A NEW CROSS THEOREM FOR SEPARATELY HOLOMORPHIC FUNCTIONS

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ABSTRACT. We prove a new cross theorem for separately holomorphic functions.

1. INTRODUCTION. MAIN RESULT

Throughout the paper we will work in the following geometric context; details may be found in [5].

We fix an integer $N \geq 2$ and let D_j be a (connected) *Riemann domain* over $\mathbb{C}^{n_j}$, $j = 1, \dots, N$. Let $\emptyset \neq A_j \subset D_j$ be *locally pluriregular*, $j = 1, \dots, N$.

We will use the following conventions: $A'_j := A_1 \times \dots \times A_{j-1}$, $j = 2, \dots, N$, $A''_j := A_{j+1} \times \dots \times A_N$, $j = 1, \dots, N-1$. Analogously, a point $a = (a_1, \dots, a_N) \in D_1 \times \dots \times D_N$ may be written as $a = (a'_j, a_j, a''_j)$, where $a'_j := (a_1, \dots, a_{j-1})$, $a''_j := (a_{j+1}, \dots, a_N)$ (with obvious exceptions for $j \in \{1, N\}$).

We define an N -fold cross

$$\mathbf{X} = \mathbf{X}((D_j, A_j)_{j=1}^N) := \bigcup_{j=1}^N A'_j \times D_j \times A''_j.$$

One may easily prove that $\mathbf{X}$ is connected.

We say that a function $f : \mathbf{X} \rightarrow \mathbb{C}$ is *separately holomorphic on $\mathbf{X}$* (we write $f \in \mathcal{O}_s(\mathbf{X})$) if for any $j \in \{1, \dots, N\}$ and $(a'_j, a''_j) \in A'_j \times A''_j$, the function $D_j \ni z_j \mapsto f(a'_j, z_j, a''_j) \in \mathbb{C}$ is holomorphic in D_j .

Let h_{A_j, D_j} denote the relative extremal function of A_j in D_j , $j = 1, \dots, N$. Recall that $h_{A, D} := \sup\{u \in \mathcal{PSH}(D) : u \leq 1, u|_A \leq 0\}$, $A \subset D$ (cf. [6], § 4.5). Put

$$\widehat{\mathbf{X}} := \{(z_1, \dots, z_N) \in D_1 \times \dots \times D_N : h_{A_1, D_1}^*(z_1) + \dots + h_{A_N, D_N}^*(z_N) < 1\},$$

where $*$ stands for the upper semicontinuous regularization. One may prove that $\widehat{\mathbf{X}}$ is connected and $\mathbf{X} \subset \widehat{\mathbf{X}}$.

The *classical cross theorem* is the following result:

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Theorem 1 ([12], [13], [15], [14], [8], [9], [10], [7], [1], [16]). *For each $f \in \mathcal{O}_s(\mathbf{X})$ there exists exactly one $\widehat{f} \in \mathcal{O}(\widehat{\mathbf{X}})$ such that $\widehat{f} = f$ on $\mathbf{X}$ and $\sup_{\widehat{\mathbf{X}}} |\widehat{f}| = \sup_{\mathbf{X}} |f| \leq +\infty$.*

The aim of this paper is to extend the above theorem to a class of more general objects, namely (N, k) -crosses $\mathbf{X}_{N,k}$ defined for $k \in \{1, \dots, N\}$ as follows:

$$\mathbf{X}_{N,k} = \mathbb{X}_{N,k}((A_j, D_j)_{j=1}^N) := \bigcup_{\substack{\alpha_1, \dots, \alpha_N \in \{0,1\} \\ \alpha_1 + \dots + \alpha_N = k}} \mathbf{x}_\alpha,$$

where

$$\mathbf{x}_\alpha := \mathbf{x}_{1,\alpha_1} \times \dots \times \mathbf{x}_{N,\alpha_N}, \quad \mathbf{x}_{j,\alpha_j} := \begin{cases} D_j, & \text{if } \alpha_j = 1, \\ A_j, & \text{if } \alpha_j = 0. \end{cases}$$

Notice that N -fold crosses are just $(N, 1)$ -crosses in the above terminology. Obviously, $\mathbf{X}_{N,N} = D_1 \times \dots \times D_N$. Thus, if $N = 2$, then in fact we have only $\mathbf{X}_{2,1}$.

Recall that the theory of extension of separately holomorphic functions had been first developed for $N = 2$. Then the N -fold case (obtained via induction) was considered as a natural generalization of $\mathbf{X}_{2,1}$. In our opinion, each of the crosses $\mathbf{X}_{N,k}$ may be considered as a *natural* generalization of $\mathbf{X}_{2,1}$. Consequently, one should try to find an analogous version of the cross theorem for all (N, k) -crosses.

We say that a function $f : \mathbf{X}_{N,k} \rightarrow \mathbb{C}$ is *separately holomorphic* ($f \in \mathcal{O}_s(\mathbf{X}_{N,k})$) if for all $a = (a_1, \dots, a_N) \in A_1 \times \dots \times A_N$ and $\alpha = (\alpha_1, \dots, \alpha_N) \in \{0, 1\}^N$ with $|\alpha| = k$, the function

$$D^\alpha := \prod_{\substack{j \in \{1, \dots, N\} : \\ \alpha_j = 1}} D_j \ni z \mapsto f(i_{a,\alpha}(z))$$

is holomorphic, where $i_{a,\alpha} : D^\alpha \rightarrow \mathbf{x}_\alpha$,

$$i_{a,\alpha}(z) := (w_1, \dots, w_N), \quad w_j := \begin{cases} z_j, & \text{if } \alpha_j = 1, \\ a_j, & \text{if } \alpha_j = 0. \end{cases}$$

Put

$$\begin{aligned} \widehat{\mathbf{X}}_{N,k} &= \widehat{\mathbb{X}}_{N,k}((A_j, D_j)_{j=1}^N) \\ &:= \left\{ (z_1, \dots, z_N) \in D_1 \times \dots \times D_N : \sum_{j=1}^N h_{A_j, D_j}^*(z_j) < k \right\}. \end{aligned}$$

Note that $\widehat{\mathbf{X}}_{N,N} = D_1 \times \dots \times D_N$.

Let $\varphi_j : D_j \rightarrow \widetilde{D}_j$ be the envelope of holomorphy (cf. [4], Definition 1.8.1). Observe that since φ_j is locally biholomorphic, the set $\widetilde{A}_j := \varphi_j(A_j) \subset \widetilde{D}_j$ is locally pluriregular, $j = 1, \dots, N$. Let

$$\widetilde{\mathbb{X}}_{N,k} := \mathbb{X}_{N,k}((\widetilde{A}_j, \widetilde{D}_j)_{j=1}^N), \quad \widehat{\widetilde{\mathbf{X}}}_{N,k} := \widehat{\mathbb{X}}_{N,k}((\widetilde{A}_j, \widetilde{D}_j)_{j=1}^N).$$

Put

$$\varphi : D_1 \times \dots \times D_N \rightarrow \widetilde{D}_1 \times \dots \times \widetilde{D}_N, \quad \varphi(z_1, \dots, z_N) := (\varphi_1(z_1), \dots, \varphi_N(z_N)).$$

Note that:

- $\varphi(\mathbf{X}_{N,k}) \subset \widetilde{\mathbf{X}}_{N,k}$,

- $\varphi(\widehat{\mathbf{X}}_{N,k}) \subset \widehat{\mathbf{X}}_{N,k}$ (because $h_{A_j, \tilde{D}_j}^* \circ \varphi_j \leq h_{A_j, D_j}^*$, $j = 1, \dots, N$).

Our main result is the following *cross theorem* for (N, k) -crosses.

Theorem 2. *For every $f \in \mathcal{O}_s(\mathbf{X}_{N,k})$ there exists exactly one $\widehat{f} \in \mathcal{O}(\widehat{\mathbf{X}}_{N,k})$ such that $\widehat{f} \circ \varphi = f$ on $\mathbf{X}_{N,k}$ and $\sup_{\widehat{\mathbf{X}}_{N,k}} |\widehat{f}| = \sup_{\mathbf{X}_{N,k}} |f|$.*

The proof will be presented in §5 and will be based on Theorem 1 and the following lemmas (which might also be useful in other applications).

Lemma 3. *Let G be a Riemann domain over $\mathbb{C}^n$, let $D \Subset G$ be a Riemann domain of holomorphy, and let $A \subset D$ be non-pluripolar. Put*

$$\Delta(\mu) := \{z \in D : h_{A,D}^*(z) < \mu\}, \quad 0 < \mu \leq 1.$$

Then

$$h_{\Delta(r), \Delta(s)}^* = \max \left\{ 0, \frac{h_{A,D}^* - r}{s - r} \right\} \text{ on } \Delta(s), \quad 0 < r < s \leq 1.$$

Lemma 4. *Assume additionally that $D_1, \dots, D_N$ are Riemann domains of holomorphy. Then*

$$h_{\widehat{\mathbf{X}}_{N,k-1}, \widehat{\mathbf{X}}_{N,k}}^*(z) = \max \left\{ 0, \sum_{j=1}^N h_{A_j, D_j}^*(z_j) - k + 1 \right\},$$

$$z = (z_1, \dots, z_N) \in \widehat{\mathbf{X}}_{N,k}, \quad k \in \{2, \dots, N\}.$$

To our knowledge the formula from Lemma 3 seems to be new. We do not know whether Lemmas 3, 4 are true for arbitrary Riemann domains.

2. BASIC PROPERTIES OF (N, k) -CROSSES

- Remark 5.*
- (a) $A_1 \times \dots \times A_N \subset \mathbf{X}_{N,k} \subset \widehat{\mathbf{X}}_{N,k}$.
 - (b) $\mathbf{X}_{N,k-1} \subset \mathbf{X}_{N,k}$, $\widehat{\mathbf{X}}_{N,k-1} \subset \widehat{\mathbf{X}}_{N,k}$, $k = 2, \dots, N$.
 - (c) $\mathbf{X}_{N,k} = (\mathbf{X}_{N-1,k-1} \times D_N) \cup (\mathbf{X}_{N-1,k} \times A_N)$, $k = 2, \dots, N-1$, $N \geq 3$.
 - (d) $\mathbf{X}_{N,k}$ and $\widehat{\mathbf{X}}_{N,k}$ are connected.
 - (e) If $(D_{j,k})_{k=1}^\infty$ is a sequence of subdomains of D_j such that $D_{j,k} \nearrow D_j$, $D_{j,k} \supset A_{j,k} \nearrow A_j$, $j = 1, \dots, N$, then $\mathbb{X}_{N,k}((A_{j,k}, D_{j,k})_{j=1}^N) \nearrow \mathbf{X}_{N,k}$ and $\widehat{\mathbb{X}}_{N,k}((A_{j,k}, D_{j,k})_{j=1}^N) \nearrow \widehat{\mathbf{X}}_{N,k}$.
 - (f) If $D_1, \dots, D_N$ are domains of holomorphy, then $\widehat{\mathbf{X}}_{N,k}$ is a domain of holomorphy.

3. PROOF OF LEMMA 3

Let

$$L := h_{\Delta(r), \Delta(s)}^*, \quad R := \max \left\{ 0, \frac{h_{A,D}^* - r}{s - r} \right\}.$$

Put $\Delta[r] := \{z \in D : h_{A,D}^*(z) \leq r\}$. It is clear that

$$(*) \quad \begin{aligned} L &= h_{\Delta(r), \Delta(s)} \geq h_{\Delta[r], \Delta(s)}^* \geq h_{\Delta[r], \Delta(s)} \geq R, \\ L &= R = 0 \text{ on } \Delta(r), \quad R = 0 \text{ on } \Delta[r]. \end{aligned}$$

Step 1. Reduction to the case $s = 1$.

Suppose that $0 < r < s < 1$. Observe that $\Delta(s)$ is a Riemann region of holomorphy. Moreover, $h_{A \cap \Delta(s), \Delta(s)}^* = (1/s)h_{A,D}^*$ on $\Delta(s)$.

Indeed, it is obvious that $h_{A \cap \Delta(s), \Delta(s)}^* \geq (1/s)h_{A,D}^*$ on $\Delta(s)$. Let $u \in \mathcal{PSH}(\Delta(s))$, $u \leq 1$, $u \leq 0$ on $A \cap \Delta(s)$. Observe that for every $z_0 \in D \cap \partial(\Delta(s))$ we have $\limsup_{z \rightarrow z_0} u(z) \leq 1 \leq (1/s)h_{A,D}^*(z_0)$. Thus, the function

$$v := \begin{cases} \max\{su, h_{A,D}^*\} & \text{on } \Delta(s), \\ h_{A,D}^* & \text{on } D \setminus \Delta(s) \end{cases}$$

is plurisubharmonic on D . It is known that there exists a pluripolar set $P \subset A$ such that $h_{A,D}^* = 0$ on $A \setminus P$ (cf. [6]). Hence, $A \setminus P \subset \Delta(s)$, $v \leq h_{A \setminus P, D}^* = h_{A,D}^*$, and therefore, $h_{A \cap \Delta(s), \Delta(s)}^* \leq (1/s)h_{A,D}^*$ on $\Delta(s)$.

In particular, $A \cap S$ is not pluripolar for every connected component S of $\Delta(s)$. Hence,

$$L = h_{\Delta(r), \Delta(s)} = h_{\{h_{A, \Delta(s)}^* < r/s\}, \Delta(s)}, \quad R = \max \left\{ 0, \frac{h_{A, \Delta(s)}^* - r/s}{1 - r/s} \right\}.$$

Thus the problem for (D, A, r, s) reduces to $(S, A \cap S, r/s, 1)$, where S is a connected component of $\Delta(s)$.

From now on we assume that $s = 1$.

Step 2. Approximation. Let $A_\nu \nearrow A$, $D_\nu \nearrow D$, where $A_\nu \subset D_\nu$ is non-pluripolar, $\nu \in \mathbb{N}$. Suppose that the formula holds for each (D_ν, A_ν, r) . Then it holds for (D, A, r) .

Indeed, we know that $h_{A_\nu, D_\nu}^* \searrow h_{A,D}^*$. Hence $\{h_{A_\nu, D}^* < r\} \nearrow \Delta(r)$. Thus $h_{\{h_{A_\nu, D}^* < r\}, D}^* \searrow h_{\Delta(r), D}^*$.

Step 3. The case where D is hyperconvex, A is compact, and $h_{A,D}^*$ is continuous.

Let $u \in \mathcal{PSH}(D)$, $u \leq 1$, $u \leq 0$ on $\Delta[r]$. Using continuity of $h_{A,D}^*$ and [6], Proposition 4.5.2, we easily conclude that $\Delta[r]$ is compact. Let $U := D \setminus \Delta[r]$. Observe that for $z_0 \in \partial U$ we get

$$\liminf_{U \ni z \rightarrow z_0} (h_{A,D}^*(z) - (1-r)u(z) - r) \geq 0.$$

Hence, by the domination principle (cf. [6], Corollary 3.7.4), $(1-r)u + r \leq h_{A,D}^*$ in U . This shows that $h_{\Delta[r], D} \leq R$. Thus, by (*), we get $h_{\Delta[r], D}^* \equiv R$ for all $0 < r < 1$. Observe that $\Delta[r_\nu] \nearrow \Delta(r)$ for $0 < r_\nu \nearrow r$. Consequently, $L \equiv R$.

Step 4. The case where D is hyperconvex and A is compact.

Let $A^{(\varepsilon)} := \bigcup_{a \in A} \widehat{\mathbb{P}}(a, \varepsilon)$, where $\widehat{\mathbb{P}}(a, \varepsilon)$ stands for the “polydisc” in the sense of the Riemann domain D ($A^{(\varepsilon)}$ is defined for small $\varepsilon > 0$). By [6], Corollary 4.5.9, we know that $h_{A^{(\varepsilon)}, D} = h_{A^{(\varepsilon)}, D}^*$ is continuous. Thus, using Step 3 and (*), we get

$$h_{\{h_{A^{(\varepsilon)}, D} \leq r\}, D} = \max \left\{ 0, \frac{h_{A^{(\varepsilon)}, D} - r}{1 - r} \right\}, \quad 0 < \varepsilon \ll 1.$$

By [6], Proposition 4.5.10, we have $h_{A^{(\varepsilon)}, D} \nearrow h_{A,D}$ as $\varepsilon \searrow 0$. In particular,

$$\{h_{A^{(\varepsilon)}, D} \leq r\} \searrow \{h_{A,D} \leq r\} \text{ as } \varepsilon \searrow 0.$$

Hence, once again by [6], Proposition 4.5.10,

$$h_{\{h_{A^{(\varepsilon)}, D} \leq r\}, D} \nearrow h_{\{h_{A,D} \leq r\}, D} \text{ as } \varepsilon \searrow 0.$$

Consequently,

$$h_{\{h_{A,D} \leq r\}, D} = \max \left\{ 0, \frac{h_{A,D} - r}{1 - r} \right\} \leq R.$$

Thus $h_{\{h_{A,D} \leq r\}, D}^* \leq R$. Observe that the set $\{h_{A,D} \leq r\} \setminus \Delta[r]$ is pluripolar. Consequently, $h_{\Delta[r], D}^* \leq R$. We finish the proof as in Step 3.

Step 5. The case where A is open.

We use Step 4 and approximation (Step 2) with $A_\nu \nearrow A$, $D_\nu \nearrow D$, where $A_\nu \Subset D_\nu$ is compact non-pluripolar and D_ν is hyperconvex, $\nu \in \mathbb{N}$.

Step 6. The case where D is hyperconvex and $A \Subset D$ is non-pluripolar.

By Step 5 we have

$$h_{\{h_{\Delta(\varepsilon), D}^* < r\}, D}^* = \max \left\{ 0, \frac{h_{\Delta(\varepsilon), D}^* - r}{1 - r} \right\}, \quad 0 < \varepsilon < 1.$$

By [2] we get

$$\frac{h_{A,D}^* - \varepsilon}{1 - \varepsilon} \leq h_{\Delta(\varepsilon), D}^* \leq h_{A,D}^*;$$

in particular, $h_{\Delta(\varepsilon), D}^* \nearrow h_{A,D}^*$ as $\varepsilon \searrow 0$. Moreover,

$$\{h_{\Delta(\varepsilon), D}^* < \frac{r - \varepsilon}{1 - \varepsilon}\} \subset \Delta(r) \subset \{h_{\Delta(\varepsilon), D}^* < r\}, \quad 0 < \varepsilon < r.$$

Consequently,

$$\begin{aligned} \max \left\{ 0, \frac{h_{\Delta(\varepsilon), D}^* - \frac{r - \varepsilon}{1 - \varepsilon}}{1 - \frac{r - \varepsilon}{1 - \varepsilon}} \right\} &= h_{\{h_{\Delta(\varepsilon), D}^* < \frac{r - \varepsilon}{1 - \varepsilon}\}, D}^* \\ &\geq h_{\Delta(r), D}^* \geq h_{\{h_{\Delta(\varepsilon), D}^* < r\}, D}^* = \max \left\{ 0, \frac{h_{\Delta(\varepsilon), D}^* - r}{1 - r} \right\}, \quad 0 < \varepsilon < r. \end{aligned}$$

Letting $\varepsilon \searrow 0$, we get the required formula.

Step 7. The general case.

We use Step 6 and approximation (Step 2) with $A_\nu \nearrow A$, $D_\nu \nearrow D$, where $A_\nu \Subset D_\nu$ is non-pluripolar and D_ν is hyperconvex, $\nu \in \mathbb{N}$.

The proof of Lemma 3 is completed.

4. PROOF OF LEMMA 4

By Remark 5(e), we may assume that $A_j \Subset D_j \Subset G_j$, where G_j is a Riemann domain over $\mathbb{C}^{n_j}$, $j = 1, \dots, N$. Fix $2 \leq k \leq N$. Let

$$h_j := h_{A_j, D_j}^*, \quad j = 1, \dots, N, \quad h(z_1, \dots, z_N) := h_1(z_1) + \dots + h_N(z_N).$$

Let

$$\begin{aligned} L_{N,k} &:= h_{\widehat{\mathbf{X}}_{N,k-1}, \widehat{\mathbf{X}}_{N,k}}^*, \quad R_{N,k}(z) = \max \left\{ 0, \sum_{j=1}^N h_{A_j, D_j}^*(z_j) - k + 1 \right\}, \\ z &= (z_1, \dots, z_N) \in \widehat{\mathbf{X}}_{N,k}. \end{aligned}$$

It is clear that $L_{N,k} \geq R_{N,k}$ and $L_{N,k} = R_{N,k} = 0$ on $\widehat{\mathbf{X}}_{N,k-1}$. Fix an $a = (a_1, \dots, a_N) \in \widehat{\mathbf{X}}_{N,k} \setminus \widehat{\mathbf{X}}_{N,k-1}$. We may assume that $h_1(a_1) \leq \dots \leq h_N(a_N)$. Suppose that $h_1(a_1) = \dots = h_s(a_s) = 0$ and $h_{s+1}(a_{s+1}), \dots, h_N(a_N) > 0$ for an

$s \in \{0, \dots, N\}$. Since $h(a) \geq k - 1$, we see that in fact $s \leq N - k \leq N - 2$. In particular, if $N = 2$, then $s = 0$.

Let $\widehat{Y}_{N-s,p} = \widehat{X}_{N-s,p}((A_j, D_j)_{j=s+1}^N)$, $p \in \{k-1, k\}$. Observe that

$$\{a_1, \dots, a_s\} \times \widehat{Y}_{N-s,p} \subset \widehat{X}_{N,p}, \quad p \in \{k-1, k\}.$$

Consequently,

$$h_{\widehat{X}_{N,k-1}, \widehat{X}_{N,k}}^*(a) \leq h_{\widehat{Y}_{N-s,k-1}, \widehat{Y}_{N-s,k}}^*(a_{s+1}, \dots, a_N).$$

Thus, if we know that $L_{N-s,k}(a_{s+1}, \dots, a_N) \leq R_{N-s,k}(a_{s+1}, \dots, a_N)$, then

$$L_{N,k}(a) \leq R_{N-s,k}(a_{s+1}, \dots, a_N) = R_{N,k}(a).$$

This reduces the proof to the case $s = 0$, i.e. $h_j(a_j) > 0$, $j = 1, \dots, N$.

Put

$$\Delta_{j,t} := \{z_j \in D_j : h_j(z_j) < t\}, \quad j = 1, \dots, N.$$

Take $0 < r_j < s_j \leq 1$, $j = 1, \dots, N$, such that $r_1 + \dots + r_N = k - 1$ and $s_1 + \dots + s_N = k$. Observe that

$$\Delta_{1,r_1} \times \dots \times \Delta_{N,r_N} \subset \widehat{X}_{N,k-1}, \quad \Delta_{1,s_1} \times \dots \times \Delta_{N,s_N} \subset \widehat{X}_{N,k}.$$

Hence, using the product property for the relative extremal function (cf. [3], Theorem 4.1) and Lemma 3, we get

$$\begin{aligned} L_{N,k}(z) &\leq h_{\Delta_{1,r_1} \times \dots \times \Delta_{N,r_N}, \Delta_{1,s_1} \times \dots \times \Delta_{N,s_N}}^*(z) \\ &= \max\{h_{\Delta_{1,r_1}, \Delta_{1,r_1}}^*(z_1), \dots, h_{\Delta_{N,r_N}, \Delta_{N,r_N}}^*(z_N)\} \\ &= \max\left\{0, \frac{h_1(z_1) - r_1}{s_1 - r_1}, \dots, \frac{h_N(z_N) - r_N}{s_N - r_N}\right\}, \\ &\quad z = (z_1, \dots, z_N) \in \Delta_{1,s_1} \times \dots \times \Delta_{N,s_N}. \end{aligned}$$

Observe that there exist numbers $s_1, \dots, s_N \in (0, 1]$ such that $s_1 + \dots + s_N = k$ and

$$h_j(a_j) < s_j < \frac{h_j(a_j)}{h(a) - k + 1}, \quad j = 1, \dots, N.$$

Indeed, since the case where $h(a) = k - 1$ is trivial, we may assume that $h(a) > k - 1$. Note that $h_j(a_j) < \frac{h_j(a_j)}{h(a) - k + 1}$, $j = 1, \dots, N$. Suppose that

$$\frac{h_j(a_j)}{h(a) - k + 1} \leq 1, \quad j = 1, \dots, \sigma, \quad \frac{h_j(a_j)}{h(a) - k + 1} > 1, \quad j = \sigma + 1, \dots, N,$$

for a $\sigma \in \{0, \dots, N\}$. Observe that

$$\sum_{j=1}^N \frac{h_j(a_j)}{h(a) - k + 1} = \frac{h(a)}{h(a) - k + 1} > k,$$

so the case $\sigma = N$ is simple. Thus, assume that $\sigma \leq N - 1$. We only need to show that

$$\left(\sum_{j=1}^{\sigma} \frac{h_j(a_j)}{h(a) - k + 1}\right) + N - \sigma > k.$$

The case where $\sigma \leq N - k$ is obvious. Thus assume that $\sigma \geq N - k + 1$. We have to show that

$$\begin{aligned} \sum_{j=1}^{\sigma} h_j(a_j) &> (h(a) - k + 1)(k - N + \sigma) \\ &= (k - 1 - N + \sigma)h(a) + \left(\sum_{j=1}^{\sigma} h_j(a_j) \right) + \left(\sum_{j=\sigma+1}^N h_j(a_j) \right) + (-k + 1)(k - N + \sigma) \end{aligned}$$

or, equivalently,

$$(k - 1 - N + \sigma)h(a) + \left(\sum_{j=\sigma+1}^N h_j(a_j) \right) < (k - 1)(k - N + \sigma).$$

We have

$$\begin{aligned} (k - 1 - N + \sigma)h(a) + \left(\sum_{j=\sigma+1}^N h_j(a_j) \right) \\ < (k - 1 - N + \sigma)k + N - \sigma \leq (k - 1)(k - N + \sigma), \end{aligned}$$

which gives the required inequality.

Now, define

$$r_j := \frac{h_j(a_j) - s_j(h(a) - k + 1)}{k - h(a)}, \quad j = 1, \dots, N.$$

Then:

- $r_j > 0$ because $s_j < \frac{h_j(a_j)}{h(a) - k + 1}$,
- $r_j < s_j$ because $h_j(a) < s_j$,
- $r_1 + \dots + r_N = k - 1$,
- $\frac{h_j(a_j) - r_j}{s_j - r_j} = h(a) - k + 1$, $j = 1, \dots, N$.

Thus

$$\begin{aligned} L_{N,k}(a) &\leq \max \left\{ 0, \frac{h_1(a_1) - r_1}{s_1 - r_1}, \dots, \frac{h_N(a_N) - r_N}{s_N - r_N} \right\} \\ &= \max \{ 0, h(a) - k + 1 \} = R_{N,k}(a). \end{aligned}$$

The proof of Lemma 4 is completed.

5. PROOF OF THEOREM 2

First we prove that for each function $f \in \mathcal{O}_s(\mathbf{X}_{N,k})$ there exists exactly one $\tilde{f} \in \mathcal{O}_s(\tilde{\mathbf{X}}_{N,k})$ such that $\tilde{f} \circ \varphi \equiv f$ and $\sup_{\tilde{\mathbf{X}}_{N,k}} |\tilde{f}| = \sup_{\mathbf{X}_{N,k}} |f|$.

Indeed, fix an $f \in \mathcal{O}_s(\mathbf{X}_{N,k})$. Take $a = (a_1, \dots, a_N)$, $b = (b_1, \dots, b_N) \in A_1 \times \dots \times A_N$ and $\alpha = (\alpha_1, \dots, \alpha_N)$, $\beta = (\beta_1, \dots, \beta_N) \in \{0, 1\}^N$ with $|\alpha| = |\beta| = k$. To simplify notation, suppose that $\alpha = (1, \dots, 1, 0, \dots, 0)$.

Observe that if $\varphi_j(a_j) = \varphi_j(b_j)$, $j = k + 1, \dots, N$, then $f(\cdot, a_{k+1}, \dots, a_N) \equiv f(\cdot, b_{k+1}, \dots, b_N)$ on $D_1 \times \dots \times D_k$.

Indeed, since $\varphi_j : D_j \rightarrow \tilde{D}_j$ is the envelope of holomorphy, for each $g_j \in \mathcal{O}(D_j)$, there exists a $\tilde{g}_j \in \mathcal{O}(\tilde{D}_j)$ such that $g_j \equiv \tilde{g}_j \circ \varphi_j$. In particular, if $\varphi(z_j) = \varphi(w_j)$,

then $g_j(z_j) = g_j(w_j)$. Take arbitrary $c_j \in A_j$, $j = 1, \dots, k$. Then

$$\begin{aligned} f(c_1, \dots, c_k, a_{k+1}, \dots, a_N) &= f(c_1, \dots, c_k, b_{k+1}, a_{k+2}, \dots, a_N) \\ &= \dots = f(c_1, \dots, c_k, b_{k+1}, \dots, b_N). \end{aligned}$$

Thus $f(\cdot, a_{k+1}, \dots, a_N) = f(\cdot, b_{k+1}, \dots, b_N)$ on $A_1 \times \dots \times A_k$. It remains to use the identity principle.

Recall that

$$(\varphi_1 \times \dots \times \varphi_k) : D_1 \times \dots \times D_k \longrightarrow \tilde{D}_1 \times \dots \times \tilde{D}_k$$

is the envelope of holomorphy (cf. [4], Proposition 1.8.15 (b)). Consequently, the function

$$\tilde{f}_\alpha(\cdot, \varphi_{k+1}(a_{k+1}), \dots, \varphi_N(a_N)) := ((\varphi_1 \times \dots \times \varphi_k)^*)^{-1}(f(\cdot, a_{k+1}, \dots, a_N))$$

is well defined on

$$\tilde{\mathbf{X}}_\alpha := \tilde{D}_1 \times \dots \times \tilde{D}_k \times \tilde{A}_{k+1} \times \dots \times \tilde{A}_N$$

with $\tilde{f}_\alpha \circ \varphi = f$ on $\tilde{\mathbf{X}}_\alpha$ and $\sup_{\tilde{\mathbf{X}}_\alpha} |\tilde{f}_\alpha| = \sup_{\mathbf{X}_\alpha} |f|$.

In particular, $\tilde{f}_\alpha \circ \varphi = f = \tilde{f}_\beta \circ \varphi$ on $A_1 \times \dots \times A_N$. Hence, by the identity principle, $\tilde{f}_\alpha = \tilde{f}_\beta$ on $\tilde{\mathbf{X}}_\alpha \cap \tilde{\mathbf{X}}_\beta$.

Thus, we may replace $((D_j, A_j)_{j=1}^N, \mathbf{X}_{N,k}, \widehat{\mathbf{X}}_{N,k})$ by $((\tilde{D}_j, \tilde{A}_j)_{j=1}^N, \tilde{\mathbf{X}}_{N,k}, \widehat{\tilde{\mathbf{X}}}_{N,k})$. So we may assume that D_j is a domain of holomorphy and $\varphi_j = \text{id}$, $j = 1, \dots, N$.

Moreover, by Remark 5(e), we may reduce the problem to the case where $A_j \in D_j \in G_j$, where G_j is a Riemann domain over $\mathbb{C}^{n_j}$, $j = 1, \dots, N$.

The case $k = N$ is trivial. The case $k = 1$ is the classical cross theorem (Theorem 1). In particular, there is nothing to prove for $N = 2$. We apply induction on N . Suppose that the result is true for $N - 1 \geq 2$.

Now, we apply finite induction on k . The case $k = 1$ is known. Suppose that the result is true for $k - 1$ with $2 \leq k \leq N - 1$.

Fix an $f \in \mathcal{O}_s(\mathbf{X}_{N,k})$ and let $C := \sup_{\mathbf{X}_{N,k}} |f|$. Recall that

$$\mathbf{X}_{N,k} = (\mathbf{X}_{N-1,k-1} \times D_N) \cup (\mathbf{X}_{N-1,k} \times A_N).$$

For each $z_N \in D_N$ the function $f(\cdot, z_N)$ belongs to $\mathcal{O}_s(\mathbf{X}_{N-1,k-1})$. By the inductive assumption there exists a $g_{z_N} \in \mathcal{O}(\widehat{\mathbf{X}}_{N-1,k-1})$ such that $g_{z_N} = f(\cdot, z_N)$ on $\mathbf{X}_{N-1,k-1}$ and $\sup_{\widehat{\mathbf{X}}_{N-1,k-1}} |g_{z_N}| \leq C$. Analogously, for each $z_N \in A_N$ there exists an $h_{z_N} \in \mathcal{O}(\widehat{\mathbf{X}}_{N-1,k})$ such that $h_{z_N} = f(\cdot, z_N)$ on $\mathbf{X}_{N-1,k}$ and $\sup_{\widehat{\mathbf{X}}_{N-1,k}} |h_{z_N}| \leq C$. Recall that $\widehat{\mathbf{X}}_{N-1,k-1} \subset \widehat{\mathbf{X}}_{N-1,k}$ and $A_1 \times \dots \times A_{N-1} \subset \mathbf{X}_{N-1,k-1} \cap \mathbf{X}_{N-1,k}$. Since the set $A_1 \times \dots \times A_{N-1}$ is not pluripolar, we get $g_{z_N} = h_{z_N}$ on $\widehat{\mathbf{X}}_{N-1,k-1}$ for $z_N \in A_N$.

Consider the 2-fold cross

$$\mathbf{Y} := \mathbb{X}(\widehat{\mathbf{X}}_{N-1,k-1}, A_N; \widehat{\mathbf{X}}_{N-1,k}, D_N) = (\widehat{\mathbf{X}}_{N-1,k-1} \times D_N) \cup (\widehat{\mathbf{X}}_{N-1,k} \times A_N)$$

and let $F : \mathbf{Y} \longrightarrow \mathbb{C}$,

$$F(z', z_N) := \begin{cases} g_{z_N}(z'), & \text{if } (z', z_N) \in \widehat{\mathbf{X}}_{N-1,k-1} \times D_N, \\ h_{z_N}(z'), & \text{if } (z', z_N) \in \widehat{\mathbf{X}}_{N-1,k} \times A_N. \end{cases}$$

Obviously, $\sup_Y |F| \leq C$. To see that $F \in \mathcal{O}_s(Y)$, we have to prove that for each $z' \in \widehat{X}_{N-1,k-1}$, the function $D_N \ni z_N \mapsto F(z', z_N)$ is holomorphic. We know that $F(\cdot, z_N)$ is holomorphic for each $z_N \in D_N$. Let

$$Z_{N-1,k-1} := \mathbb{X}_{N-1,k-1}((A_j, D_j)_{j=2}^N).$$

Analogously as above, for each $z_1 \in D_1$ there exists a $\varphi_{z_1} \in \mathcal{O}(\widehat{Z}_{N-1,k-1})$ such that $\varphi_{z_1} = f(z_1, \cdot)$ on $Z_{N-1,k-1}$. Thus

$$\begin{aligned} F(z_1, \dots, z_N) &= f(z_1, \dots, z_N) = \varphi_{z_1}(z_2, \dots, z_N), \\ (z_1, \dots, z_N) &\in (X_{N-1,k-1} \times D_N) \cap (D_1 \times Z_{N-1,k-1}) \supset A_1 \times \dots \times A_{N-1} \times D_N. \end{aligned}$$

Consequently, $F(z', \cdot) \in \mathcal{O}(D_N)$ for $z' \in A_1 \times \dots \times A_{N-1}$ and hence, using Terada's theorem (cf. e.g. [11]), we conclude that $F \in \mathcal{O}(\widehat{X}_{N-1,k-1} \times D_N)$.

Now, by the classical cross theorem (Theorem 1) with $N = 2$, there exists an $\widehat{f} \in \mathcal{O}(\widehat{Y})$ such that $\widehat{f} = F$ on Y (in particular, $\widehat{f} = f$ on $X_{N,k}$) and $\sup_Y |\widehat{f}| \leq C$. Recall that

$$\widehat{Y} = \{(z', z_N) \in \widehat{X}_{N-1,k} \times D_N : h_{\widehat{X}_{N-1,k-1}, \widehat{X}_{N-1,k}}^*(z') + h_{A_N, D_N}^*(z_N) < 1\}.$$

Thus, it remains to apply Lemma 4.

The proof of Theorem 2 is completed.

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