SINGULAR INTEGRALS ALONG N DIRECTIONS IN $\mathbb{R}^2$

CIPRIAN DEMETER

(Communicated by Michael T. Lacey)

ABSTRACT. We prove optimal bounds in $L^2(\mathbb{R}^2)$ for the maximal operator obtained by taking a singular integral along N arbitrary directions in the plane. We also give a new proof for the optimal L^2 bound for the single scale Kakeya maximal function in the plane.

1. INTRODUCTION

Our main result is the following:

Theorem 1.1. *Let Σ_N be a set of N unit vectors in $\mathbb{R}^2$. Let m be a Hörmander-Mikhlin multiplier; that is,*

$$|\partial^\alpha m(\xi)| \lesssim_\alpha \frac{1}{|\xi|^\alpha}, \quad \xi \neq 0$$

for each $\alpha \geq 0$. Define

$$T_N^* f(x, y) := \sup_{v \in \Sigma_N} \left| \int \hat{f}(\xi, \eta) m(v_1 \xi + v_2 \eta) e^{i(x\xi + y\eta)} d\xi d\eta \right|.$$

Then

$$\|T_N^*\|_{2 \rightarrow 2} \lesssim \log N.$$

In Section 3 we also prove that the bound $\log N$ is optimal. It is well known (see for example Proposition 2, page 245 in [11]) that the distribution $\hat{m}$ coincides with a function K outside the origin, which means that if f is compactly supported and (x, y) is outside the support of f , then we also have the following kernel representation:

$$T_N^* f(x, y) = \sup_{v \in \Sigma_N} \left| \int f((x, y) + tv) K(t) dt \right|.$$

Thus, Theorem 1.1 is the singular integral analogue of the following bound for the Kakeya maximal function due to Nets Katz:

Theorem 1.2 ([6]). *Let Σ_N be a set of N unit vectors in $\mathbb{R}^2$. Define*

$$M_N^* f(x, y) := \sup_{v \in \Sigma_N} \sup_{\epsilon > 0} \left| \frac{1}{2\epsilon} \int_{|t| < \epsilon} f((x, y) + tv) dt \right|.$$

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Then

$$\|M_N^*\|_{2 \rightarrow 2} \lesssim \log N.$$

The techniques behind Theorem 1.1 also give the following.

Theorem 1.3. *Let Σ be a lacunary set of unit vectors in $\mathbb{R}^2$, and denote by Σ_N the first N of them. Let m be a Hörmander-Mikhlin multiplier; that is,*

$$|\partial^\alpha m(\xi)| \lesssim_\alpha \frac{1}{|\xi|^\alpha}, \quad \xi \neq 0$$

for each $\alpha \geq 0$. Define

$$T_N^* f(x, y) := \sup_{v \in \Sigma_N} \left| \int \hat{f}(\xi, \eta) m(v_1 \xi + v_2 \eta) e^{i(x\xi + y\eta)} d\xi d\eta \right|.$$

Then

$$\|T_N^*\|_{2 \rightarrow 2} \lesssim \sqrt{\log N}.$$

This result is sharp. Indeed, it is proved in [5] that in the case of the Hilbert transform $m(\xi) = \operatorname{sgn}(\xi)$, $\|T_N^*\|_{2 \rightarrow 2} \gtrsim \sqrt{\log N}$, for any set Σ_N consisting of N (not necessarily lacunary) directions. This is in sharp contrast with the case of the maximal function M_Σ^* , which satisfies $\|M_\Sigma^*\|_{2 \rightarrow 2} \lesssim 1$, for each lacunary set Σ (see [10]).

In Section 4 we give a new proof for the following fact proved in [7].

Theorem 1.4. *Let Σ_N be a set of N unit vectors in $\mathbb{R}^2$. Define the single scale Keakeya operator*

$$M_N^0 f(x, y) := \sup_{v \in \Sigma_N} \left| \int_{|t| < 1} f((x, y) + tv) dt \right|.$$

Then

$$\|M_N^0\|_{2 \rightarrow 2} \lesssim \sqrt{\log N}.$$

The proofs in both [6] and [7] rely on stopping time arguments similar to John Nirenberg's inequality in the context of product BMO. Perhaps surprisingly, our arguments in this paper avoid product theory and are essentially L^2 based.

The proof of Theorem 1.1 is very similar to the proof of Theorem 1.1 in [4]; in particular, it relies on the deep inequality of Chang, Wilson and Wolff. The second major ingredient we use is a result of Lacey and Li [8] for single annulus operators, which is essentially equivalent with Carleson's theorem [2].

In the case $K(t) = p.v. \frac{1}{t}$, the bound

$$\|H_N^*\|_{2 \rightarrow 2} \lesssim \log N$$

for the operator

$$\begin{aligned} & H_N^* f(x, y) \\ &:= \sup_{v \in \Sigma_N} \left| p.v. \int f((x, y) + tv) \frac{dt}{t} \right| = \sup_{v \in \Sigma_N} \left| \int \hat{f}(\xi, \eta) \operatorname{sgn}(v_1 \xi + v_2 \eta) e^{i(x\xi + y\eta)} d\xi d\eta \right| \end{aligned}$$

immediately follows from the Rademacher-Menshov theorem. We do not see a way to extend this type of approach to the general case.

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2. DISCRETE VERSUS CONTINUOUS SQUARE FUNCTION

To simplify notation, we will often replace (x, y) with x .

Recall the conditional expectation with respect to the σ -algebra consisting of dyadic squares of side length 2^{-j} ,

$$E_j f(x) := \sum_{Q: |Q|=2^{-2j}} \langle f, \frac{1_Q}{|Q|} \rangle 1_Q(x)$$

and let $\Delta_j = E_{j+1} - E_j$ be the martingale difference. Denote by

$$\Delta(f)(x) = \left(\sum_{k \geq 0} |\Delta_k f(x)|^2 \right)^{1/2}$$

the discrete square function. We will need the following fundamental inequality.

Lemma 2.1 (The Chang-Wilson-Wolff inequality, [3]). *There exist constants c_1, c_2 such that for all $\lambda > 0$ and $0 < \epsilon < 1$ one has*

$$\begin{aligned} |\{x \in \mathbb{R}^2 : |f(x) - E_0 f(x)| > 2\lambda, \Delta(f)(x) < \epsilon\lambda\}| \\ \leq c_2 e^{-\frac{c_1}{\epsilon^2}} |\{x \in \mathbb{R}^2 : \sup_{k \geq 0} |E_k f(x)| > \epsilon\lambda\}|. \end{aligned}$$

Define for each $v = (v_1, v_2) \in \Sigma_N$,

$$T_v f(x, y) := \int \hat{f}(\xi, \eta) m(v_1 \xi + v_2 \eta) e^{i(x\xi + y\eta)} d\xi d\eta.$$

Let $\phi \in C^\infty(\mathbb{R}^2)$ be supported in $1/2 < |\xi| < 2$ so that

$$\sum_{k \in \mathbb{Z}} \phi(2^{-k} \xi) = 0, \quad \xi \neq 0.$$

Choose β to be a Schwartz function such that $\hat{\beta}$ is supported in $|x| \leq 1/4$ and β is nonzero in $1/4 \leq |\xi| \leq 4$ and $\beta(0) = 0$. Finally, choose a Schwartz function ψ such that $\psi(\xi)\beta^2(\xi) = 1$ for ξ in the support of ϕ .

For each linear bounded operator $T : L^2 \rightarrow L^2$ denote by $T_k f = T(S_k f) : L^2 \rightarrow L^2$, where $\widehat{S_k f}(\xi) = \hat{f}\phi(2^{-k}\xi)$.

We need the following variant of Lemma 3.1 from [4].

Lemma 2.2. *Let T be a linear bounded multiplier operator $T : L^2 \rightarrow L^2$; that is,*

$$\widehat{Tf} = m\hat{f},$$

for some $m \in L^\infty(\mathbb{R}^2)$. Then there exists $c_3 > 0$ independent of N, T and x such that for almost every $x \in \mathbb{R}^2$ and each $f \in L^2(\mathbb{R}^2)$,

$$\Delta(Tf)(x) \leq c_3 \left(\sum_{k \in \mathbb{Z}} |M_2(M(T_k f)(x))|^2 \right)^{1/2},$$

where

$$M_2 g(x) = (M(g^2)(x))^{1/2},$$

Mg being the standard Hardy-Littlewood maximal function.

Proof. Define

$$\begin{aligned} \widehat{B_k f}(\xi) &= \beta(2^{-k}\xi) \hat{f}(\xi), \\ \widehat{L_k f}(\xi) &= \psi(2^{-k}\xi) \hat{f}(\xi). \end{aligned}$$

Note that

$$\Delta_k T f = \sum_{n \in \mathbb{Z}} (\Delta_k B_{k+n})(B_{k+n} L_{k+n}) T_{k+n} f.$$

Sublemma 4.2 from [4] implies that (uniformly) for each x ,

$$|\Delta_k B_{k+n} g(x)| \lesssim 2^{-|n|/2} M_2 g(x).$$

Since also

$$|B_{k+n} L_{k+n} g(x)| \lesssim M g(x),$$

uniformly over k, n , the result follows from the triangle inequality. $\square$

Lemma 2.3. *Let T be a linear bounded multiplier operator $T : L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)$ associated with the multiplier m . Assume in addition that m is zero on the ball with radius 2^{2N} . Then there exists $c_4 > 0$ independent of N , T and x such that for almost every $x \in \mathbb{R}^2$ and each $f \in L^2(\mathbb{R}^2)$,*

$$|E_0(Tf)(x)| \leq c_4 2^{-N} \left(\sum_{k \in \mathbb{Z}} |M_2(M(T_k f)(x))|^2 \right)^{1/2}.$$

Proof. The argument follows as in the previous lemma by using the fact that (part of Sublemma 4.2 in [4])

$$|E_0 B_n g(x)| \lesssim 2^{-|n|/2} M_2 g(x),$$

for each $n \geq 0$. $\square$

3. PROOF OF THEOREM 1.1 AND THEOREM 1.3

We have proved in the previous section that

$$\sup_{v \in \Sigma_N} |\Delta(T_v)f(x)| \leq G(f)(x) := c_3 \left(\sum_{k \in \mathbb{Z}} |M_2(M(\sup_{v \in \Sigma_N} |T_v(S_k f)|)(x))|^2 \right)^{1/2}.$$

The following lemma is a variant of similar lemmas from [1] and [6].

Lemma 3.1. *Let T be a sublinear operator on some measure space (X, m) . Assume*

$$\begin{aligned} \|T\|_{L^1 \rightarrow L^{1,\infty}} &\lesssim N, \\ \|T\|_{L^2 \rightarrow L^{2,\infty}} &\lesssim 1, \\ \|T\|_{L^p \rightarrow L^p} &\lesssim N, \end{aligned}$$

for some $2 < p < \infty$. Then

$$\|T\|_{L^2 \rightarrow L^2} \lesssim \sqrt{\log N}.$$

Proof. We need to prove that

$$\int_0^\infty \lambda |\{Tf > 3\lambda\}| d\lambda \lesssim \log N \|f\|_2^2.$$

Let

$$\nu_f(\alpha) := |\{ |f| > \alpha \}|.$$

Split

$$f = f_1^\lambda + f_2^\lambda + f_3^\lambda,$$

where

$$\begin{aligned} f_1^\lambda &= \begin{cases} f, & \text{if } |f| > N\lambda, \\ 0 & \text{otherwise,} \end{cases} \\ f_2^\lambda &= \begin{cases} f, & \text{if } N\lambda \geq |f| > N^{-q}\lambda, \\ 0 & \text{otherwise,} \end{cases} \\ f_3^\lambda &= \begin{cases} f, & \text{if } N^{-q}\lambda \geq |f|, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where $q(p-2) = p$. Now,

$$\begin{aligned} \int_0^\infty \lambda |\{Tf_1^\lambda > \lambda\}| d\lambda &\lesssim N \int_0^\infty \|f_1^\lambda\|_1 d\lambda = N \int_0^\infty \int_{N\lambda}^\infty \nu_f(\alpha) d\alpha d\lambda \\ &= N \int_0^\infty \nu_f(\alpha) \int_0^{\alpha/N} d\lambda d\alpha = \|f\|_2^2, \\ \int_0^\infty \lambda |\{Tf_2^\lambda > \lambda\}| d\lambda &\lesssim \int_0^\infty \lambda^{-1} \|f_2^\lambda\|_2^2 d\lambda = \int_0^\infty \lambda^{-1} \int_{N^{-q}\lambda}^{N\lambda} \alpha \nu_f(\alpha) d\alpha d\lambda \\ &= \int_0^\infty \alpha \nu_f(\alpha) \int_{\alpha/N}^{N^q\alpha} \lambda^{-1} d\lambda d\alpha \lesssim (\log N) \|f\|_2^2, \\ \int_0^\infty \lambda |\{Tf_3^\lambda > \lambda\}| d\lambda &\lesssim N^p \int_0^\infty \lambda^{1-p} \|f_3^\lambda\|_p^p d\lambda = N^p \int_0^\infty \lambda^{1-p} \int_0^{N^{-q}\lambda} \alpha^{p-1} \nu_f(\alpha) d\alpha d\lambda \\ &= N^p \int_0^\infty \alpha^{p-1} \nu_f(\alpha) \int_{N^q\alpha}^\infty \lambda^{1-p} d\lambda d\alpha \lesssim \|f\|_2^2. \quad \square \end{aligned}$$

Proposition 3.2.

$$\|G(f)\|_2 \leq C \sqrt{\log N} \|f\|_2,$$

with C independent of N .

Proof. It suffices to prove that for each k ,

$$\|A_k(f) := \sup_{v \in \Sigma_N} |T_v(S_k f)|\|_2 \leq C \sqrt{\log N} \|S_k f\|_2.$$

It was proved in [8] that A_k maps L^2 to $L^{2,\infty}$ and L^p to itself for $2 < p < \infty$, with implicit bounds independent of N . That is a deep result, essentially equivalent with Carleson's theorem [2].

Since each T_v maps L^1 to $L^{1,\infty}$,

$$\|A_k(f)\|_{1,\infty} \lesssim N \|f\|_1.$$

The result now follows from Lemma 3.1 above. $\square$

Proof of Theorem 1.1. By using a limiting argument, we can assume that m is supported in a finite union of dyadic annuli $2^j \leq |\xi| < 2^{j+1}$. By rescaling, we can assume m is actually supported away from the ball of radius 2^{2N} (for example). Let $\epsilon_N = \sqrt{c_1 \log N}$. For each $\lambda > 0$,

$$\{x : \sup_{v \in \Sigma_N} |T_v f(x)| > 4\lambda\} \subset E_{\lambda,1} \cup E_{\lambda,2} \cup E_{\lambda,3},$$

where

$$\begin{aligned} E_{\lambda,1} &= \{x : \sup_{v \in \Sigma_N} |T_v f(x) - E_0 T_v f(x)| > 2\lambda, Gf(x) \leq \epsilon_N \lambda\}, \\ E_{\lambda,2} &= \{x : Gf(x) > \epsilon_N \lambda\}, \\ E_{\lambda,3} &= \{x : \sup_{v \in \Sigma_N} |E_0 T_v f(x)| > 2\lambda\}. \end{aligned}$$

By Lemma 2.1,

$$\begin{aligned} |E_{\lambda,1}| &\leq \sum_{v \in \Sigma_N} |\{x : |T_v f(x) - E_0 T_v f(x)| > 2\lambda, \Delta(T_v f)(x) \leq \epsilon_N \lambda\}| \\ &\lesssim \frac{1}{N} \sum_{v \in \Sigma_N} |\{x : M(T_v f)(x) > \epsilon_N \lambda\}|. \end{aligned}$$

Using this, Lemma 2.3 and Proposition 3.2, we get that

$$\int_0^\infty \lambda \sum_{i=1}^3 |E_{\lambda,i}| d\lambda \lesssim (\log N)^2 \|f\|_2^2.$$

Theorem 1.1 now follows. $\square$

Proof of Theorem 1.3. Use the same argument as in the proof of Theorem 1.1. It is implicit from [9] that

$$\|A_k(f) := \sup_{v \in \Sigma} |T_v(S_k f)|\|_2 \leq C \|S_k f\|_2.$$

This is due to the fact that the Kakeya maximal function associated with each lacunary set Σ satisfies $\|M_\Sigma^*\|_{2 \rightarrow 2} \lesssim 1$ (see [10]). $\square$

The following result shows that the bound in Theorem 1.1 is optimal.

Proposition 3.3. *Let Σ_N be a set of N equidistributed unit vectors in $\mathbb{R}^2$. Define*

$$T_N^* f(x, y) := \sup_{v \in \Sigma_N} \left| \lim_{\epsilon \rightarrow \infty} \int_{|t| > \epsilon} f((x, y) + tv) \frac{dt}{t} \right|.$$

Then,

$$\|T_N^*\|_{2 \rightarrow 2} \gtrsim (\log N).$$

Proof. This is a standard example, so we only briefly sketch the details. Define $f(x, y) = \frac{1}{\|(x, y)\|} \mathbf{1}_{1 \leq \|(x, y)\| \leq N/100}$.

Let $100 \leq \|(x, y)\| \leq N/100$. If $v := \frac{(x, y)}{\|(x, y)\|}$, then

$$\begin{aligned} \left| \lim_{\epsilon \rightarrow \infty} \int_{|t| > \epsilon} f((x, y) + tv) \frac{dt}{t} \right| &= \frac{1}{\|(x, y)\|} \left| \log \frac{N^2 - \|(x, y)\|^2}{N^2} - \ln(\|(x, y)\|^2 - 1) \right| \\ &\geq \frac{\log \|(x, y)\|}{2\|(x, y)\|}. \end{aligned}$$

Let $w \in \Sigma_N$ be such that $\|v - w\| \leq \frac{10}{N}$. It easily follows that

$$\left| \lim_{\epsilon \rightarrow \infty} \int_{|t| > \epsilon} f((x, y) + tv) \frac{dt}{t} - \lim_{\epsilon \rightarrow \infty} \int_{|t| > \epsilon} f((x, y) + tw) \frac{dt}{t} \right| < \frac{\log \|x\|}{10\|x\|},$$

simply from the fact that $\|tv - tw\| \leq 1/5$ if $|t| < N/50$ and the fact that $|f(x, y) - f(x', y')| \leq \frac{2\|(x, y) - (x', y')\|}{\|(x, y)\|^2}$ if $1 \leq \|(x, y)\|, \|(x', y')\| \leq N/100$ and $\|(x, y) - (x', y')\| \leq 1$. Thus,

$$T_N^* f(x, y) \geq \lim_{\epsilon \rightarrow \infty} \int_{|t| > \epsilon} f((x, y) + tw) \frac{dt}{t} \geq \frac{\log \|(x, y)\|}{4\|(x, y)\|},$$

and hence

$$\|T_N^* f\|_2 \gtrsim (\log N)^{3/2}.$$

Since $\|f\|_2 \lesssim (\log N)^{1/2}$, the proposition follows. $\square$

4. PROOF OF THEOREM 1.4

First, let us remark that there is a proof along the lines of the argument for Theorem 1.1. We however choose to give a different, self-contained argument, one that in particular does not appeal to the Chang-Wilson-Wolff inequality.

It suffices to prove the bound for

$$M_0 F(x, y) := \sup_{v \in \Sigma_N} \left| \int F(x + tv_1, y + tv_2) \psi(t) dt \right|,$$

where ψ is a positive function such that $\hat{\psi}$ is supported in $[0, 1]$.

Decompose

$$F = F_0 + \sum_{n \geq 1} F_n,$$

such that

$$\|F\|_2^2 = \|F_0\|_2^2 + \sum_{n \geq 1} \|F_n\|_2^2.$$

Here F_0 is the Fourier restriction to the unit ball B (that is, $\widehat{F_0} = \hat{F}1_B$), while F_n is the Fourier restriction to the annulus $2^{n-1} \leq |(\xi, \eta)| \leq 2^n$. Note that

$$M_0 F_0(x, y) = \sup_{v \in \Sigma_N} \left| \int \widehat{F_0}(\xi, \eta) \phi(\xi, \eta) \hat{\psi}(v_1 \xi + v_2 \eta) e^{i(x\xi + y\eta)} d\xi d\eta \right| \lesssim M F_0(x, y),$$

where M is the Hardy-Littlewood maximal function and

$$1_B \leq \phi \leq 1_{2B}$$

is smooth. This is the case since $K_v(x, y) := (\phi(\xi, \eta) \hat{\psi}(v_1 \xi + v_2 \eta))(x, y)$ easily satisfies

$$|K_v(x, y)| \lesssim (1 + \|(x, y)\|)^{-3},$$

with bound independent of v , and $M_0 F_0(x, y) = \sup_{v \in \Sigma_N} |F_0 * K_v(x, y)|$.

Next, we analyze the case $n \geq 1$.

Definition 4.1 (Grids). A collection of intervals on the unit circle S^1 is called a grid if for any two intervals from the collection that intersect, one of them should contain the other one.

The standard dyadic grid $\mathcal{G}^0$ is the collection of dyadic intervals on S^1 , obtained by identifying S^1 with $[0, 1)$. In addition, for $i \in \{1/3, 2/3\}$, define $\mathcal{G}^i$ to consist of the projection on S^1 of the intervals

$$\{[2^j k, 2^j(k + (-1)^j i)] : 2^j \leq 1, k \in \mathbb{Z}\}.$$

It is easy to see that each $\mathcal{G}^i$ is a grid. Moreover, each interval I on the circle is contained inside an interval of similar length from one of these grids.

Denote by Ω_n any partition of the annulus $2^{n-1} \leq |(\xi, \eta)| \leq 2^n$ in n sectors with equal area. For each $\omega \in \Omega_n$, call $d(\omega)$ the direction of the radius that splits ω into two sectors with equal area. For each $\omega \in \Omega_n$, let $\tilde{\omega}$ be the sector in the larger annulus $2^{n-2} \leq |(\xi, \eta)| \leq 2^{n+1}$, with twice the aperture of ω and having the same bisector $d(\omega)$. Also let $A(\omega)$ be the projection of the sector ω on S^1 . Denote by $B(\omega)$ some interval from some grid $\mathcal{G}^i$ that contains $10A(\omega) + \frac{\pi}{2}$,¹ and whose length is comparable to that of $A(\omega)$, which in turn is comparable to 2^{-n} .

Decompose further

$$F_n = \sum_{\omega \in \Omega_n} F_\omega,$$

where $\widehat{F_\omega} = \hat{F}1_\omega$. Note that

$$\int F_\omega(x + tv_1, y + tv_2)\psi(t)dt = \int \hat{F}_\omega(\xi, \eta)\hat{\psi}(v_1\xi + v_2\eta)e^{i(x\xi + y\eta)}d\xi d\eta$$

is nonzero only if there exists $(\xi, \eta) \in \omega$ such that $0 \leq v_1\xi + v_2\eta \leq 1$. This implies that $v \in 10A(\omega) + \frac{\pi}{2} \subset B(\omega)$.

Let θ_ω be a bump function adapted to and supported on $\tilde{\omega}$ and which equals 1 on ω . Note that

$$\int F_\omega(x + tv_1, y + tv_2)\psi(t)dt = \int \hat{F}_\omega(\xi, \eta)\theta_\omega(\xi, \eta)\hat{\psi}(v_1\xi + v_2\eta)e^{i(x\xi + y\eta)}d\xi d\eta.$$

Lemma 4.2. *If $v, v' \in B(\omega)$ and $\omega \in \Omega_n$, then*

$$\begin{aligned} & \left| \int F_\omega(x + tv_1, y + tv_2)\psi(t)dt - \int F_\omega(x + tv'_1, y + tv'_2)\psi(t)dt \right| \\ (1) \quad & \lesssim 2^n \|v - v'\| M^*F(x, y), \end{aligned}$$

where

$$M^*F(x, y) = \sup_{\epsilon, \delta > 0} \frac{1}{\epsilon\delta} \int_{|t| < \epsilon} \int_{|s| < \delta} |F|(x + t, y + s) dt ds.$$

Proof. Indeed, by rotation invariance it suffices to assume that $d(\omega) = (0, -1)$. Then, $B(\omega)$ is an interval of length roughly 2^{-n} containing $(1, 0)$. Thus, if $v, v' \in B(\omega)$, then $|v_2|, |v'_2| \lesssim 2^{-n}$. This immediately shows that for each $\alpha, \beta \geq 0$,

$$|\partial_\xi^\alpha \partial_\eta^\beta [\theta_\omega(\xi, \eta)(\hat{\psi}(v_1\xi + v_2\eta) - \hat{\psi}(v'_1\xi + v'_2\eta))]| \lesssim 2^{n(1-\beta)} \|v - v'\|.$$

By using this, integration by parts and the fact that θ_ω is supported in $\tilde{\omega}$, it follows that

$$\begin{aligned} |K_\omega(x, y) &:= \int \theta_\omega(\xi, \eta)[\hat{\psi}(v_1\xi + v_2\eta) - \hat{\psi}(v'_1\xi + v'_2\eta)]e^{i(x\xi + y\eta)}d\xi d\eta| \\ &\lesssim 2^{2n} \|v - v'\| \min\{1, |x|^{-2}, |y2^n|^{-2}\} \lesssim 2^{2n} \|v - v'\| (1 + |x| + 2^n |y|)^{-2}. \end{aligned}$$

The lemma follows. $\square$

Define

$$\tilde{F}_\omega(x, y, v) := \int F_\omega(x + tv_1, y + tv_2)\psi(t)dt,$$

and $\Omega = \bigcup_{n \geq 1} \Omega_n$. Thus,

$$M_0\left(\sum_{n \geq 1} F_n\right)(x, y) = \sup_{v \in \Sigma_N} \left| \sum_{n \geq 1} \sum_{\omega \in \Omega} \tilde{F}_\omega(x, y, v) \right|.$$

¹ $10A(\omega)$ is the interval on S^1 with the same center as $A(\omega)$ and 10 times its length.

By splitting the sum above into three parts, we can assume without loss of generality that all intervals $B(\omega)$ belong to a fixed grid.

For $1 \leq \kappa \leq \log N$, let Ω^κ consist of those $\omega \in \Omega$ such that

$$(2) \quad 2^{\kappa-1} < \#(B(\omega) \cap \Sigma_N) \leq 2^\kappa.$$

Let

$$T_\kappa F(x, y) = \sup_{v \in \Sigma_N} \left| \sum_{n \geq 1} \sum_{\omega \in \Omega^\kappa} \tilde{F}_\omega(x, y, v) \right|.$$

It suffices to prove that $\|T_\kappa\|_2 \lesssim 1$ for each κ , by orthogonality and the fact that

$$T_\kappa F = T_\kappa \left(\sum_{\omega \in \Omega^\kappa} F_\omega \right).$$

Definition 4.3. A cluster is a subset $\mathbf{C}$ of Ω^κ such that $\bigcap_{\omega \in \mathbf{C}} B(\omega) \neq \emptyset$. Let $t(\mathbf{C})$ be the center of the smallest $B(\omega)$ in the cluster.

The grid property and (2) imply that Ω^κ can be split into clusters such that the intervals $B(\omega)$ and $B(\omega')$ are pairwise disjoint if ω, ω' belong to distinct clusters. Thus,

$$\sup_{v \in \Sigma_N} \left| \sum_{\omega \in \Omega^\kappa} \tilde{F}_\omega(x, y, v) \right| = \sup_{\mathbf{C}} \sup_{v \in \Sigma_N} \left| \sum_{\omega \in \mathbf{C}} \tilde{F}_\omega(x, y, v) \right|.$$

Note also that $\sum_{\omega \in \mathbf{C}} F_\omega$ and $\sum_{\omega \in \mathbf{C}' } F_\omega$ are orthogonal, for two distinct clusters $\mathbf{C}$ and $\mathbf{C}'$. Using these two facts, it suffices to prove that for each cluster $\mathbf{C}$,

$$\left\| \sup_{v \in \Sigma_N} \left| \sum_{\omega \in \mathbf{C}} \tilde{F}_\omega(x, y, v) \right| \right\|_2 \lesssim \|F\|_2.$$

Note that for each $v \in \Sigma_N$,

$$\begin{aligned} \sum_{\omega \in \mathbf{C}} \tilde{F}_\omega(x, y, v) &= \sum_{\substack{\omega \in \mathbf{C} \\ v \in B(\omega)}} \tilde{F}_\omega(x, y, v) \\ &= \sum_{\substack{\omega \in \mathbf{C} \\ v \in B(\omega)}} \tilde{F}_\omega(x, y, t(\mathbf{C})) + O(M^* F(x, y)) \quad (\text{by Lemma 4.2}) \\ &= P_m \left(\sum_{\omega \in \mathbf{C}} \tilde{F}_\omega(x, y, t(\mathbf{C})) \right) + O(M^* F(x, y)) \end{aligned}$$

for some appropriate $m = m(v, \mathbf{C})$, where P_m denotes the Fourier projection on the ball with radius 2^m . The last equality above follows since $\tilde{F}_\omega(x, y, v)$ is supported in frequency in ω . Recall that the maximal operator $\sup_m |P_m F|$ is bounded on L^2 , by invoking standard maximal function arguments. Also, the operator $\sum_{\omega \in \mathbf{C}} \tilde{F}_\omega(x, y, t(\mathbf{C}))$ has a bounded multiplier and thus it is bounded on L^2 . This ends the proof.

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