [4] that $\Theta_4 \neq \mathcal{F}_4^c$. Thus, Lemma 2.1 cannot be generalized for the matrices of bigger size.

3. APPLICATION. LINEAR COMBINATIONS OF FOUR UNITARIES

For $\beta_1, \dots, \beta_4 \in \mathbb{C}$, consider the $*$ -algebra given by the generators and relations

$$\mathcal{U}_{\beta_1, \beta_2, \beta_3, \beta_4} = \mathbb{C}\langle u_1, u_2, u_3, u_4 \mid \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3 + \beta_4 u_4 = 0 \rangle.$$

As an application of Lemma 2.1, we get the following theorem.

Theorem 3.1. *Let $\beta_1, \dots, \beta_4 \in \mathbb{C}$. The $*$ -algebra $\mathcal{U}_{\beta_1, \beta_2, \beta_3, \beta_4}$ has a representation on a Hilbert space H if and only if it has a one-dimensional representation.*

Proof. Let $\pi : \mathcal{U}_{\beta_1, \beta_2, \beta_3, \beta_4} \rightarrow B(H)$ be a representation. Then the unitary operators $U_k = \pi(u_k)$ ($k = 1, 2, 3, 4$), acting in a Hilbert space H , satisfy the linear relation

$$\beta_1 U_1 + \dots + \beta_4 U_4 = 0.$$

To simplify notation, we will rewrite this relation in the form

$$Z = \alpha_1 U + \alpha_2 V + \alpha_3 W$$

with U, V, W, Z unitary operators. Without loss of generality, we can assume that $\alpha_1, \alpha_2, \alpha_3$ are real. We need to show that there are $\phi_1, \dots, \phi_4 \in \mathbb{R}$ such that

$$(3.1) \quad e^{i\phi_4} = \alpha_1 e^{i\phi_1} + \alpha_2 e^{i\phi_2} + \alpha_3 e^{i\phi_3}.$$

Let u, v, w, z be the unit vectors which are the first columns of the operators U, V, W, Z which are given by (possibly infinite) matrices in some orthonormal basis. Then $z = \alpha u + \beta v + \gamma w$. Thus,

$$\langle \alpha_1 u + \alpha_2 v + \alpha_3 w, \alpha_1 u + \alpha_2 v + \alpha_3 w \rangle = 1.$$

Hence,

$$(3.2) \quad \operatorname{Re}(\alpha_1 \alpha_2 \langle u, v \rangle + \alpha_1 \alpha_3 \langle u, w \rangle + \alpha_2 \alpha_3 \langle v, w \rangle) = \frac{1}{2}(1 - \alpha_1^2 - \alpha_2^2 - \alpha_3^2).$$

Consider the set $X_{\alpha_1, \alpha_2, \alpha_3}$ consisting of the real numbers

$$2 \operatorname{Re}(\alpha_1 \alpha_2 \langle u_1, u_2 \rangle + \alpha_1 \alpha_3 \langle u_1, u_3 \rangle + \alpha_2 \alpha_3 \langle u_2, u_3 \rangle),$$

where u_1, u_2, u_3 runs over all triples of unit vectors in a separable Hilbert space. Since $(H, \operatorname{Re} \langle \cdot, \cdot \rangle)$ defines a *real* Hilbert space, the set $X_{\alpha_1, \alpha_2, \alpha_3}$ is the same as the set

$$\{2(\alpha_1 \alpha_2 \langle u_1, u_2 \rangle + \alpha_1 \alpha_3 \langle u_1, u_3 \rangle + \alpha_2 \alpha_3 \langle u_2, u_3 \rangle) \mid (u_1, u_2, u_3) \in \mathcal{U}_3\},$$

where $\mathcal{U}_3$ consists of triples of unit vectors in a separable real Hilbert space.

The set consisting of the matrices

$$\begin{pmatrix} 1 & \langle u_1, u_2 \rangle & \langle u_1, u_3 \rangle \\ \langle u_1, u_2 \rangle & 1 & \langle u_2, u_3 \rangle \\ \langle u_1, u_3 \rangle & \langle u_2, u_3 \rangle & 1 \end{pmatrix}$$

coincides with the real positive semidefinite matrices with 1's on the main diagonal, which is a convex set. Thus, $X_{\alpha_1, \alpha_2, \alpha_3}$ must be a closed convex bounded subset of $\mathbb{R}$. Let $a = \min X$, $b = \max X$, i.e. $X_{\alpha_1, \alpha_2, \alpha_3} = [a, b]$.

It was proved in [12] that for any set of coefficients $(a_{ij})_{i < j} \in \mathbb{R}^{n(n-1)/2}$ one has

$$\begin{aligned} & \min \left\{ x_1 + \cdots + x_n \mid \begin{pmatrix} x_1 & a_{12} & \cdots & a_{1n} \\ a_{12} & x_2 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & x_n \end{pmatrix} \geq 0 \right\} \\ &= -\min \left\{ 2 \sum_{\substack{i, j=1 \\ i < j}}^n a_{ij} q_{ij} \mid \begin{pmatrix} 1 & q_{12} & \cdots & q_{1n} \\ q_{12} & 1 & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{1n} & q_{2n} & \cdots & 1 \end{pmatrix} \geq 0 \right\}. \end{aligned}$$

In particular, we have that

$$\begin{aligned} & 2 \max \{ \alpha_1 \alpha_2 \langle u_1, u_2 \rangle + \alpha_1 \alpha_3 \langle u_1, u_3 \rangle + \alpha_2 \alpha_3 \langle u_2, u_3 \rangle \mid (u_1, u_2, u_3) \in \mathcal{U}_3 \} \\ &= \min \left\{ y_1 + y_2 + y_3 \mid \begin{pmatrix} 1 & \alpha_1 \alpha_2 & -\alpha_1 \alpha_3 \\ -\alpha_1 \alpha_2 & 1 & -\alpha_2 \alpha_3 \\ -\alpha_1 \alpha_3 & -\alpha_2 \alpha_3 & 1 \end{pmatrix} \geq 0 \right\} \\ &= -2 \min \left\{ -\alpha_1 \alpha_2 e^{i(\phi_2 - \phi_1)} - \alpha_1 \alpha_3 e^{i(\phi_3 - \phi_1)} - \alpha_2 \alpha_3 e^{i(\phi_3 - \phi_2)} \right\}. \end{aligned}$$

The last equality above follows from (2.1). Thus,

$$\begin{aligned} b &= 2 \max \{ \alpha_1 \alpha_2 \langle u_1, u_2 \rangle + \alpha_1 \alpha_3 \langle u_1, u_3 \rangle + \alpha_2 \alpha_3 \langle u_2, u_3 \rangle \mid (u_1, u_2, u_3) \in \mathcal{U}_3 \} \\ &= 2 \max \left\{ \alpha_1 \alpha_2 e^{i(\phi_2 - \phi_1)} + \alpha_1 \alpha_3 e^{i(\phi_3 - \phi_1)} + \alpha_2 \alpha_3 e^{i(\phi_3 - \phi_2)} \right\}. \end{aligned}$$

Analogously,

$$\begin{aligned} a &= 2 \min \{ \alpha_1 \alpha_2 \langle u_1, u_2 \rangle + \alpha_1 \alpha_3 \langle u_1, u_3 \rangle + \alpha_2 \alpha_3 \langle u_2, u_3 \rangle \mid (u_1, u_2, u_3) \in \mathcal{U}_3 \} \\ &= -2 \max \{ -\alpha_1 \alpha_2 \langle u_1, u_2 \rangle - \alpha_1 \alpha_3 \langle u_1, u_3 \rangle - \alpha_2 \alpha_3 \langle u_2, u_3 \rangle \mid (u_1, u_2, u_3) \in \mathcal{U}_3 \} \\ &= -2 \max \left\{ -\alpha_1 \alpha_2 e^{i(\phi_2 - \phi_1)} - \alpha_1 \alpha_3 e^{i(\phi_3 - \phi_1)} - \alpha_2 \alpha_3 e^{i(\phi_3 - \phi_2)} \right\} \\ &= 2 \min \left\{ \alpha_1 \alpha_2 e^{i(\phi_2 - \phi_1)} + \alpha_1 \alpha_3 e^{i(\phi_3 - \phi_1)} + \alpha_2 \alpha_3 e^{i(\phi_3 - \phi_2)} \right\}. \end{aligned}$$

Since the set

$$Y_{\alpha_1, \alpha_2, \alpha_3} = \left\{ 2\alpha_1\alpha_2e^{i(\phi_2-\phi_1)} + \alpha_1\alpha_3e^{i(\phi_3-\phi_1)} + \alpha_2\alpha_3e^{i(\phi_3-\phi_2)} \mid \phi_1, \phi_2, \phi_3 \in \mathbb{R} \right\}$$

is closed and convex, it is equal to $[a, b]$. Hence, $1 - \alpha_1^2 - \alpha_2^2 - \alpha_3^2 \in Y_{\alpha_1, \alpha_2, \alpha_3}$. It follows that there are complex numbers $e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3}$ which after substitution for u, v, w , correspondingly, in (3.2) give an identity. Thus, $z = \alpha_1e^{i\phi_1} + \alpha_2e^{i\phi_2} + \alpha_3e^{i\phi_3}$ has modulus one and $e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3}, z$ satisfy (3.1). $\square$

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DEPARTMENT OF MATHEMATICAL SCIENCES, CHALMERS UNIVERSITY OF TECHNOLOGY, SE-412 96 GÖTEBORG, SWEDEN

E-mail address: popovych@univ.kiev.ua