SUB-BERGMAN SPACES IN THE UNIT BALL OF $\mathbb{C}^n$

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ABSTRACT. Let $\Phi(z) = (\varphi_1(z), \dots, \varphi_l(z))$ be holomorphic from the unit ball of $\mathbb{C}^n$ into the unit ball of $\mathbb{C}^l$. We denote by $B_\alpha(z, w)$ the weighted Bergman kernel. We give a condition for the kernel $(1 - \Phi(z)\overline{\Phi(w)})B_\alpha(z, w)$ to be a reproducing kernel and we study the related Hilbert space.

1. INTRODUCTION

The aim of this note is to give a characterization of some reproducing kernels of spaces of holomorphic functions on the unit ball of $\mathbb{C}^n$. These kernels are obtained from the weighted Bergman kernel $B_\alpha(z, w)$ and bounded holomorphic functions $\varphi_1(z), \dots, \varphi_l(z)$.

We denote by $\mathbb{B}$ the unit ball of $\mathbb{C}^n$ and by $r(z) = \sum_{i=1}^n |z_i|^2 - 1$ a defining function. Let $\alpha > -1$ and let $dV_\alpha(z) = (-r(z))^\alpha dV(z)$. We denote by $\mathbb{A}_\alpha^2$ the weighted Bergman space of $\mathbb{B}$ with respect to the measure $dV_\alpha(z)$ [5]. It is a Hilbert space and its reproducing kernel is the weighted Bergman kernel $B_\alpha(z, w) = \frac{c_\alpha}{(1 - z\overline{w})^{n+1+\alpha}}$. Let $\Phi(z) = (\varphi_1(z), \dots, \varphi_l(z))$ be holomorphic functions on the unit ball. We consider the kernel

$$(1.1) \quad K_\alpha(z, w) = \left(1 - \Phi(z)\overline{\Phi(w)}\right) B_\alpha(z, w).$$

We have the following result:

Theorem 1.1. *Let $\alpha > -1$ and let $\Phi(z) = (\varphi_1(z), \dots, \varphi_l(z))$ be holomorphic. The kernel $K_\alpha(z, w)$ is a reproducing kernel if and only if $\sup_{\mathbb{B}} |\Phi(w)| \leq 1$.*

Let φ be a holomorphic function; the reproducing property of kernels of type $(1 - \varphi(z)\overline{\varphi(w)})B(z, w)$ was studied by F. Beatrous and J. Burbea [2] for spaces of holomorphic functions on domains of $\mathbb{C}^n$ and by S. Saitoh for abstract kernels of Hilbert spaces of functions [7]. In the case of the unit disc of $\mathbb{C}$, when φ is a finite Blaschke product, the characterization of the related Hilbert space was obtained by K. Zhu in [10] and [11] for Bergman spaces and by S. Sultanic for weighted Bergman spaces [9]. The same question was introduced by L. De Branges and J. Rovnyak in the context of Hardy spaces [8]. The following theorem gives the Hilbert space associated to the kernel $K_\alpha(z, w)$ in the case of the unit ball of $\mathbb{C}^n$.

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For $-n \leq \alpha < -1$, $B_\alpha(z, w)$ is the reproducing kernel of the diagonal weighted Besov space $\mathcal{B}_\alpha^2$. A holomorphic function f belongs to $\mathcal{B}_\alpha^2$ if and only if $(I + R)^m f$ is in $\mathbb{A}_{2m+\alpha}^2$, where $R = \sum_{k=1}^n z_k \partial_{z_k}$ is the radial derivative and $m > (-\alpha - 1)/2$ [13]. In this case we have the following:

Theorem 1.2. *Let $\Phi(z) = (\varphi_1(z), \dots, \varphi_l(z))$ be holomorphic and non-constant. The functional Hilbert space associated to $K_\alpha(z, w)$ is, with an equivalent norm,*

- (1) $\mathbb{A}_{\alpha-1}^2$ when $\alpha > 0$,
- (2) the Hardy space $\mathbb{H}^2$ when $\alpha = 0$,
- (3) $\mathcal{B}_{\alpha-1}^2$ when $-1 < \alpha < 0$

if and only if there exists $C > 0$ such that $1 - |\Phi(z)|^2 \leq C(-r(z))$ for z in $\mathbb{B}$.

Let us mention an example with $n = l \geq 2$ which motivates this note. Let a be a point of $\mathbb{B}_n$ and denote by Φ_a the related holomorphic automorphism. A straightforward computation gives $K_\alpha(z, w) = \frac{cB_{\alpha-1}(z, w)}{(1 - z\bar{a})(1 - a\bar{w})}$ and the Hilbert space is $\mathbb{A}_{\alpha-1}^2$ equipped with the norm $\|f\|^2 = \int_{\mathbb{B}} |f(w)|^2 |1 - a\bar{w}|^2 dV_{\alpha-1}(w)$.

In the case where $a = 0$, the kernel is $B_{\alpha-1}(z, w)$ and the Hilbert space is $\mathbb{A}_{\alpha-1}^2$ with its standard norm.

As far as we know, automorphisms are the only applications which satisfy the condition of the theorem in the case where $n = l \geq 2$.

2. SUB-BERGMAN SPACES

Recall that the Bergman projection B_α is the orthogonal projection from $L^2(dV_\alpha)$ onto $\mathbb{A}_\alpha^2$. Given a bounded function φ , the Toeplitz operator of symbol φ is defined by $T_\varphi f = B_\alpha(\varphi f)$. We consider the positive operator H given as follows:

Definition 2.1. Let $\Phi(z) = (\varphi_1(z), \dots, \varphi_l(z))$ be holomorphic. We set

$$Hf = \sum_{i=1}^l \varphi_i T_{\bar{\varphi}_i} f, \quad f \in \mathbb{A}_\alpha^2.$$

We first calculate the norm of H .

Proposition 2.2. *Let H be as above. Then $\|H\| = \sup_{\mathbb{B}} |\Phi(w)|^2$.*

Proof. Let f be in $\mathbb{A}_\alpha^2$. Then

$$\|Hf\|_{\mathbb{A}_\alpha^2} \leq \sum_{i=1}^l \|\varphi_i B_\alpha(\bar{\varphi}_i f)\|_{\mathbb{A}_\alpha^2} \leq \sup_{\mathbb{B}} |\Phi(w)| \sum_{i=1}^l \|B_\alpha(\bar{\varphi}_i f)\|_{\mathbb{A}_\alpha^2}.$$

Since B_α is a projection, $\|B_\alpha(\bar{\varphi}_i f)\|_{\mathbb{A}_\alpha^2} \leq \|\varphi_i f\|_{\mathbb{A}_\alpha^2}$ and then

$$\sum_{i=1}^l \|B_\alpha(\bar{\varphi}_i f)\|_{\mathbb{A}_\alpha^2} \leq \sup_{\mathbb{B}} |\Phi(w)| \|f\|_{\mathbb{A}_\alpha^2}.$$

Let $f_w(z) = \frac{B_\alpha(z, w)}{\sqrt{B_\alpha(w, w)}}$, with w in $\mathbb{B}$. Then $Hf_w(z) = \sum_{i=1}^l \varphi_i(z) \bar{\varphi}_i(w) f_w(z)$ and $\langle Hf_w, f_w \rangle_\alpha = |\Phi(w)|^2$, where $\langle \cdot, \cdot \rangle_\alpha$ is the inner product of $\mathbb{A}_\alpha^2$. The relation $\langle Hf_w, f_w \rangle_\alpha \leq \|H\| \|f\|_{\mathbb{A}_\alpha^2}^2$ and the fact that $\|f_w\|_{\mathbb{A}_\alpha^2} = 1$ finish the proof. $\square$

Proof of Theorem 1.1. Assume that $\sup_{\mathbb{B}} |\Phi(w)|^2 \leq 1$. Then H is a contraction and the operator $(I - H)^{1/2}$ is a well defined positive operator. It is well known that $(I - H)^{1/2} (\mathbb{A}_\alpha^2)$ is a sub-Hilbert space with reproducing kernel given by $(I - H)B_\alpha(z, w) = K_\alpha(z, w)$. See [8] and [10] for details.

Assume now that $K_\alpha(z, w)$ is a reproducing kernel; then for any finite sequence $(w_j)_{j=1..m}$ of points of the ball, the matrix $(K_\alpha(w_j, w_k))_{j,k=1..m}$ is positive definite [1]. If we consider a point w in the ball, the positivity gives $K_\alpha(w, w) = (1 - |\Phi(w)|^2) B_\alpha(w, w) \geq 0$, and then $|\Phi(w)|^2 \leq 1$. $\square$

Proof of Theorem 1.2. The proof involves the Douglas criterion [4]; we give here the statement.

Douglas criterion: Let $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}$ be Hilbert spaces. Let A from $\mathcal{H}_1$ into $\mathcal{H}$ and B from $\mathcal{H}_2$ into $\mathcal{H}$ be bounded operators.

There exists $\lambda > 0$ such that $AA^* \ll \lambda BB^*$ if and only if $\text{Im}(A) \subset \text{Im}(B)$.

In the definition, the relation $AA^* \ll \lambda BB^*$ means that $\lambda BB^* - AA^*$ is a positive operator.

We consider the identity map $\Phi_0(z) = z$ and we denote by H_0 its associated operator given by Definition 2.1. The kernel of the Hilbert space $(I - H_0)^{1/2} (\mathbb{A}_\alpha^2)$ is $(1 - z\bar{w}) B_\alpha(z, w) = B_{\alpha-1}(z, w)$. Then $(I - H_0)^{1/2} (\mathbb{A}_\alpha^2) = \mathbb{A}_{\alpha-1}^2$ by the uniqueness of the reproducing kernel.

If we suppose that $K_\alpha(z, w)$ is a reproducing kernel of $\mathbb{A}_{\alpha-1}^2$, then by the Douglas criterion there exists $\lambda > 0$ such that $I - H \ll \lambda(I - H_0)$. Let f be in $\mathbb{A}_\alpha^2$. Then $\langle (I - H)f, f \rangle_\alpha \leq \lambda \langle (I - H_0)f, f \rangle_\alpha$. Take w in $\mathbb{B}$. The function $f_w(z) = B_\alpha(z, w)$ gives $1 - |\Phi(w)|^2 \leq \lambda'(-r(w))$. $\square$

In the case where $\alpha \geq 1$, the sufficient condition follows from the next result:

Proposition 2.3. *Let $\Phi(z) = (\varphi_1(z), \dots, \varphi_l(z))$ be holomorphic. If $1 - |\Phi(z)|^2 \leq C(-r(z))$, there exists $\lambda > 0$ such that*

$$\frac{1}{\lambda}(I - H_0) \ll I - H \ll \lambda(I - H_0).$$

Proof of Proposition 2.3. Let $\alpha \geq 1$. The right “inequality” follows from two technical lemmas: $\square$

Lemma 2.4. *There exists $C > 0$ such that for f in $\mathbb{A}_\alpha^2$,*

$$\langle (I - H)f, f \rangle_\alpha \leq C \|f\|_{\mathbb{A}_{\alpha+1}^2}^2.$$

Proof. Take f in $\mathbb{A}_\alpha^2$. Then

$$(2.1) \quad \langle (I - H)f, f \rangle_\alpha = \int_{\mathbb{B}} \int_{\mathbb{B}} K_\alpha(z, w) f(w) \overline{f(z)} (-r(w))^\alpha (-r(z))^\alpha dV(w) dV(z).$$

Let $K(z, w) = K_\alpha(z, w)(-r(w))^{-1/2+\alpha/2}(-r(z))^{-1/2+\alpha/2}$ and denote by K the integral operator given by

$$(2.2) \quad Kg(z) = \int_{\mathbb{B}} K(z, w) g(w) dV(w), \quad g \in L^2(\mathbb{B}).$$

Assume for the moment that K is bounded in $L^2(\mathbb{B})$. We consider the function $g(z) = f(z)(-r(z))^{1/2+\alpha/2}$; then

$$\begin{aligned} \langle (I - H)f, f \rangle_\alpha &= \int_{\mathbb{B}} K g(z) \overline{g(z)} dV(z) \\ &\leq \|K(g)\|_{L^2(\mathbb{B})} \|f\|_{\mathbb{A}_{\alpha+1}^2} \leq C \|f\|_{\mathbb{A}_{\alpha+1}^2}^2. \end{aligned}$$

It remains to study the boundedness of K . This follows from the Schur criterion [12].

We only have to prove that there exists $C > 0$ such that $|K(z, w)| \leq \frac{C}{|1 - z\bar{w}|^{n+1}}$.

Recall that (see [6, Prop. 8.1.4])

$$(2.3) \quad \frac{|1 - \Phi(z)\overline{\Phi(w)}|^2}{(1 - |\Phi(z)|^2)(1 - |\Phi(w)|^2)} \leq \frac{|1 - z\bar{w}|^2}{(-r(z))(-r(w))}.$$

Therefore there exists $C > 0$ such that $|1 - \Phi(z)\overline{\Phi(w)}| \leq C|1 - z\bar{w}|$, and then $|K(z, w)|$ has the desired pointwise estimate. $\square$

We finish the proof of the proposition with the second technical lemma:

Lemma 2.5. *Let $\alpha > -1$. There exists $C > 0$ such that*

$$\frac{1}{C} \|f\|_{\mathbb{A}_{\alpha+1}^2}^2 \leq \langle (I - H_0)f, f \rangle_\alpha \leq C \|f\|_{\mathbb{A}_{\alpha+1}^2}^2, \quad f \in \mathbb{A}_{\alpha+1}^2.$$

Proof. The proof is direct. Let $f(z) = \sum_m a_m z^m$. Recall that

$$\|z^m\|_{\mathbb{A}_\alpha^2}^2 = c_\alpha \frac{\Gamma(m_1 + 1) \cdots \Gamma(m_n + 1)}{\Gamma(|m| + n + 1 + \alpha)},$$

where $|m| = m_1 + \cdots + m_n$ [6]. Let us remark that $(1 - z\bar{w})B_\alpha(z, w) = cB_{\alpha-1}(z, w)$. Then

$$(I - H_0)f(z) = c \int_{\mathbb{B}} B_{\alpha-1}(z, w) f(w) (-r(w))^\alpha dV(w) = c \sum_m \frac{a_m z^m}{|m| + n + \alpha}$$

and

$$\langle (I - H_0)f, f \rangle_\alpha = c \sum_m |a_m|^2 \frac{\Gamma(m_1 + 1) \cdots \Gamma(m_n + 1)}{(|m| + n + \alpha) \Gamma(|m| + n + 1 + \alpha)}.$$

Notice that

$$\|f\|_{\mathbb{A}_{\alpha+1}^2}^2 = c \sum_m |a_m|^2 \|z^m\|_{\mathbb{A}_{\alpha+1}^2}^2 = c \sum_m |a_m|^2 \frac{\Gamma(m_1 + 1) \cdots \Gamma(m_n + 1)}{\Gamma(|m| + n + 2 + \alpha)}.$$

The equivalence follows from the relation $\Gamma(z + 1) = z\Gamma(z)$.

For the left “inequality” of Proposition 2.3, we consider the auxiliary operator $\tilde{H}$ given by $\tilde{H}f = T_{|\Phi|^2}(f) = B_\alpha(|\Phi|^2 f)$, $\alpha > -1$. Let us remark that $\|\tilde{H}\| \leq \sup_{\mathbb{B}} |\Phi(w)|^2$ and $\langle Hf, f \rangle_\alpha = \sum_{i=1}^l \langle \varphi_i f, B_\alpha(\varphi_i f) \rangle_\alpha \leq \sum_{i=1}^l \langle \varphi_i f, \varphi_i f \rangle_\alpha = \langle \tilde{H}f, f \rangle_\alpha$; then

$$(2.4) \quad I - \tilde{H} \ll I - H.$$

We have the following result: $\square$

Lemma 2.6. *Let $\tilde{H}$ be as above. There exists $\lambda > 0$ such that $I - H_0 \ll \lambda(I - \tilde{H})$.*

Proof. Let us remark that $1 - |\Phi(0)|^2 \leq 2|1 - \Phi(z)\overline{\Phi(0)}|$. It follows from the relation (2.3) that there exists $C > 0$ such that $-r(z) \leq C(1 - |\Phi(z)|^2)$ and $\|f\|_{\mathbb{A}_{\alpha+1}^2}^2 \leq C\langle (I - \tilde{H})f, f \rangle_\alpha$. It remains to apply Lemma 2.5 to achieve the proof.

The proof is similar for $-1 < \alpha \leq 1$, but a minor modification is required to use the Schur criterion. We recall that for holomorphic functions u and v [6],

$$\begin{aligned}
 & \int_{\mathbb{B}} u(w) \overline{v(w)} (-r(w))^\alpha dV(w) \\
 &= \int_{\mathbb{B}} (R + (n+1+\alpha)I) u(w) \overline{v(w)} (-r(w))^{\alpha+1} dV(w) \\
 (2.5) \quad &= \int_{\mathbb{B}} u(w) (\overline{R} + (n+1+\alpha)I) \overline{v(w)} (-r(w))^{\alpha+1} dV(w).
 \end{aligned}$$

The kernel of $(I - H_0)^{1/2} (\mathbb{A}_\alpha^2)$ is given by $(1 - z\overline{w}) B_\alpha(z, w) = cB_{\alpha-1}(z, w)$; thus $(I - H_0)^{1/2} (\mathbb{A}_\alpha^2)$ is the weighted Bergman space $\mathbb{A}_{\alpha-1}^2$ when $0 < \alpha < 1$, the Hardy space $\mathbb{H}^2$ for $\alpha = 0$ and the weighted Besov space $\mathcal{B}_{\alpha-1}$ for $-1 < \alpha < 0$.

For f in $\mathbb{A}_\alpha^2$,

$$(I - H)f(z) = \int_{\mathbb{B}} (\overline{R} + (n+1+\alpha)I) K_\alpha(z, w) f(w) (-r(w))^{\alpha+1} dV(w)$$

and the relation (2.1) becomes

$$\langle (I - H)f, f \rangle_\alpha = \int_{\mathbb{B}} (R + (n+1+\alpha)I) (I - H)f(z) \overline{f(z)} (-r(z))^{\alpha+1} dV(z).$$

Let

$$\begin{aligned}
 K(z, w) &= (-r(z))^{1/2+\alpha/2} (-r(w))^{1/2+\alpha/2} \\
 &\quad \times (R + (n+1+\alpha)I) (\overline{R} + (n+1+\alpha)I) K_\alpha(z, w)
 \end{aligned}$$

and let K be the integral operator given by (2.2). Then

$$\langle (I - H)f, f \rangle_\alpha = \int_{\mathbb{B}} K g(z) \overline{g(z)} dV(z)$$

with $g(z) = f(z) (-r(z))^{1/2+\alpha/2}$. Let us remark that

$$\begin{aligned}
 K(z, w) &= \sum_{s,t=0}^1 (I + (n+1)R)^s (I + (n+1)\overline{R})^t (1 - \Phi(z)\overline{\Phi(w)}) \\
 &\quad \times (I + (n+1)R)^{1-s} (I + (n+1)\overline{R})^{1-t} B_\alpha(z, w).
 \end{aligned}$$

Then there exists $C > 0$ such that

$$\begin{aligned}
 |K(z, w)| &\leq C(-r(z))^{1/2+\alpha/2} (-r(w))^{1/2+\alpha/2} \left(\frac{|1 - \Phi(z)\overline{\Phi(w)}|}{|1 - z\overline{w}|^{n+1+\alpha+2}} \right. \\
 &\quad \left. + \frac{\sum_i \left| \frac{\partial \varphi_i(w)}{\partial w_j} \right| + \sum_i \left| \frac{\partial \varphi_i(z)}{\partial z_j} \right|}{|1 - z\overline{w}|^{n+1+\alpha+1}} + \frac{\sum_i \left| \frac{\partial \varphi_i(w)}{\partial w_j} \right| \left| \frac{\partial \varphi_i(z)}{\partial z_j} \right|}{|1 - z\overline{w}|^{n+1+\alpha}} \right).
 \end{aligned}$$

Recall that $|1 - \Phi(z)\overline{\Phi(w)}| \leq C|1 - z\overline{w}|$ and let us remark that the condition on Φ gives $\sup_{\mathbb{B}} \left| \frac{\partial \varphi_i(w)}{\partial w_j} \right| \leq C$; then

$$|K(z, w)| \leq \frac{C(-r(z))^{\alpha/2+1/2}(-r(w))^{\alpha/2+1/2}}{|1 - z\overline{w}|^{n+\alpha+2}}.$$

Let us remark that $(-r(z))$ and $(-r(w))$ are bounded by a constant times $|1 - z\overline{w}|$. Then $|K(z, w)| \leq \frac{C}{|1 - z\overline{w}|^{n+1}}$. The operator K is bounded in $L^2(\mathbb{B})$ and the relation (2.1) gives $\langle (I - H)f, f \rangle_\alpha \leq C\|f\|_{\mathbb{A}_{\alpha+1}^2}^2$. Lemma 2.5 gives $I - H \leq \lambda(I - H_0)$. $\square$

Theorem 1.2 characterizes the sub-Hilbert space $(I - H)^{1/2}(\mathbb{A}_\alpha)$: this space is the analogue of the unit ball of $\mathbb{C}^n$ for the complementary space [3]. We consider the space $(I - \tilde{H})^{1/2}(\mathbb{A}_\alpha)$; it is a consequence of the relation (2.4) and Lemma 2.6 that there exists $C > 0$ such that

$$\frac{1}{C}(I - H_0) \ll I - \tilde{H} \ll I - H.$$

Under the condition on Φ given in Theorem 1.2, we have

$$(I - \tilde{H})^{1/2}(\mathbb{A}_\alpha) = (I - H)^{1/2}(\mathbb{A}_\alpha).$$

REFERENCES

1. N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc. **68** (1950), 337–404. MR0051437 (14:479c)
2. F. Beatrous and J. Burbea, *Positive-definiteness and its applications to interpolation problems for holomorphic functions*, Trans. Amer. Math. Soc. **284** (1984), 247–270. MR742424 (85e:32020)
3. L. de Branges and J. Rovnyak, *Square summable power series*, Holt, Rinehart and Winston, New York, 1966. MR0215065 (35:5909)
4. R. Douglas, *On majorization, factorization, and range inclusion of operators on Hilbert space*, Proc. Amer. Math. Soc. **1** (1966), 413–415. MR0203464 (34:3315)
5. S. Krantz, *Function theory of several complex variables*, John Wiley and Sons, New York, 1982. MR635928 (84c:32001)
6. W. Rudin, *Function theory in the unit ball of $\mathbb{C}^n$* , Classics in Mathematics **241**, Springer-Verlag, New York, 1980. MR601594 (82i:32002)
7. S. Saitoh, *Theory of reproducing kernels and its applications*, Pitman Research Notes in Mathematics **189**, Wiley, New York, 1988. MR983117 (90f:46045)
8. D. Sarason, *Sub-Hardy Hilbert spaces in the unit disk*, John Wiley & Sons, Inc., New York, 1994. MR1289670 (96k:46039)
9. S. Sultanic, *Sub-Bergman Hilbert spaces*, J. of Math. Anal. Appl. **234** (2005), 639–649. MR2262497 (2007h:46042)
10. K. Zhu, *Sub-Bergman Hilbert spaces on the unit disk*, Indiana Univ. Math. J. **45** (1996), 165–176. MR1406688 (97f:46043)
11. K. Zhu, *Sub-Bergman Hilbert spaces on the unit disk II*, Journal of Funct. Anal. **202** (2003), 327–341. MR1990528 (2004g:46040)

12. K. Zhu, *Operator theory in function spaces*, Marcel Dekker, New York, 1990. MR1074007 (92c:47031)
13. K. Zhu, *Spaces of holomorphic functions in the unit ball*, Springer-Verlag, New York, 2004. MR2115155 (2006d:46035)

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