THE DISTRIBUTION OF ZEROS OF A CLASS OF JACOBI POLYNOMIALS

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ABSTRACT. Polynomials whose coefficients are successive derivatives of a class of generalized Laguerre polynomials evaluated at $x = 0$ are shown to be stable. These polynomials can be expressed in terms of Jacobi polynomials. The authors also prove that a related family of polynomials, depending on a parameter, possess only real and negative zeros. A special class of stability-preserving operators is also investigated.

1. INTRODUCTION

A real polynomial, $p(z)$, is said to be *stable* (or a *Hurwitz polynomial*) if all the zeros of p lie in the open left half-plane, $\operatorname{Re} z < 0$. It is frequently said that the interest in the study of stable polynomials stems from the fact that if the zeros of the eigenequation of a differential equation lie in the left half-plane, then the solution of the differential equation is stable. But in fact stability analysis plays a pivotal role in many areas of pure and applied mathematics. Here we merely cite a few topics and recent papers that have inspired our interest in stability theory and motivated the present work: the Routh-Hurwitz problem (M. Marden [19, Ch. IX]), matrix analysis (O. Holtz [15]), stability-preserving operators (T. Craven and G. Csordas [10]), differential equations and numerical analysis (D. Gottlieb [12] and D. Gottlieb and L. Lustman [13]), the Jacobi Tau class of spectral methods and eigenvalue problems (F. Waleffe [27] and M. Charalambides and F. Waleffe [4], [5]).

The characterization of stable polynomials that is most useful for our purposes is given by the *Hermite-Biehler Theorem* ([13], [24, p. 197]) in terms of positive pairs, defined below.

Definition 1. Two real polynomials Ω_1 and Ω_2 of degree n and $n - 1$ (or n), respectively, form a *positive pair* if: (a) the zeros $z_1, \dots, z_n$ of Ω_1 and $z'_1, \dots, z'_{n-1}$ (or $z'_1, \dots, z'_n$) of Ω_2 are real, negative and distinct; (b) the zeros strictly interlace (or alternate) as follows: $z_1 < z'_1 < \dots < z'_{n-1} < z_n < 0$ (or $z'_1 < z_1 < \dots < z'_n < z_n < 0$); (c) the highest coefficients of Ω_1 and Ω_2 are of like sign.

Theorem 2 (Hermite-Biehler Theorem). *A real polynomial $p(z) = \Omega_1(z^2) + z\Omega_2(z^2)$ is stable if and only if $\Omega_1(z)$ and $\Omega_2(z)$ form a positive pair.*

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The Hermite-Biehler theorem ([13], [24, p. 197]) states that the even and odd parts of stable polynomials form positive pairs. This supplies us with a powerful tool to prove reality and negativity of the zeros of certain polynomials. In addition, we will also make use of the following characterization of positive pairs.

Lemma 3 (D. Wagner [26, Lemma 2.4]). *Let $P(z)$ and $Q(z)$ be real standard polynomials (i.e., the leading coefficients of the real polynomials P and Q are positive) with only negative zeros. Then $P(z)$ interlaces (or alternates) $Q(z)$ (in the sense of Definition 1) if and only if for all $\mu > 0$ both $Q(z) + \mu P(z)$ and $Q(z) + \mu z P(z)$ have only real negative zeros.*

An outline of this paper is as follows. We prove that polynomials whose coefficients are successive derivatives of a class of generalized Laguerre polynomials (evaluated at $x = 0$) are stable (Theorem 4). This result is sharp (cf. Remark 1; see also Conjecture 1 and Open Problem 1). More general combinations of these polynomials are also stable (Theorem 8). We also show that a related family of polynomials possess only real negative zeros (Theorem 5 and Theorem 9). With the aid of the aforementioned results, together with the Hermite-Biehler Theorem and Lemma 3, we show that some members of the above family (depending on a parameter) form positive pairs (Theorem 10). In Section 3, we consider a special class of stability-preserving operators, known as multiplier sequences (Theorem 11). With the aid of n -sequences (Definition 12) and a recent result about the zero distribution of hypergeometric polynomials (Theorem 14), we prove that a certain family of stable polynomials have only real negative zeros.

2. NEW CLASSES OF STABLE POLYNOMIALS AND POSITIVE PAIRS

In this section, we present a new class of stable polynomials obtained with the aid of the *generalized Laguerre polynomials*, $L_n^{(\alpha)}(x)$, where $L_n^{(\alpha)}$ can be defined via the Rodrigues formula (cf. [23, pp. 203–204]),

$$(1) \quad L_n^{(\alpha)}(x) := \frac{x^{-\alpha} e^x}{n!} \frac{d^n}{dx^n} (e^{-x} x^{n+\alpha}) = \sum_{j=0}^n (-1)^j \binom{n+\alpha}{n-j} \frac{x^j}{j!}.$$

Since $\frac{d^k}{dx^k} L_n^{(\alpha)}(x) = (-1)^k L_{n-k}^{(\alpha+k)}(x)$ (see, for example, [23, p. 203]), it follows from

$$(1) \text{ that } \left(\frac{d^k}{dx^k} L_n^{(\alpha)}(x) \right)_{x=0} = (-1)^k \binom{n+\alpha}{n-k} \text{ and whence}$$

$$(2) \quad \sum_{k=0}^n (-1)^k z^k D^k L_n^{(\alpha)}(0) = \sum_{k=0}^n \binom{n+\alpha}{n-k} z^k, \quad z \in \mathbb{C}, \quad D := \frac{d}{dx},$$

a relation we will use in the following theorem.

Theorem 4. *For each $n \geq 1$, the polynomial*

$$(3) \quad p_n^{(\alpha)}(z) = \sum_{k=0}^n (-1)^k z^k D^k L_n^{(\alpha)}(0) = \sum_{k=0}^n \binom{n+\alpha}{n-k} z^k \quad (D := \frac{d}{dx})$$

is stable for $-1 < \alpha \leq 1$, where $L_n^{(\alpha)}$ denotes the generalized Laguerre polynomial of degree n (cf. (1)).

Proof. Consider the polynomial

$$(4) \quad f_n(x; z) = \sum_{k=0}^n z^k D^k L_n^{(\alpha)}(x) \quad (z \in \mathbb{C}, x \in \mathbb{R}),$$

and note that for $z \neq 0$, f_n satisfies the differential equation

$$(5) \quad \frac{1}{z} \left(f_n - L_n^{(\alpha)} \right) = \frac{df_n}{dx}.$$

Multiply (5) by $\frac{1}{x} \overline{f_n}$, and then add to this its complex conjugate to obtain

$$(6) \quad \left(\frac{1}{z} + \frac{1}{\bar{z}} \right) |f_n|^2 \frac{1}{x} - \left(\frac{\overline{f_n}}{xz} + \frac{f_n}{x\bar{z}} \right) L_n^{(\alpha)} = \frac{d|f_n|^2}{dx} \frac{1}{x}.$$

Next, multiply both sides of (6) by the (generalized) Laguerre weight $w(x) = x^\alpha e^{-x}$, and for any $\varepsilon > 0$, integrate over (ε, ∞) :

$$(7) \quad \left(\frac{1}{z} + \frac{1}{\bar{z}} \right) \int_\varepsilon^\infty |f_n|^2 x^{\alpha-1} e^{-x} dx - \int_\varepsilon^\infty \left(\frac{\overline{f_n}}{xz} + \frac{f_n}{x\bar{z}} \right) L_n^{(\alpha)} x^\alpha e^{-x} dx \\ = \int_\varepsilon^\infty \frac{d|f_n|^2}{dx} x^{\alpha-1} e^{-x} dx.$$

Now, let z_j be any zero of $f_n(0; z)$ and note that $z_j \neq 0$. Since $f_n(0; z_j) = 0$, it follows that $\frac{f_n(x; z_j)}{x}$ is a polynomial of degree $n-1$ in x . Hence, as $\varepsilon \rightarrow 0^+$, the orthogonality property of the Laguerre polynomials for $\alpha > -1$ (cf. [23, p. 206]) implies that the second term of (7) vanishes. Furthermore, an integration by parts yields

$$(8) \quad \frac{2 \operatorname{Re}(z_j)}{|z_j|^2} \int_0^\infty |f_n|^2 x^{\alpha-1} e^{-x} dx = - \int_0^\infty |f_n|^2 (\alpha - 1 - x) x^{\alpha-2} e^{-x} dx,$$

where the existence of the improper integrals can be justified as was done above. Since $-1 < \alpha \leq 1$, $x \geq 0$ and $\alpha - 1 - x \leq 0$, it follows from (8) that $\operatorname{Re}(z_j) > 0$. We have thus shown that the real part of an arbitrary zero of $f_n(0; z)$ is positive and consequently, the zeros of the polynomial

$$(9) \quad p_n^{(\alpha)}(z) = f_n(0; -z) = \sum_{k=0}^n (-1)^k z^k D^k L_n^{(\alpha)}(0) \quad (-1 < \alpha \leq 1)$$

all have negative real parts; that is, $p_n^{(\alpha)}$ is a stable polynomial. $\square$

We note that the polynomials $p_n^{(\alpha)}$ are hypergeometric polynomials which can be expressed in terms of Jacobi polynomials, where one of the parameters depends on n (see, for example, [25, p. 62, eq. 4.21.2]):

$$p_n^{(\alpha)}(z) = \binom{n+\alpha}{n}_2 F_1(-n, 1; \alpha+1; -z) = P_n^{(\alpha, -n-\alpha)}(1+2z).$$

Before we proceed to establish some more general results and consequences of Theorem 4, we pause here for a moment (see Remarks 1-3) and examine the need for the assumptions on the parameter α in Theorem 4, provide some examples, formulate a conjecture, propose an open problem related to Theorem 4, and call attention to the Laplace transform representation of the polynomials $p_n^{(\alpha)}$.

Remark 1. We remark that Theorem 4 is sharp in the sense that the restriction on the parameter α is necessary. Indeed, if α lies outside the interval $(-1, 1]$, then, in general, the polynomial $p_n^{(\alpha)}$ need not be stable. For example, with the aid of, say, Mathematica or Maple, one can readily verify that when $n = 5$ and $\alpha = 20$, the polynomial $p_5^{(20)}$ is not stable. Note that if $\alpha = -1$, then the requisite integrability condition in the proof of Theorem 2 is not satisfied. However, when $\alpha = -1$, then $p_n^{(-1)}(z) = z(1+z)^{n-1}$, $n \geq 2$, and whence $p_n^{(-1)}$ is only *quasi-stable*; that is, the zeros of $p_n^{(-1)}$ lie in the closed half-plane, $\operatorname{Re} z \leq 0$. In addition, we formulate the following conjecture that is based on our experiments and numerical work. Our preliminary work further suggests that the closer α is to 1, where $\alpha > 1$, the larger n has to be to produce an unstable polynomial $p_n^{(\alpha)}$.

Conjecture 1. *For each $\alpha > 1$, there is a positive integer n such that the polynomial $p_n^{(\alpha)}$ (cf. (3) of Theorem 4) is unstable.*

Open Problem 1. *If p is a polynomial of degree n with zeros $\{z_k\}_{k=1}^n$, where $\operatorname{Re} z_k < 0$, $k = 1, 2, \dots, n$, the abscissa of stability of p is defined as $\sigma(p) := \max_{1 \leq k \leq n} \operatorname{Re} z_k$. Determine the abscissa of stability of $p_n^{(\alpha)}$ (cf. (3) of Theorem 4) for $-1 < \alpha \leq 1$.*

Remark 2. We also note that if the real polynomial p is stable, where $\deg p = n \geq 2$ and $p(0) \neq 0$, then so is the *reverse* polynomial $p^*(z) := z^n p(\frac{1}{z})$. Indeed, z_k , $\operatorname{Re} z_k < 0$, is a zero of p if and only if $\frac{1}{z_k}$ is a zero of p^* and $\operatorname{Re} \frac{1}{z_k} = \frac{\operatorname{Re} z_k}{|z_k|^2} < 0$. Consequently, it follows from Theorem 4 (cf. (3)) that the polynomial

$$b_n^{(\alpha)}(z) := (p_n^{(\alpha)}(z))^* = \sum_{k=0}^n \binom{n+\alpha}{k} z^k$$

is stable whenever $-1 < \alpha \leq 1$. Now observe that $b_n^{(\alpha)}$, the n^{th} partial sum of the binomial series $(1+z)^{n+\alpha}$, $|z| < 1$, is a stable polynomial for $-1 < \alpha \leq 1$. We hasten to note that if $m \in \mathbb{N}$, $m \neq n$, then, in general, the m^{th} partial sum of the binomial series $(1+z)^{n+\alpha}$ is not a stable polynomial. For example, one can show that the 7^{th} partial sum of $(1+z)^{12}$ is not a stable polynomial. In Section 3, we will reexamine this curious phenomenon in conjunction with the theory of multiplier sequences and n -sequences.

Remark 3. We also note here that Theorem 4 can be reformulated using some simple linear operators acting on the vector space, $\mathbb{R}[x]$, of all real polynomials. Let $T := \{\frac{1}{k!}\}_{k=0}^\infty$ and $\Lambda := \{k!\}_{k=0}^\infty$. Define the action of T (and similarly the action of Λ) on monomials by $T[x^k] := \frac{1}{k!}x^k$ and then extend the definition to $\mathbb{R}[x]$ by linearity. We note that T preserves stability (see Section 3) while, in general, Λ does not. Indeed, by way of illustration, consider the stable polynomial $f(x) = (1+x)^6$. Now, with the aid of some numerical work (or otherwise) we can demonstrate that $\Lambda[(1+x)^6] = \sum_{k=0}^6 \binom{6}{k} k! x^k$ is not a stable polynomial. In contrast, we know from Theorem 4 that $\Lambda[L_n^{(\alpha)}(-x)]$ is a stable polynomial whenever $-1 < \alpha \leq 1$. Or equivalently, we observe that, for $k = 0, 1, 2, \dots$,

$$k! = \Gamma(k+1) = \int_0^\infty e^{-t} t^k dt \quad \text{and} \quad L_n^{(\alpha)}(-zt) = \sum_{j=0}^n \binom{n+\alpha}{n-j} \frac{t^j z^j}{j!} \quad (t > 0),$$

and thus, by Theorem 4, if $-1 < \alpha \leq 1$, then

$$p_n^{(\alpha)}(z) = \Lambda[L_n^{(\alpha)}(-z)] = \int_0^\infty e^{-t} L_n^{(\alpha)}(-zt) dt = \sum_{k=0}^n \binom{n+\alpha}{n-k} z^k$$

is a stable polynomial. Can one deduce directly from this elegant formula that $p_n^{(\alpha)}$ is a stable polynomial?

With the aid of the Hermite-Biehler Theorem and Theorem 4, we next obtain a family of polynomials all of whose zeros are real and negative.

Theorem 5. *Let $L_n^{(\alpha)}$ denote the generalized Laguerre polynomial of degree n , $n \geq 2$. If $-1 < \alpha \leq 2$, then the zeros of the polynomial*

$$(10) \quad q_n^{(\alpha)}(z) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} z^k D^{2k} L_n^{(\alpha)}(0) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+\alpha}{n-2k} z^k$$

are all real and negative. Moreover, if $-1 < \alpha \leq 1$, then the polynomials $q_n^{(\alpha)}$ and $q_{n-1}^{(\alpha+1)}$ form a positive pair.

Proof. Applying the Hermite-Biehler Theorem (cf. Theorem 2) to the stable polynomial $p_n^{(\alpha)}$, $-1 < \alpha \leq 1$ (cf. (3) of Theorem 4), we find that the zeros of the polynomials

$$(11) \quad \Omega_{1,n}^{(\alpha)}(z) := \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} z^k D^{2k} L_n^{(\alpha)}(0) \quad \text{and} \quad \Omega_{2,n}^{(\alpha)}(z) := \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} z^k D^{2k+1} L_n^{(\alpha)}(0)$$

are all real and negative. Since $\Omega_{1,n}^{(\alpha)} = q_n^{(\alpha)}$, the theorem is verified for $-1 < \alpha \leq 1$. Next, using the relation $DL_n^{(\alpha)}(0) = -L_{n-1}^{(\alpha+1)}(0)$ ($D = d/dx$), we obtain that

$$(12) \quad \Omega_{2,n}^{(\alpha)}(z) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} z^k D^{2k+1} L_n^{(\alpha)}(0) = - \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} z^k D^{2k} L_{n-1}^{(\alpha+1)}(0) = -q_{n-1}^{(\alpha+1)}(z)$$

has only real negative zeros for $-1 < \alpha \leq 1$. Therefore, the change of indices, $n \rightarrow n+1$ and $\alpha \rightarrow \alpha-1$, shows that all the zeros of the polynomial $q_n^{(\alpha)}$ are real and negative for $0 < \alpha \leq 2$. Since $\Omega_{1,n}^{(\alpha)} = q_n^{(\alpha)}$ and $\Omega_{2,n}^{(\alpha)} = -q_{n-1}^{(\alpha+1)}$ (cf. (12)), the last assertion of the theorem is an immediate consequence of Theorem 2. This completes the proof. $\square$

It is well-known ([12], [13]) that polynomial approximations to solutions of differential equations are the basic ingredient of many numerical methods. In [4], M. Charalambides and F. Waleffe point out the advantages of approximations based on orthogonal polynomials. In particular, these authors show that the characteristic polynomials for the Gegenbauer Tau method obey a three-term recurrence relation plus a constant term which vanishes for Legendre Tau and Galerkin cases. Inspired by the importance of the aforementioned results, we now proceed to show that the polynomials defined in Theorem 5 (cf. (10)) also obey a three-term recurrence relation plus a constant term.

Theorem 6. The polynomials $q_n^{(\alpha)}$, $-1 < \alpha \leq 2$, defined by (10) of Theorem 5,

$$(13) \quad q_n^{(\alpha)}(z) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} z^k D^{2k} L_n^{(\alpha)}(0) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+\alpha}{n-2k} z^k,$$

obey the following recurrence relation:

$$(14) \quad zq_n^{(\alpha)}(z) = q_{n+2}^{(\alpha)}(z) - 2q_{n+1}^{(\alpha)}(z) + q_n^{(\alpha)}(z) - C_n^{(\alpha)},$$

where

$$(15) \quad C_n^{(\alpha)} = \binom{n+\alpha}{n} \frac{\alpha(\alpha-1)}{(n+1)(n+2)}.$$

Proof. The generalized Laguerre polynomials satisfy the relation ([23, p. 202]), $L_n^{(\alpha)}(x) = -\frac{d}{dx} L_{n+1}^{(\alpha)}(x) + \frac{d}{dx} L_n^{(\alpha)}(x)$ and hence, in terms of the second derivatives, we have

$$(16) \quad L_n^{(\alpha)}(x) = \frac{d^2}{dx^2} L_{n+2}^{(\alpha)}(x) - 2\frac{d^2}{dx^2} L_{n+1}^{(\alpha)}(x) + \frac{d^2}{dx^2} L_n^{(\alpha)}(x).$$

Substituting (16) into (13) and setting $q_0^{(\alpha)}(z) = 1$, we obtain

$$(17) \quad zq_n^{(\alpha)}(z) = q_{n+2}^{(\alpha)}(z) - 2q_{n+1}^{(\alpha)}(z) + q_n^{(\alpha)}(z) - C_n^{(\alpha)}, \quad \text{where}$$

$$C_0^{(\alpha)} = L_2^{(\alpha)}(0) = \binom{2+\alpha}{2},$$

$$C_1^{(\alpha)} = L_3^{(\alpha)}(0) - 2L_2^{(\alpha)}(0) = \binom{3+\alpha}{3} - 2\binom{2+\alpha}{2} = \frac{(\alpha-3)(\alpha+1)(\alpha+2)}{6}, \quad \text{and by induction}$$

$$C_n^{(\alpha)} = L_{n+2}^{(\alpha)}(0) - 2L_{n+1}^{(\alpha)}(0) + L_n^{(\alpha)}(0) = \cdots = \binom{n+\alpha}{n} \frac{\alpha(\alpha-1)}{(n+1)(n+2)}.$$

Since $C_n^{(\alpha)}$ is zero if $\alpha = 0$ or $\alpha = 1$, we see that the polynomials $q_n^{(0)}$ and $q_n^{(1)}$ satisfy a three-term recurrence relation. $\square$

Remark 4. We note here, parenthetically, that an elementary induction argument shows that for $n = 1, 2, 3, \dots$,

$$(18) \quad q_n^{(0)}(z) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{n-2k} z^k = \frac{(1+\sqrt{z})^n + (1-\sqrt{z})^n}{2},$$

$$(19) \quad q_n^{(1)}(z) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{n-2k} z^k = \frac{(1+\sqrt{z})^{n+1} - (1-\sqrt{z})^{n+1}}{2\sqrt{z}},$$

with due regard for the removable singularity at $z = 0$. Now, if $c_n := a^n + b^n$, $a, b \in \mathbb{C}$, $n \in \mathbb{N}$, then $c_{n+2} - 2c_{n+1} + c_n = (-1+a)^2 a^n + (-1+b)^2 b^n$. Substituting $a \rightarrow (1+\sqrt{z})$ and $b \rightarrow (1-\sqrt{z})$ in this expression, a calculation shows that $q_n^{(0)}$ satisfies a three-term recurrence relation. An analogous argument demonstrates, without appealing to the properties of the Laguerre polynomials, that the polynomials $q_n^{(1)}$ also satisfy a three-term recurrence relation.

As we shall see below, the family of polynomials described in Theorem 5 enjoys an additional significant property; polynomials of successive degrees form positive pairs. In order to prove this, we need to establish a number of results and we begin here with the following useful preparatory lemma.

Lemma 7. Let $\lambda, \mu \geq 0$. For fixed α , $-1 < \alpha \leq 1$, set

$$(20) \quad f_n(x; z) := \sum_{k=0}^n z^k D^k L_n^{(\alpha)}(x) + (\lambda + \mu z^2) \sum_{k=0}^{n-1} z^k D^k L_{n-1}^{(\alpha)}(x) \quad (x \in \mathbb{R}, z \in \mathbb{C}),$$

where $L_n^{(\alpha)}$ denotes the generalized Laguerre polynomial of degree $n \geq 2$ (cf. (1)). If z_j is a zero of the polynomial $f_n(0; z)$, then

$$(21) \quad I_1 := \int_0^\infty \left(\frac{\overline{f_n}}{xz_j} + \frac{f_n}{x\overline{z_j}} \right) L_{n-1}^{(\alpha)} x^\alpha e^{-x} dx = -\frac{1}{n} \left(\frac{1}{z_j} + \frac{1}{\overline{z_j}} \right) \frac{\Gamma(n+\alpha)}{(n-1)!}$$

and

$$(22) \quad I_2 := \int_0^\infty \left(\frac{z_j \overline{f_n}}{x} + \frac{\overline{z_j} f_n}{x} \right) L_{n-1}^{(\alpha)} x^\alpha e^{-x} dx = -\frac{1}{n} (z_j + \overline{z_j}) \frac{\Gamma(n+\alpha)}{(n-1)!},$$

where Γ denotes the gamma function and $f_n = f_n(x; z_j)$.

Proof. From (20) it follows that

$$(23) \quad f_n(x; z_j) = L_n^{(\alpha)}(x) + h_{n-1}(x), \quad (h_{n-1}(x) \in \mathbb{C}[x], \deg(h_{n-1}) \leq n-1),$$

and since $f_n(0; z_j) = 0$, the polynomial $f_n(x; z_j)$ can also be expressed as

$$(24) \quad f_n(x; z_j) = x \sum_{k=0}^{n-1} c_k L_k^{(\alpha)}(x) = x c_{n-1} L_{n-1}^{(\alpha)}(x) + \tilde{h}_{n-1}(x),$$

where $\tilde{h}_{n-1}(x) \in \mathbb{C}[x], \deg(\tilde{h}_{n-1}) \leq n-1$. Next, we use the recurrence relation satisfied by the Laguerre polynomials (cf. [23, p. 202]),

$$n L_n^{(\alpha)}(x) = (2n-1+\alpha-x) L_{n-1}^{(\alpha)}(x) - (n-1+\alpha) L_{n-2}^{(\alpha)}(x),$$

in conjunction with (24) and obtain that

$$(25) \quad f_n(x; z_j) = -n c_{n-1} L_{n-1}^{(\alpha)}(x) + \hat{h}_{n-1}(x), \quad (\hat{h}_{n-1}(x) \in \mathbb{C}[x], \deg(\hat{h}_{n-1}) \leq n-1).$$

A comparison of (23) and (25) shows that $c_{n-1} = -\frac{1}{n}$. From (24) and the fact that $f_n(0; z_j) = 0$, we deduce that, for $x \neq 0$,

$$(26) \quad \frac{f_n(x; z_j)}{x} = -\frac{1}{n} L_{n-1}^{(\alpha)}(x) + r_{n-2}(x), \quad (r_{n-2}(x) \in \mathbb{C}[x], \deg(r_{n-2}) \leq n-2).$$

Thus, $\deg\left(\frac{\overline{f_n}(x; z_j)}{xz_j}\right) = \deg\left(\frac{f_n(x; z_j)}{x\overline{z_j}}\right) = n-1$. Now, using the orthogonality properties of the generalized Laguerre polynomials $L_n^{(\alpha)}$ with respect to the weight function $w(x) = x^\alpha e^{-x}$ (see, for example, [23, p. 206, formulae (7) and (8)]), we obtain

$$\int_0^\infty r_{n-2}(x) L_{n-1}^{(\alpha)} x^\alpha e^{-x} dx = 0$$

and

$$I_1 = \int_0^\infty \left(\frac{\overline{f_n}}{xz_j} + \frac{f_n}{x\overline{z_j}} \right) L_{n-1}^{(\alpha)} x^\alpha e^{-x} dx = -\frac{1}{n} \left(\frac{1}{z_j} + \frac{1}{\overline{z_j}} \right) \frac{\Gamma(n+\alpha)}{(n-1)!}.$$

Finally, the foregoing argument also shows that

$$(27) \quad \int_0^\infty \left(\frac{\overline{f_n}}{x\overline{z_j}} + \frac{f_n}{xz_j} \right) L_{n-1}^{(\alpha)} x^\alpha e^{-x} dx = -\frac{1}{n} \left(\frac{1}{z_j} + \frac{1}{\overline{z_j}} \right) \frac{\Gamma(n+\alpha)}{(n-1)!}.$$

Hence, multiplying both sides of (27) by $|z_j|^2$ yields the desired result:

$$I_2 = \int_0^\infty \left(\frac{z_j \overline{f_n}}{x} + \frac{\overline{z_j} f_n}{x} \right) L_{n-1}^{(\alpha)} x^\alpha e^{-x} dx = -\frac{1}{n} (z_j + \overline{z_j}) \frac{\Gamma(n+\alpha)}{(n-1)!}. \quad \square$$

Theorem 8. Let $p_n^{(\alpha)}$ denote the polynomial defined by (3) (see Theorem 4). If $\lambda, \mu \geq 0$, then the polynomial

$$(28) \quad P_n(z; \alpha, \lambda, \mu) := p_n^{(\alpha)}(z) + (\lambda + \mu z^2) p_{n-1}^{(\alpha)}(z)$$

is stable for $-1 < \alpha \leq 1$.

Proof. For fixed $\lambda, \mu \geq 0$ and $\alpha, -1 < \alpha \leq 1$, consider

$$(29) \quad f_n(x; z) := \sum_{k=0}^n z^k D^k L_n^{(\alpha)}(x) + (\lambda + \mu z^2) \sum_{k=0}^{n-1} z^k D^k L_{n-1}^{(\alpha)}(x) \quad (x \in \mathbb{R}, z \in \mathbb{C}),$$

where $L_n^{(\alpha)}$ denotes the generalized Laguerre polynomial of degree $n \geq 2$ (cf. (1)). Note that for $z \neq 0$, f_n satisfies the differential equation

$$(30) \quad \frac{1}{z} \left(f_n - L_n^{(\alpha)} - (\lambda + \mu z^2) L_{n-1}^{(\alpha)} \right) = \frac{df_n}{dx}.$$

Now, let z_j be any zero of $f_n(0; z)$ and note that $z_j \neq 0$. Next, substitute z_j for z in (30), multiply both sides of (30) by $\frac{\overline{f_n(x; z_j)}}{x} x^\alpha e^{-x}$, add the complex conjugate of the resulting expression and then integrate:

$$(31) \quad \begin{aligned} & \left(\frac{1}{z_j} + \frac{1}{\overline{z_j}} \right) \int_0^\infty |f_n|^2 x^{\alpha-1} e^{-x} dx - \int_0^\infty \left(\frac{\overline{f_n}}{z_j x} + \frac{f_n}{\overline{z_j} x} \right) (L_n^{(\alpha)} + \lambda L_{n-1}^{(\alpha)}) x^\alpha e^{-x} dx \\ & - \mu \int_0^\infty \left(\frac{z_j \overline{f_n}}{x} + \frac{\overline{z_j} f_n}{x} \right) L_{n-1}^{(\alpha)} x^\alpha e^{-x} dx = \int_0^\infty \frac{d|f_n|^2}{dx} x^{\alpha-1} e^{-x} dx, \end{aligned}$$

where $f_n = f_n(x; z_j)$. Since $f_n(0; z_j) = 0$, $\frac{f_n(x; z_j)}{x}$ and its complex conjugate are polynomials of degree $n-1$ in x . Thus, by orthogonality of $L_n^{(\alpha)}$ with respect to the weight function $w(x) = x^\alpha e^{-x}$,

$$I_0 = \int_0^\infty \left(\frac{\overline{f_n}}{x z_j} + \frac{f_n}{x \overline{z_j}} \right) L_n^{(\alpha)} x^\alpha e^{-x} dx = 0.$$

Therefore, with the aid of (21) and (22) in Lemma 2 (cf. I_1 and I_2), we can express (31) in the form

$$\left(\frac{1}{z_j} + \frac{1}{\overline{z_j}} \right) \int_0^\infty |f_n|^2 x^{\alpha-1} e^{-x} dx - \lambda I_1 - \mu I_2 = \int_0^\infty \frac{d|f_n|^2}{dx} x^{\alpha-1} e^{-x} dx,$$

or, using the explicit expressions for I_1 and I_2 and simplifying, we have

$$\begin{aligned} & \left(\frac{1}{z_j} + \frac{1}{\overline{z_j}} \right) \left[\int_0^\infty |f_n|^2 x^{\alpha-1} e^{-x} dx + \frac{1}{n} \frac{\Gamma(n+\alpha)}{(n-1)!} (\lambda + \mu |z_j|^2) \right] \\ & = \int_0^\infty \frac{d|f_n|^2}{dx} x^{\alpha-1} e^{-x} dx. \end{aligned}$$

Further simplification, together with an integration by parts, yields

$$(32) \quad \frac{2 \operatorname{Re}(z_j)}{|z_j|^2} \left[\int_0^\infty |f_n|^2 x^{\alpha-1} e^{-x} dx + \frac{\Gamma(n+\alpha)}{n!} (\lambda + \mu |z_j|^2) \right] \\ = - \int_0^\infty |f_n|^2 (\alpha - 1 - x) x^{\alpha-2} e^{-x} dx.$$

If $-1 < \alpha \leq 1$ and $x \geq 0$, then $\alpha - 1 - x \leq 0$ and whence $\operatorname{Re}(z_j) > 0$, by (32). Therefore, if $\lambda, \mu \geq 0$, then the polynomial

$$(33) \quad f_n(0; -z) = \sum_{k=0}^n (-1)^k z^k D^k L_n^{(\alpha)}(0) + (\lambda + \mu z^2) \sum_{k=0}^{n-1} (-1)^k z^k D^k L_{n-1}^{(\alpha)}(0) \\ = \sum_{k=0}^n \binom{n+\alpha}{n-k} z^k + (\lambda + \mu z^2) \sum_{k=0}^{n-1} \binom{n-1+\alpha}{n-1-k} z^k \\ = p_n^{(\alpha)}(z) + (\lambda + \mu z^2) p_{n-1}^{(\alpha)}(z) \\ = P_n(z; \alpha, \lambda, \mu)$$

is stable for $-1 < \alpha \leq 1$. □

From the above theorem we deduce the following result.

Theorem 9. *Let*

$$(34) \quad q_n^{(\alpha)}(z) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} z^k D^{2k} L_n^{(\alpha)}(0) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+\alpha}{n-2k} z^k.$$

If $\lambda, \mu \geq 0$, then the zeros of the polynomial

$$(35) \quad Q_n(z; \alpha, \lambda, \mu) := q_n^{(\alpha)}(z) + (\lambda + \mu z) q_{n-1}^{(\alpha)}(z) \quad (n \geq 2)$$

are all real and negative for $-1 < \alpha \leq 2$.

Proof. Fix $\lambda, \mu \geq 0$. By Theorem 8, the polynomial

$$(36) \quad P_n(z; \alpha, \lambda, \mu) := p_n^{(\alpha)}(z) + (\lambda + \mu z^2) p_{n-1}^{(\alpha)}(z)$$

is stable for $-1 < \alpha \leq 1$. Now a calculation shows that $P_n(z; \alpha, \lambda, \mu)$ can be expressed as

$$(37) \quad P_n(z; \alpha, \lambda, \mu) = \Omega_{1,n}^{(\alpha)}(z^2) + z \Omega_{2,n-1}^{(\alpha+1)}(z^2),$$

where

$$(38) \quad \Omega_{1,n}^{(\alpha)}(z) := q_n^{(\alpha)}(z) + (\lambda + \mu z) q_{n-1}^{(\alpha)}(z) \quad \text{and}$$

$$(39) \quad \Omega_{2,n-1}^{(\alpha+1)}(z) := q_{n-1}^{(\alpha+1)}(z) + (\lambda + \mu z) q_{n-2}^{(\alpha+1)}(z).$$

Hence, by the Hermite-Biehler Theorem, the zeros of $\Omega_{1,n}^{(\alpha)}(z) = Q_n(z; \alpha, \lambda, \mu)$ are all real and negative for $-1 < \alpha \leq 1$. Since the zeros of $\Omega_{2,n-1}^{(\alpha+1)}$ are also all real and negative for $-1 < \alpha \leq 1$, the change of indices, $n \rightarrow n+1$ and $\alpha \rightarrow \alpha-1$, shows that the zeros of $Q_n(z; \alpha, \lambda, \mu)$ are all real and negative for $0 < \alpha \leq 2$. □

As an immediate consequence of Theorem 9 and Lemma 3, we obtain the following principal result of this section.

Theorem 10. *If $-1 < \alpha \leq 2$, then the polynomials $q_n^{(\alpha)}$ and $q_{n-1}^{(\alpha)}$ (defined in (34) of Theorem 9) form a positive pair.*

Remark 5. Asymptotic results for Jacobi polynomials with negative parameters have been obtained recently. From these results one may derive conclusions about the distribution of zeros when $n \rightarrow \infty$ (see, for example, A. B. J. Kuijlaars and A. Martinez-Finkelshtein [16], A. B. J. Kuijlaars, A. Martinez-Finkelshtein and R. Orive [17], A. Martinez-Finkelshtein and R. Orive [20]).

3. SCHOLIA. MULTIPLIER SEQUENCES, n -SEQUENCES AND STABILITY

Linear operators that preserve stability play an important role in stability analysis. Recently, in a series of papers ([1], [2], [3]), J. Borcea and P. Brändén obtained a complete characterization of all linear operators that preserve stability of univariate as well as multivariate polynomials. Here, our focus will be on some special classes of stability-preserving linear operators as they pertain to the results obtained in Section 2 (cf. Remark 3). By Theorem 4 and Remark 2, the polynomial $b_n^{(\alpha)}(z) = \sum_{k=0}^n \binom{n+\alpha}{k} z^k$ is stable for $-1 < \alpha \leq 1$. Since the Hadamard product of stable polynomials is stable (J. Garloff and D.G. Wagner [14]), it follows that the polynomial

$$(40) \quad B_n^{(\alpha)}(z) := b_n^{(\alpha)}(z) * (1+z)^n := \sum_{k=0}^n \binom{n}{k} \binom{n+\alpha}{k} z^k$$

is stable for $-1 < \alpha \leq 1$. But more is true, for we will show that if $-1 < \alpha$, then the polynomials $B_n^{(\alpha)}$, $n = 1, 2, 3, \dots$, have only real and negative zeros.

Remark 6. If $\alpha = m$ is a non-negative integer, then it is known, and in any case it is easy to prove, that $B_n^{(m)}$, $\deg B_n^{(m)}(z) = \min(m, n)$, has only real negative zeros. Rather than elaborate here on an elementary proof, we justify the aforementioned claim by noting that it is an immediate consequence of the Schur-Szegő-Malo Composition Theorem [21, Satz 7.5, p. 29]. We remark that the polynomials $B_n^{(\alpha)}$ are hypergeometric polynomials which can be expressed in terms of Jacobi polynomials, where one of the parameters depends on n (see, for example, [25, p. 62, eq. 4.21.2]):

$$B_n^{(\alpha)}(z) = {}_2F_1(-n, -n - \alpha; 1; z) = P_n^{(0, -2n - \alpha - 1)}(1 - 2z).$$

In order to motivate our proof, we will digress here for a moment and succinctly summarize some facts from the theory of multiplier sequences. To begin with, we recall the following terminology. A sequence $T = \{\gamma_k\}_{k=0}^{\infty}$ of non-negative real numbers is called a *multiplier sequence* if, whenever the real polynomial $p(z) = \sum_{k=0}^n a_k z^k$ has *only* real zeros, the polynomial $T[p(z)] := \sum_{k=0}^n \gamma_k a_k z^k$ also has *only* real zeros. (For a survey of results pertaining to multiplier sequences, see [8] and the references contained therein.) The *algebraic characterization* of multiplier sequences [22, p. 111] asserts that $T = \{\gamma_k\}_{k=0}^{\infty}$, $\gamma_k \geq 0$, is a multiplier sequence if and only if the polynomial

$$(41) \quad g_m(z) := T[(1+z)^m] = \sum_{j=0}^m \binom{m}{j} \gamma_j z^j \quad \text{has only real zeros for all } m = 1, 2, 3, \dots$$

The polynomials $g_m(z)$ are called the *Jensen polynomials* associated with T . The *transcendental characterization* of multiplier sequences may be expressed as follows. Let $\varphi(z) := \sum_{k=0}^{\infty} \gamma_k \frac{z^k}{k!}$, $\gamma_k \geq 0$, be a real entire function of order at most one. Then the Taylor coefficients $\{\gamma_k\}_{k=0}^{\infty}$ form a multiplier sequence if and only if $\varphi(z)$ has only real negative zeros ([18, Ch. VIII], [22, p. 115], [9] or [7]). Another fundamental result states that if φ is an entire function of order at most one having only real negative zeros, then the sequence $T = \{\varphi(k)\}_{k=0}^{\infty}$ is a multiplier sequence by a theorem of Laguerre (cf. [21, Satz 3.2] or [7, Theorem 1.4]). One of the simplest stability-preserving operators is the differentiation operator, $D := d/dz$, by virtue of the Gauss-Lucas Theorem ([19, p. 12], [9]). The next theorem says that multiplier sequences are also stability-preserving operators.

Theorem 11 ([9]). *Let $T = \{\gamma_k\}_{k=0}^{\infty}$ be a non-negative multiplier sequence. Then $T[p(z)] := \sum_{k=0}^n a_k \gamma_k z^k$ is a stable polynomial whenever $p(z) = \sum_{k=0}^n a_k z^k$ is a stable polynomial.*

The paradigm example of a non-trivial multiplier sequence is the sequence $T = \{\frac{1}{k!}\}_{k=0}^{\infty} = \{\frac{1}{\Gamma(k+1)}\}_{k=0}^{\infty}$. Since $p_n^{(\alpha)}(z) = \sum_{k=0}^n \binom{n+\alpha}{k} z^k$, $-1 < \alpha \leq 1$, is stable, it follows from Theorem 11 that the polynomial $T[p_n^{(\alpha)}(z)] = \sum_{k=0}^n \binom{n+\alpha}{k} \frac{z^k}{k!}$, $-1 < \alpha \leq 1$, is also a stable polynomial. However, simple examples show that the polynomial $T[p_n^{(\alpha)}(z)]$, where $T = \{\frac{1}{k!}\}_{k=0}^{\infty}$ need not have only real zeros. (For example, set $n = 7$ and $\alpha = -1/2$.) These considerations and examples suggest that we generalize the notion of a multiplier sequence as follows.

Definition 12. For a fixed positive integer n , let $T = \{\gamma_0, \gamma_1, \dots, \gamma_n\}$, $\gamma_k \geq 0$, $k = 0, 1, \dots, n$. Then T is called an *n-sequence* if for every real polynomial $p(z)$, $\deg p(z) \leq n$, with only real zeros, the polynomial $T[p(z)]$ also has only real zeros.

Thus, it is clear from the definition that every multiplier sequence is an *n-sequence* (for all $n \in \mathbb{N}$), but that, in general, an *n-sequence* need not be a multiplier sequence. Today, there is no known transcendental characterization of *n-sequences*, but there is a useful algebraic characterization that is similar to the one cited above for multiplier sequences.

Proposition 13 ([6]). *Let $T = \{\gamma_0, \gamma_1, \dots, \gamma_n\}$, $\gamma_k \geq 0$. Then T is an *n-sequence* if and only if the polynomial $T[(1+z)^n]$ has only real zeros.*

With the terminology adopted here, our goal is to prove that, for $-1 < \alpha$, the sequence $T = \{\binom{n+\alpha}{k}\}_{k=0}^n$ is an *n-sequence*. Thus, in light of the above proposition, we need to show that the polynomial $B_n^{(\alpha)}$ (cf. (40)) has only real negative zeros. For the reader's convenience we also recall here a few pertinent facts about the Gauss hypergeometric function, ${}_2F_1$, defined by

$$(42) \quad {}_2F_1(a, b; c; z) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \quad |z| < 1,$$

where a , b and c are complex parameters and $(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1) = \Gamma(\alpha+k)/\Gamma(\alpha)$ is the Pochhammer symbol. If $\alpha = -n$ is a negative integer, the series terminates and reduces to a polynomial of degree n , called a *hypergeometric polynomial*. Since $(-n)_k = (-1)^k n!/(n-k)!$ if $0 \leq k \leq n$, $(-n)_k = 0$ if $k \geq n+1$,

we find that

$$(43) \quad {}_2F_1(-n, -n - \alpha; 1; z) = 1 + \sum_{k=1}^n \frac{(-n)_k (-n - \alpha)_k}{k!} \frac{z^k}{k!} = \sum_{k=0}^n \binom{n}{k} \binom{n + \alpha}{k} z^k.$$

Using the Hilbert-Klein formulae [25, Theorem 7.2, p. 145] and the connection between hypergeometric polynomials, ${}_2F_1$, and the Jacobi polynomials, K. Driver and K. Jordaan [11] recently obtained the following result concerning the zero distribution of hypergeometric polynomials.

Theorem 14 ([11, Theorem 3.2 (iv), (v)]). *Fix $b, c \in \mathbb{R}$, where $c > 0$, and set $F(z) = {}_2F_1(-n, b; c; z)$.*

- (i) *If $b < -n$, then all the zeros of $F(z)$ are real and negative.*
- (ii) *If $-n < b < 0$, $-j < b < -j + 1$, $j = 1, 2, \dots, n$, then $F(z)$ has j real negative zeros. The remaining $(n - j)$ zeros of F are all non-real if $(n - j)$ is even, while if $(n - j)$ is odd, F has $(n - 1 - j)$ non-real zeros and one additional zero in $(1, \infty)$.*

Preliminaries aside, we are now in a position to show that the polynomial $B_n^{(\alpha)}$ has only real negative zeros. If α is not an integer, this establishes the existence of (what appears to be) a new family of n -sequences.

Theorem 15. *Let $n \geq 2$. If $-1 < \alpha$, then the polynomial $B_n^{(\alpha)}$ has only real negative zeros and whence the sequence $T = \{\binom{n+\alpha}{k}\}_{k=0}^n$ is an n -sequence.*

Proof. First we note that for $\alpha > -1$ and $0 \leq k \leq n$, $\binom{n+\alpha}{k} \geq 0$. Thus, by Proposition 13, it suffices to show that the polynomial

$$(44) \quad B_n^{(\alpha)}(z) := \sum_{k=0}^n \binom{n}{k} \binom{n+\alpha}{k} z^k$$

has only real zeros. Set $c = 1$ and $b = -n - \alpha$. Now $B_n^{(\alpha)}(z) = {}_2F_1(-n, -n - \alpha; 1; z)$ (cf. (43)). If $\alpha > 0$, then $b < -n$, and hence by Theorem 14 (i), $B_n^{(\alpha)}$ has only real negative zeros. If $-1 < \alpha < 0$, then $b = -n - \alpha < -n + 1$, and whence by part (ii) of Theorem 14 (with $j = n$), $B_n^{(\alpha)}$ has only real negative zeros. Finally, $B_n^{(0)}$ has only real negative zeros, since the sequence $\{\binom{n}{k}\}_{k=0}^n$ is a multiplier sequence (cf. Remark 5). $\square$

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