BARBASHIN-KRASOVSKII THEOREM FOR STOCHASTIC DIFFERENTIAL EQUATIONS

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ABSTRACT. A system of stochastic differential equations $dX(t) = f(X)dt + \sum_{i=1}^k g_i(X)dW_i(t)$ which has a zero solution $X = 0$ is considered. It is assumed that there exists a positive definite function $V(x)$ such that the corresponding operator LV is nonpositive. It is proved that if the set $\{M : LV = 0\}$ does not include entire semitrajectories of the system almost surely, then the zero solution is asymptotically stable in probability.

1. INTRODUCTION

The main method for investigating the stability and asymptotic stability in probability of the zero solution of the systems of stochastic differential equations is Lyapunov's Direct (or Second) Method.

Lyapunov's initial results and the initial work involving Lyapunov vector-functions pertained to finite-dimensional dynamical systems determined by ordinary differential equations. However, it was quickly recognized that the Second Method can be extended to also be applicable to finite-dimensional dynamical systems determined by difference equations and to infinite-dimensional dynamical systems defined on Hilbert, Banach and even general metric spaces. (Such dynamical systems are determined, e.g., by functional differential equations, partial differential equations, Volterra integro-differential equations and others.)

Very often, when we are interested in investigation of asymptotic stability of the zero solution, in applications it is possible to construct the Lyapunov function when its derivative is nonpositive only. In this case Barbashin and Krasovskii proved a well-known result [2] on asymptotic stability. This result was extended in [9], [10] and [11] to the case of periodic and almost periodic systems.

Has'minskii [7, 8] proved a generalization of Lyapunov theorems in the case of stochastic differential equations. Has'minskii also proved a theorem [8] on asymptotic stability in probability using the nondegeneracy condition requirement for the matrix, corresponding to the coefficients of Wiener processes. Many results concerning the stability and asymptotic stability in probability of the zero solution of stochastic differential equations were obtained also by Kushner [14, 15], Mao

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[16, 17] and others. In recent years the method of Lyapunov's functions for stochastic differential equations has been developed in [1, 3, 4, 5, 12, 13, 18, 19] and others.

In this paper, we obtain the criteria of the asymptotic stability in probability of the zero solution of a system of stochastic differential equations in the case when $LV \leq 0$ and the set $\{M : LV = 0\}$ does not include entire semitrajectories of the system (2.1) almost surely. This result generalizes the Barbashin-Krasovskii theorem [2] to the case of stochastic differential equations.

2. DEFINITIONS AND PRELIMINARY RESULTS

Consider the system of autonomous stochastic differential equations

$$(2.1) \quad dX(t) = f(X)dt + \sum_{i=1}^k g_i(X)dW_i(t), \quad f(0) \equiv 0, \quad g_i(0) \equiv 0,$$

where $X = (X_1, \dots, X_n) \in R^n$, $t \in [s, +\infty)$ is time. We assume that $f(X), g_i(X)$ are vectors in R^n and $W_i(t)$ are independent Wiener processes. We assume that the functions $f(X)$ and $g_i(X)$ satisfy the Lipschitz condition in X in every bounded in X domain, i.e.,

$$(2.2) \quad \sum_{i=1}^k |g_i(x) - g_i(y)| + |f(x) - f(y)| < C|x - y|.$$

Denote $\|X\| = \sqrt{X_1^2 + \dots + X_n^2}$ and

$$B_H := \{x \in R^n : \|x\| < H\}.$$

Denote by $X^{s,x}(t)$ the solution of (2.1) with initial condition $X^{s,x}(s) = x$. Let us define the $n \times k$ matrix $g(x) = (g_1(x), \dots, g_k(x))$ and the $n \times n$ matrix $A(x) = [a_{ij}(x)] = g(x)g^T(x)$, where $g^T(x)$ is a transpose matrix of $g(x)$. Set $\tau_{S_H}(t) = \min(\tau_{S_H}, t)$, where τ_{S_H} is the first exit time from the B_H of the sample path of the process $X(t)$ defined by equation (2.1).

Let

$$LV(x) = \sum_{i=1}^n f_i(x) \frac{\partial V}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 V}{\partial x_i \partial x_j}.$$

Definition 2.1. The trivial solution of system (2.1) is said to be stable in probability if for every $\varepsilon > 0$ and $s \in R_+$

$$\lim_{x \rightarrow 0} P\{\sup_{t > s} \|X^{s,x}(t)\| > \varepsilon\} = 0.$$

Definition 2.2. The trivial solution of system (2.1) is said to be asymptotically stable in probability if it is stable in probability and

$$\lim_{x \rightarrow 0} P\{\lim_{t \rightarrow \infty} \|X^{s,x}(t)\| = 0\} = 1.$$

Definition 2.3. According to [6, 21] we say that a function $g : R_+ \rightarrow R_+$ is Hahn's function ($g \in \mathcal{K}$) if it is continuous, monotonically increasing and $g(0) = 0$.

Definition 2.4 ([21]). A function $V : R^n \rightarrow R$ is said to be positive definite if there exists $\alpha \in \mathcal{K}$ such that $V(x) \geq \alpha(\|x\|)$.

Definition 2.5 ([21]). A function $V : R^n \rightarrow R$ is said to have an infinitesimal upper limit if there exists $\beta \in \mathcal{K}$ such that $V(x) \leq \beta(\|x\|)$.

Lemma 2.1 ([8]). *Let $V(x)$ be a function in class $C_2^0(B_H)$, bounded in the domain B_H , and suppose that $LV(x) \leq 0$ in this domain. Then the process $V(X^{s,x}(\tau_{S_H}(t)))$ is a supermartingale, so that*

$$EV(X^{s,x}(\tau_{S_H}(t))) \leq V(x)$$

for $x \in B_H$.

Theorem 2.1 ([8]). *Assume that in the domain B_H there exists a positive definite function $V(x) \in C_2^0(B_H)$ which satisfies the condition*

$$LV \leq 0$$

for $x \neq 0$. Then the trivial solution $X(t) \equiv 0$ of the system (2.1) is stable in probability.

Theorem 2.2 ([8]). *If in the domain B_H there exists a function $V(x) \in C_2^0(B_H)$ which satisfies the following conditions:*

- (1) $\alpha(\|x\|) \leq V(x) \leq \beta(\|x\|)$,
- (2) $LV(x) \leq -\gamma(\|x\|)$,

where $\alpha, \beta, \gamma \in \mathcal{K}$, then the trivial solution $X(t) \equiv 0$ of system (2.1) is asymptotically stable in probability.

3. MAIN RESULTS

Theorem 3.1. *Suppose that there exists a positive definite function $V(x) \in C_2^0(B_H)$ such that LV satisfies almost surely the following conditions in the domain $B_H \setminus \{0\}$:*

- (1) $LV < 0$ outside of the set M ,
- (2) $LV = 0$ on M ,

where the set M does not include entire semitrajectories $X^{s,x}(t)$ of the system (2.1) for $s < t < \infty$ (with the exception of $X \equiv 0$). Then the zero solution of equations (2.1) is asymptotically stable in probability.

Proof. Fix any $\varepsilon > 0$. Let K_x denote the set of sample paths of $X^{s,x}(t)$ such that $\tau_{S_\varepsilon} = \infty$. Since the function V satisfies the assumptions of Theorem 2.1, the solution $X(t) \equiv 0$ is stable in probability, and therefore

$$(3.1) \quad P(K_x) \rightarrow 1, \quad \text{as } x \rightarrow 0.$$

In other words,

$$(3.2) \quad P(\|X^{s,x}(t)\| < \varepsilon) \rightarrow 1, \quad \text{as } x \rightarrow 0 \quad \text{for all } \varepsilon > 0.$$

Let us show that the solution $X(t) \equiv 0$ is asymptotically stable in probability. Since $LV \leq 0$, using Lemma 2.1 we have

$$V_1(X^{s,x}(t)) := EV(X^{s,x}(\tau_{S_H}(t))) \leq V(x) \quad \text{for all } t \geq s,$$

and the deterministic function $V_1(X^{s,x}(t))$ is a nonnegative, bounded function of t . Since $LV \leq 0$, Lemma 3.3.1 from [8] implies that the function $V_1(X^{s,x}(t))$ is also a nonincreasing function of t . Therefore this function has some limit V_0 as $t \rightarrow \infty$; moreover

$$(3.3) \quad V_1(X^{s,x}(t)) \geq V_0 \quad \text{for } t \geq s.$$

Suppose that $V_0 \neq 0$. Using (3.2) and the fact that area $\|X^{s,x}(t)\| < \varepsilon$ is bounded, we can conclude that there exists a sequence of points $X_{(k)}^{s,x} = X^{s,x}(s + kt_1)$ ($k = k_1, k_2, \dots$; $t_1 = \text{const} > 0$) which is convergent almost surely to the point q , with

coordinates $X^* = (X_1^*, X_2^*, \dots, X_n^*)$, belonging to the area $\|X^{s,x}(t)\| \leq \varepsilon$. Using the continuity of the function V_1 we have $V_1(X_1^*, X_2^*, \dots, X_n^*) = V_0$.

Consider now the solutions of (2.1) $X^{s,q}(\tau_{S_\varepsilon}(t))$ and $X^{s,k_i}(\tau_{S_\varepsilon}(t))$, starting at moment $t = s$ from the points q and $X_{(k_i)}^{s,x}$ respectively. Since the solution $X^{s,q}(\tau_{S_\varepsilon}(t))$ for $s \leq t < \infty$ almost surely does not belong to the set M completely, there should exist some time interval when $LV < 0$ along this solution. This means, using Lemma 3.3.1 from [8], that there exists a time moment $T > s$ such that

$$(3.4) \quad V_1(X^{s,q}(T)) = V_2 < V_0.$$

Since the sequence $X^{s,k_i}(\tau_{S_\varepsilon}(t))$ converges to the point q with probability 1, according to [20] we have the inequality

$$E(X^{s,q}(T) - X^{s,k_i}(T))^2 \leq CE\|X^* - X^{s,x}(s + k_it_1)\|^2 \rightarrow 0 \quad \text{as } k_i \rightarrow \infty,$$

where $C = \text{const} > 0$. Hence, we have

$$E(X^{s,q}(T) - X^{s,k_i}(T))^2 \rightarrow 0 \quad \text{as } k_i \rightarrow \infty.$$

Using the stability in probability of the solutions $X^{s,q}(t)$ and $X^{s,k_i}(t)$, we see that $(X^{s,q}(T) - X^{s,k_i}(T))^2$ is nonnegative and bounded, and by Skorokhod [22], there exists a probability space (Ω^s, F^s, P^s) such that almost surely P^s we have

$$(X^{s,q}(T) - X^{s,k_i}(T))^2 \rightarrow 0 \quad \text{as } k_i \rightarrow \infty.$$

Therefore, using (3.4) and continuity of the function V_1 , we have

$$(3.5) \quad \lim_{k_i \rightarrow \infty} V_1(X^{s,k_i}(T)) = V_2.$$

Since the functions $f(X)$ and $g_i(X)$ in (2.1) are independent of time t , on the probability space (Ω^s, F^s, P^s) with probability 1 we have

$$X^{s,k_i}(T) = X^{s,x}(T + s + k_it_1);$$

hence we can rewrite (3.5) in the following form:

$$(3.6) \quad \lim_{k_i \rightarrow \infty} V_1(X^{s,x}(T + s + k_it_1)) = V_2 < V_0.$$

But (3.6) contradicts (3.3); therefore our assumption that $V_0 \neq 0$ is wrong, and hence $V_0 = 0$. This means that

$$\lim_{t \rightarrow \infty} V_1(X^{s,x}(t)) = 0.$$

Hence

$$\lim_{t \rightarrow \infty} \|X^{s,x}(t)\| = 0$$

with probability 1. □

Example 3.1. Consider the system of stochastic differential equations

$$(3.7) \quad \begin{aligned} dy &= -y(z^2 + 1)dt + y\sqrt{z^2 + 1}dW_1(t), \\ dz &= -zdt + ydW_2(t). \end{aligned}$$

This system has the trivial solution

$$(3.8) \quad y = 0, \quad z = 0.$$

Let us show that this solution is asymptotically stable in probability. Denote

$$V := \frac{1}{2}(y^2 + z^2).$$

We get

$$LV = -y^2(z^2 + 1) + \frac{1}{2}y^2(z^2 + 1) - z^2 + \frac{1}{2}y^2 = -z^2(\frac{1}{2}y^2 + 1) \leq 0.$$

Note that we cannot apply Theorem 2.2, because condition 2 of this theorem is not fulfilled. But we can apply Theorem 3.1, since $LV = 0$ on the set $M = \{(y, z) : z = 0\}$. We see from the second equation of (3.7) that the set M does not contain almost surely any nontrivial solution of the system (3.7). Therefore, solution (3.8) of system (3.7) is asymptotically stable in probability.

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