THE THRESHOLD FUNCTION FOR VANISHING OF THE TOP HOMOLOGY GROUP OF RANDOM d -COMPLEXES

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ABSTRACT. For positive integers n and d , and the probability function $0 \leq p(n) \leq 1$, we let $Y_{n,p,d}$ denote the probability space of all at most d -dimensional simplicial complexes on n vertices, which contain the full $(d-1)$ -dimensional skeleton, and whose d -simplices appear with probability $p(n)$. In this paper we determine the threshold function for vanishing of the top homology group in $Y_{n,p,d}$, for all $d \geq 1$.

1. THRESHOLDS FOR VANISHING OF THE $(d-1)$ ST HOMOLOGY GROUP OF RANDOM d -COMPLEXES

In 1959 Erdős and Rényi defined a natural model for random graphs which has since become classical. In this model, which we call¹ $Y_{n,p,1}$, the random graph has n vertices, and the edges are chosen uniformly and independently at random with probability p . Usually, one is interested in questions concerning various statistics on this probability space, in the situation when n goes to infinity, and p is a function of n . One of the main results of Erdős-Rényi concerning $Y_{n,p,1}$ was the discovery of the threshold function for the connectivity of the graph. More precisely, reformulated in our language, they have shown the following theorem.

Theorem 1.1 (Erdős-Rényi Theorem, [4]). *Assume that $w(n)$ is any function $w : \mathbb{N} \rightarrow \mathbb{R}$, such that $\lim_{n \rightarrow \infty} w(n) = \infty$, and $p = p(n)$ is the probability depending on n . Then we have*

- (1) *if $p(n) = \frac{\log n - w(n)}{n}$, then $\lim_{n \rightarrow \infty} \text{Prob}(\tilde{\beta}_0(Y_{n,p,1}; \mathbb{Z}/2\mathbb{Z}) > 0) = 1$;*
- (2) *if $p(n) = \frac{\log n + w(n)}{n}$, then $\lim_{n \rightarrow \infty} \text{Prob}(\tilde{\beta}_0(Y_{n,p,1}; \mathbb{Z}/2\mathbb{Z}) = 0) = 1$.*

More recently, the two-dimensional analog $Y_{n,p,2}$ of the Erdős-Rényi model was considered by Linial-Meshulam in [11], and, further, the d -dimensional model $Y_{n,p,d}$, for $d \geq 3$, was considered by Meshulam-Wallach in [13].

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¹The standard notation for the Erdős-Rényi model is $G(n, p)$. Here we follow the notation introduced by Linial-Meshulam in [11] and by Meshulam-Wallach in [13].

In these generalizations, the graphs are replaced with simplicial complexes of dimension at most d , on n vertices, where all simplices of dimension $d-1$ or less are required to be in the complex, and the simplices of dimension d are chosen uniformly and independently at random with probability p . The combined work of Linial-Meshulam and Meshulam-Wallach yields threshold functions for the vanishing of the $(d-1)$ th homology group of $Y_{n,p,d}$ with coefficients in a finite abelian group. Specifically, the following is known.

Theorem 1.2 (Linial-Meshulam, [11]; Meshulam-Wallach, [13]). *Assume that $w(n)$ is any function $w : \mathbb{N} \rightarrow \mathbb{R}$, such that $\lim_{n \rightarrow \infty} w(n) = \infty$, and $p = p(n)$ is the probability depending on n , and F is a finite abelian group. Then we have*

- (1) *if $p(n) = \frac{d \log n - w(n)}{n}$, then $\lim_{n \rightarrow \infty} \text{Prob}(H_{d-1}(Y_{n,p,d}; F) \neq 0) = 1$;*
- (2) *if $p(n) = \frac{d \log n + w(n)}{n}$, then $\lim_{n \rightarrow \infty} \text{Prob}(H_{d-1}(Y_{n,p,d}; F) = 0) = 1$.*

Curiously, the methods of [11, 13] do not easily extend to the case of integer coefficients, and finding the threshold functions for the vanishing of $H_{d-1}(Y_{n,p,d}; \mathbb{Z})$ remains open even for the case $d = 2$.

On the other hand, the threshold for vanishing of the fundamental group of $\Delta \in Y_{n,p,2}$ is well understood due to work of Babson, Hoffman, and Kahle. The following deep result can be found in [2].

Theorem 1.3 (Babson, Hoffman, Kahle, [2, Theorem 1.3]). *If $w(n)$ is a function, such that $\lim_{n \rightarrow \infty} w(n) = \infty$, and $p(n) \geq \sqrt{\frac{3 \log n + w(n)}{n}}$, then $\text{Prob}(\pi_1(Y_{n,p,2}) = 0) = 1$.*

Since the simplicial complexes Δ in $Y_{n,p,d}$ have dimension at most d , and are, on the other hand, required to contain a full $(d-1)$ -dimensional skeleton, we have $H_i(\Delta; F) = 0$, for all $i \neq d-1, d$, where F is an arbitrary abelian group. In this paper we complement the study undertaken by Linial-Meshulam and Meshulam-Wallach by computing the threshold functions for the vanishing of the top dimensional homology.

We note here that for fixed n and d , $\text{Prob}(H_{d-1}(Y_{n,p,d}; F) = 0)$ is increasing monotone in p , whereas $\text{Prob}(H_d(Y_{n,p,d}; F) = 0)$ is decreasing monotone in p . This is because, on the one hand, adding a random d -simplex either increases the d th Betti number by 1, or decreases the $(d-1)$ th Betti number by 1. On the other hand, sampling with probability $q > p$ can be done in consecutive steps: first sample with probability p , then if the simplex has not been chosen, sample again, but this time with probability $\frac{q-p}{1-p} > 0$.

To place this paper in a more general context, we mention that there has recently been quite a bit of exciting work on topologically constructed probability spaces; we refer in particular to [6, 7, 8, 9, 12, 14] and the references therein.

2. TERMINOLOGY AND THE FORMULATION OF THE MAIN RESULT

We start by recalling some standard notation. For a positive integer n , we let Δ_n denote the full $(n-1)$ -dimensional simplex. Given a simplicial complex Δ , and a nonnegative integer d , we let $\Delta^{(d)}$ denote the d -dimensional skeleton of Δ , and we let $\Delta(d)$ denote the set of the d -simplices of Δ . Furthermore, for an arbitrary abelian group F , we let $B_{d-1}(\Delta; F)$ denote the subspace of $C_{d-1}(\Delta; F)$ generated by the boundaries of the d -simplices from Δ , and we let $Z_d(\Delta; F)$ denote the subspace

of $C_d(\Delta; F)$ consisting of the cycles. Finally, $H_d(\Delta; F)$ denotes the d th homology group of Δ with coefficients in F , whereas $\beta_d(\Delta; F)$ denotes the corresponding d th Betti number. For a d -chain $\sigma \in C_d(\Delta; F)$ we let $\text{supp } \sigma$ denote the subset of $\Delta(d)$ consisting of all d -simplices appearing with nonzero coefficients in σ . We also assume familiarity with the Bachmann-Landau notation for the asymptotic behavior of functions.

For positive integers n and d , and a real number $0 \leq p \leq 1$, we let $Y_{n,p,d}$ denote the probability space of all at most d -dimensional simplicial complexes on n vertices, which contain the full $(d-1)$ -dimensional skeleton, and whose d -simplices appear independently with probability p . Formally, the underlying set of $Y_{n,p,d}$ consists of all simplicial complexes Δ , such that $\Delta_n^{(d-1)} = \Delta^{(d-1)}$, and $\Delta(d+1) = \emptyset$; clearly there are $2^{\binom{n}{d+1}}$ of them. The probability associated to each Δ is equal to $p^{|\Delta(d)|}(1-p)^{\binom{n}{d+1}-|\Delta(d)|}$. When the values n , p , and d are fixed, and S is some set of simplices of Δ_n , we shall write $\text{Prob}(S)$ to denote the probability that all of the simplices from S are present in the simplicial complex sampled from $Y_{n,p,d}$.

To work with the probability space $Y_{n,p,d}$ we shall use the following notation. We write $\Delta \in Y_{n,p,d}$ when we sample a simplicial complex from $Y_{n,p,d}$. For any integer i , and any abelian group F , we write $\beta_i(Y_{n,p,d}; F)$ to denote the expectation of the i th Betti number in the probability space $Y_{n,p,d}$. We also write $\text{Prob}(\beta_i(Y_{n,p,d}; F) = 0)$ and $\text{Prob}(\beta_i(Y_{n,p,d}; F) > 0)$ to denote the probabilities that the i th Betti number of $\Delta \in Y_{n,p,d}$ is equal to 0, correspondingly is strictly larger than 0. Similarly, for an arbitrary abelian group F , we write $\text{Prob}(H_i(Y_{n,p,d}; F) = 0)$ and $\text{Prob}(H_i(Y_{n,p,d}; F) \neq 0)$ to denote the probabilities that the i th homology group of $\Delta \in Y_{n,p,d}$ is trivial, correspondingly nontrivial.

To keep our argument as simple as possible, we shall initially restrict ourselves to the case $F = \mathbb{Z}/2\mathbb{Z}$. The adjustments needed to handle the general case will follow in Section 5.

Theorem 2.1. *The probability $p(n) = \Theta\left(\frac{1}{n}\right)$ is the threshold probability for vanishing of the top homology of the random simplicial d -complex. More precisely, assume that $p = p(n) = w(n)/n$, and $d \geq 1$. Then we have*

- (1) *if $\lim_{n \rightarrow \infty} w(n) = 0$, then $\lim_{n \rightarrow \infty} \text{Prob}(H_d(Y_{n,p,d}; \mathbb{Z}/2\mathbb{Z}) = 0) = 1$;*
- (2) *if $\lim_{n \rightarrow \infty} w(n) = \infty$, then $\lim_{n \rightarrow \infty} \text{Prob}(H_d(Y_{n,p,d}; \mathbb{Z}/2\mathbb{Z}) > 0) = 1$.*

Since $H_d(Y_{n,p,d}; \mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^{\beta_d(Y_{n,p,d}; \mathbb{Z}/2\mathbb{Z})}$, and the statement of Theorem 2.1 concerns vanishing/nonvanishing of the homology, for brevity we will use the Betti number notation in all of our proofs.

It appears interesting to note that Theorem 2.1 together with Theorem 1.2 implies that there is an interval of values of p , such that adding a random d -simplex with probability from that interval has a bounded away from 0 probability of decreasing β_{d-1} , as well as a bounded away from 0 probability of increasing β_d .

Before proceeding with the proof, we need two more pieces of notation.

Definition 2.2. For an arbitrary positive integer d , let Σ_d denote the subset of $C_{d-1}(\Delta_n; \mathbb{Z}/2\mathbb{Z}) \times 2^{\Delta_n^{(d)}} \times \mathbb{Z}_{\geq 0}$ defined by the following: $(\sigma, S, \lambda) \in \Sigma_d$ if and only if $|\text{supp } \sigma| > (\lambda - 1)(d + 1)$.

In particular, $(0, S, \lambda) \in \Sigma_d$ implies $\lambda = 0$.

Definition 2.3. For $(\sigma, S, \lambda) \in \Sigma_d$, we define $\rho(\sigma, S, \lambda)$ to be the probability that $\Delta \in Y_{n,p,d}$ satisfies the following two conditions:

- (1) Δ contains σ in its boundary set, i.e., $\sigma \in B_{d-1}(\Delta; \mathbb{Z}/2\mathbb{Z})$;
- (2) the sets $\Delta(d)$ and S are disjoint.

So, informally speaking, a collection of the simplices from Δ can be used to complement σ to a d -cycle, avoiding the d -simplices from S .

For future reference, we record a few simple properties of $\rho(-, -, -)$.

Lemma 2.4.

- (1) For any $(\sigma, S_1, \lambda), (\sigma, S_2, \lambda) \in \Sigma_d$, the set inclusion $S_2 \subseteq S_1$ implies $\rho(\sigma, S_1, \lambda) \leq \rho(\sigma, S_2, \lambda)$;
- (2) we have $\rho(\sigma, S, \lambda_1) = \rho(\sigma, S, \lambda_2)$, whenever $(\sigma, S, \lambda_1), (\sigma, S, \lambda_2) \in \Sigma_d$;
- (3) whenever $(\sigma, \Delta_n(d), \lambda) \in \Sigma_d$, and $\sigma \neq 0$, we have $\rho(\sigma, \Delta_n(d), \lambda) = 0$;
- (4) we have $\rho(\sigma, S, \lambda) = 0$, for all $(\sigma, S, \lambda) \in \Sigma_d$, such that $\partial\sigma \neq 0$.

Proof. The first condition holds simply because in $Y_{n,p,d}$ it is no more probable that a simplicial complex satisfies a (possibly) more stringent set of conditions. The second condition is straightforward. The third condition is true since in this case $\Delta(d)$ must be empty. Finally, the fourth condition holds since the square of the differential in a chain complex is equal to 0. $\square$

3. PROOF OF THE FIRST PART OF THEOREM 2.1

We start with the first part of Theorem 2.1, which is more difficult (and more interesting). Its proof relies on the following lemma, which might also be useful in its own right.

Lemma 3.1. *Let us fix positive integers n and d , and a probability $1 \geq p \geq 0$, such that $d \geq 2$, $n \geq d + 1$, and $pn < 1$. Set $w := pn$. For any $(\sigma, S, \lambda) \in \Sigma_d$ we have*

$$(3.1) \quad \rho(\sigma, S, \lambda) \leq c(d, \lambda) p^\lambda / (1 - w)^\lambda,$$

where $c(d, \lambda) = (d + 1)^\lambda \lambda!$.

The case $S = \emptyset$ is of special interest to us, and we adopt the abbreviated notation $\rho(\sigma, \lambda) := \rho(\sigma, \emptyset, \lambda)$.

Proof of Lemma 3.1. By Lemma 2.4(4) we can always assume that $\partial\sigma = 0$, as otherwise the left hand side of (3.1) is equal to 0.

We shall use induction on λ . The base of induction is $\lambda = 0$. In this case $c(d, 0) = 1$ for all d , and the right hand side of (3.1) is equal to 1; hence the inequality is trivially satisfied.

To prove the induction step, let us now assume that $\lambda \geq 1$, and that the inequality (3.1) has been shown for all $\tilde{\lambda}$, such that $0 \leq \tilde{\lambda} \leq \lambda - 1$. Since $(\sigma, S, \lambda) \in \Sigma_d$, we have $\sigma \neq 0$. Having fixed the value of λ , we now run another induction procedure; this one is downwards on the cardinality of S . The base $|S| = \binom{n}{d}$ is provided by Lemma 2.4(3), since the left hand side of (3.1) is then equal to 0. We now make the induction step in $|S|$.

Let us choose a $(d - 1)$ -simplex $e \in \text{supp } \sigma$. If $\sigma \in B_{d-1}(\Delta; \mathbb{Z}/2\mathbb{Z})$, then there must exist a d -simplex $\tau \in \Delta(d)$ such that $e \in \partial\tau$. Let Ω denote the set of all d -simplices $\tau \in \Delta_n(d)$ such that $e \in \partial\tau$. Clearly, we have $|\Omega| = n - d$. We represent Ω as a disjoint union $\Omega = A \cup B \cup C$, where the sets A , B , and C are defined as

follows:

$$\begin{aligned} A &:= \{\tau \in \Omega \setminus S \mid |\text{supp}(\sigma + \partial\tau)| > (\lambda - 1)(d + 1)\}, \\ B &:= \{\tau \in \Omega \setminus S \mid |\text{supp}(\sigma + \partial\tau)| \leq (\lambda - 1)(d + 1)\}, \\ C &:= \Omega \cap S. \end{aligned}$$

Since some simplex from $A \cup B$ must be picked in Δ we have the inequality

$$(3.2) \quad \rho(\sigma, S, \lambda) \leq \sum_{\tau \in A \cup B} \text{Prob}(\tau) \rho(\sigma + \partial\tau, S \cup \{\tau\}, \lambda_\tau),$$

where for each τ the value λ_τ is chosen so that $(\sigma + \partial\tau, S \cup \{\tau\}, \lambda_\tau) \in \Sigma_d$. In fact, we shall see shortly that one can always choose λ_τ to be λ or $\lambda - 1$. Substituting p for $\text{Prob}(\tau)$, breaking the sum on the right hand side of (3.2) into two, and using the fact that $(\sigma + \partial\tau, S \cup \{\tau\}, \lambda) \in \Sigma_d$ for all $\tau \in A$, we obtain

$$(3.3) \quad \rho(\sigma, S, \lambda) \leq p \sum_{\tau \in A} \rho(\sigma + \partial\tau, S \cup \{\tau\}, \lambda) + p \sum_{\tau \in B} \rho(\sigma + \partial\tau, S \cup \{\tau\}, \lambda_\tau).$$

Let α denote the first summand, and let β denote the second summand on the right hand side of (3.3). We shall estimate these terms separately.

First, since $|S \cup \{\tau\}| > |S|$, by the induction assumption (on $|S|$) we have

$$(3.4) \quad \begin{aligned} \alpha &\leq p |A| c(d, \lambda) p^\lambda / (1 - w)^\lambda < p n c(d, \lambda) p^\lambda / (1 - w)^\lambda \\ &= w c(d, \lambda) p^\lambda / (1 - w)^\lambda. \end{aligned}$$

Let us next consider the summand β . To start with, if $\tau \in B$, then $\text{supp } \partial\tau$ contains at least one simplex from $\text{supp } \sigma$ other than e , and it is uniquely determined by that simplex (together with e). It follows that $|B| \leq |\text{supp } \sigma| - 1$. Assume now that $\tau \in B$. In that case we have

$$(3.5) \quad \begin{aligned} (\lambda - 1)(d + 1) &\geq |\text{supp}(\sigma + \partial\tau)| \geq |\text{supp } \sigma| - |\text{supp } \partial\tau| \\ &= |\text{supp } \sigma| - (d + 1) > (\lambda - 2)(d + 1), \end{aligned}$$

implying that $(\sigma + \partial\tau, S \cup \{\tau\}, \lambda - 1) \in \Sigma_d$, and that $|\text{supp } \sigma| \leq (\lambda + 1)(d + 1)$. Hence, by the induction assumption (on λ) we have the estimate

$$(3.6) \quad \begin{aligned} \beta &\leq p |B| c(d, \lambda - 1) p^{\lambda-1} / (1 - w)^{\lambda-1} \\ &\leq (|\text{supp } \sigma| - 1) c(d, \lambda - 1) p^\lambda / (1 - w)^{\lambda-1} \\ &< \lambda (d + 1) c(d, \lambda - 1) p^\lambda / (1 - w)^{\lambda-1}. \end{aligned}$$

Substituting the estimates from (3.4) and (3.6) into (3.3), we obtain

$$(3.7) \quad \rho(\sigma, S, \lambda) < (w c(d, \lambda) + \lambda (d + 1) c(d, \lambda - 1) (1 - w)) p^\lambda / (1 - w)^\lambda.$$

This yields the desired inequality (3.1) for the constant $c(d, \lambda)$ recursively defined by the equation

$$c(d, \lambda) := w c(d, \lambda) + \lambda (d + 1) c(d, \lambda - 1) (1 - w),$$

that is,

$$c(d, \lambda) := (d + 1) \lambda c(d, \lambda - 1).$$

Since $c(d, 0) = 1$, we arrive at

$$(3.8) \quad c(d, \lambda) = (d + 1)^\lambda \lambda!,$$

which finishes the proof of the lemma. $\square$

We are now ready to proceed with the proof of the first part of our main theorem.

Proof of Theorem 2.1(1). Let us first settle the case $d = 1$, as can be done completely explicitly, without referring to Lemma 3.1. Clearly, for the first Betti number of $\Delta \in Y_{n,p,1}$ to be nontrivial the graph Δ must contain cycles. For $l = 3, \dots, n$, let z_l denote the number of the l -cycles in a complete graph on n vertices. Then we have

$$(3.9) \quad \text{Prob}(\beta_1(Y_{n,p,1}; \mathbb{Z}/2\mathbb{Z})) \leq \sum_{\text{cycles } c} \text{Prob}(c) = \sum_{l=3}^n z_l p^l.$$

Substituting $z_l = \frac{1}{2} \binom{n}{l} (l-1)!$ into (3.9) we obtain

$$(3.10) \quad \begin{aligned} \text{Prob}(\beta_1(Y_{n,p,1}; \mathbb{Z}/2\mathbb{Z})) &\leq \sum_{l=3}^n \binom{n}{l} (l-1)! p^l / 2 \\ &= \sum_{l=3}^n \frac{n(n-1) \dots (n-l+1)}{2l} p^l < \sum_{l=3}^n n^l p^l = w^3 + \dots + w^n \\ &= w^3(1 + w + \dots + w^{n-3}) < \frac{w^3}{1-w}. \end{aligned}$$

In particular,

$$\lim_{n \rightarrow \infty} \text{Prob}(\beta_1(Y_{n,p,1}; \mathbb{Z}/2\mathbb{Z})) \leq \lim_{n \rightarrow \infty} \frac{w(n)^3}{1-w(n)} = 0.$$

For the rest of the proof we assume that $d \geq 2$. For an arbitrary d -simplex t , let A_t denote the event in $Y_{n,p,d}$ that the chosen complex Δ has a nontrivial homology cycle which has a representative τ satisfying $t \in \text{supp } \tau$. Let t_0 denote the d -simplex with vertices $\{1, \dots, d+1\}$. Clearly, due to symmetry, $\text{Prob}(A_t) = \text{Prob}(A_{t_0})$, for all $t \in \Delta_n(d)$, and so we have

$$(3.11) \quad \begin{aligned} \text{Prob}(\beta_d(Y_{n,p,d}; \mathbb{Z}/2\mathbb{Z}) > 0) &\leq \sum_{t \in \Delta_n(d)} \text{Prob}(t) \text{Prob}(A_t) \\ &= p \binom{n}{d+1} \text{Prob}(A_{t_0}) < p n^{d+1} \text{Prob}(A_{t_0}) = w n^d \text{Prob}(A_{t_0}). \end{aligned}$$

We shall next estimate $\text{Prob}(A_{t_0})$. As a precursor of the general argument we consider the case $d = 2$. In this case t_0 is the triangle with vertex set $\{1, 2, 3\}$. Let e denote the edge with vertices 1 and 2. In order for the event A_{t_0} to occur, we must pick some triangle s_i with the vertex set $\{1, 2, i\}$, where $i = 4, \dots, n$. Hence we have the inequality

$$(3.12) \quad \text{Prob}(A_{t_0}) \leq \sum_{i=4}^n \text{Prob}(s_i) \rho(\partial(t_0 + s_i), \{s_i\}, 2).$$

Since $|\text{supp}(\partial(t_0 + s_i))| = 4 > 1 \cdot 3 = (\lambda - 1)(d + 1)$, by Lemma 3.1 we have

$$\rho(\partial(t_0 + s_i), \{s_i\}, 2) \leq \frac{3^2 \cdot 2! \cdot p^2}{(1-w)^2} = \frac{18p^2}{(1-w)^2}.$$

Combining this with (3.11) and (3.12), and the fact that $\text{Prob}(s_i) = p$, we obtain

$$\text{Prob}(\beta_d(Y_{n,p,2}; \mathbb{Z}/2\mathbb{Z}) > 0) \leq w n^2 \sum_{i=4}^n p \frac{18p^2}{(1-w)^2} < \frac{18w n^3 p^3}{(1-w)^2} = \frac{18w^4}{(1-w)^2};$$

hence $\lim_{n \rightarrow \infty} \text{Prob}(\beta_d(Y_{n,p,2}; \mathbb{Z}/2\mathbb{Z}) > 0) = 0$ if $\lim_{n \rightarrow \infty} w(n) = 0$.

Let us now consider the case $d \geq 3$. The argument is along the same lines as for $d = 2$, but with more technical estimates, as it does not suffice anymore to just add one d -simplex to t_0 . Let $e_1, \dots, e_{d+1}$ denote the $(d-1)$ -dimensional faces of t_0 taken in an arbitrary order. For the event A_{t_0} to occur, for each $i \in [d+1]$, we must pick at least one d -simplex different from t_0 whose boundary contains e_i .

Assume $T = \{t_1, \dots, t_{d+1}\}$ is such a collection of d -simplices; i.e., for all $i \in [d+1]$, we have $t_i \in \Delta(d) \setminus \{t_0\}$ and $e_i \in \text{supp}(\partial t_i)$. For any $i, j \in [d+1]$, $i \neq j$, we have $t_i \neq t_j$, since the only d -simplex whose boundary contains both e_i and e_j is t_0 . We consider the d -chain $\tau := \sum_{i=0}^{d+1} t_i$.

Every d -simplex t_i has a unique vertex v_i which does not belong to e_i . We define a set partition $\pi = \pi_1 \cup \dots \cup \pi_m$ on T by putting t_i and t_j to the same block if $v_i = v_j$.

Claim. We have

$$(3.13) \quad |\text{supp}(\partial \tau)| > (m-2)(d+1).$$

Proof of the claim. Clearly, the set $\text{supp}(\partial \tau)$ consists of all the elements in $\bigcup_{i=0}^{d+1} \text{supp}(\partial t_i)$ which belong to the odd-numbered sets in that union. By construction, all the elements of $\text{supp}(\partial t_0)$ belong to exactly one other set in that union, so all these cancel out.

Potentially, we have $d(d+1)$ remaining elements. There will be no cancellation between the elements of $\text{supp}(\partial t_i)$ and $\text{supp}(\partial t_j)$ if t_i and t_j belong to different blocks in π . If they belong to the same block, then there is exactly one cancellation, namely of the $(d-1)$ -simplices $\{v\} \cup (t_i \cap t_j)$, where v is the vertex corresponding to the block of π containing t_i and t_j . Furthermore, all these cancellations are disjoint from each other, since there are precisely two $(d-1)$ -simplices in ∂t_0 containing $t_i \cap t_j$. We conclude that

$$(3.14) \quad |\text{supp}(\partial \tau)| = d(d+1) - \sum_{i=1}^m 2 \binom{|\pi_i|}{2} = d(d+1) - \sum_{i=1}^m |\pi_i|(|\pi_i| - 1) \\ = d(d+1) + \sum_{i=1}^m |\pi_i| - \sum_{i=1}^m |\pi_i|^2 = (d+1)^2 - \sum_{i=1}^m |\pi_i|^2.$$

Since the sum $\sum_{i=1}^m |\pi_i|$ is fixed and all the terms in that sum are positive integers, the maximum of $\sum_{i=1}^m |\pi_i|^2$ is achieved by the values $|\pi_1| = \dots = |\pi_{m-1}| = 1$, $|\pi_m| = d+1 - (m-1)$. Hence (3.14) yields

$$(3.15) \quad |\text{supp}(\partial \tau)| \geq (d+1)^2 - (m-1) - (d+1 - (m-1))^2 \\ = (d+1)^2 - (m-1) - (d+1)^2 + 2(d+1)(m-1) - (m-1)^2 \\ = (m-1)(2d+2-m) \geq (m-1)(2d+2-(d+1)) > (m-2)(d+1),$$

thereby proving (3.13). $\square$

Since for A_{t_0} to occur some constellation T must be present in our complex, we have an estimate

$$(3.16) \quad \text{Prob}(A_{t_0}) \leq \sum_{\pi} (n-d-1)^m p^{d+1} \rho(\partial \tau, \text{supp } \tau, m-1),$$

where the sum is taken over all partitions $\pi = \pi_1 \cup \dots \cup \pi_m$, the factor $(n-d-1)^m$ records choosing the m vertices corresponding to the blocks of π , the factor p^{d+1} records the probability of choosing the set T , which is uniquely determined by the choice of these vertices, and the term $\rho(\partial\tau, \text{supp } \tau, m-1)$ is well-defined by the claim which we just proved and the fact that $m \geq 1$. Using the inequality (3.1), we arrive at

$$(3.17) \quad \begin{aligned} \text{Prob}(A_{t_0}) &\leq \sum_{\pi} (n-d-1)^m p^{d+1} (d+1)^{m-1} (m-1)! p^{m-1} / (1-w)^{m-1} \\ &< \frac{(d+1)^d d!}{(1-w)^d} p^d \sum_{\pi} n^m p^m = \frac{(d+1)^d d!}{(1-w)^d} p^d \sum_{\pi} w^m \\ &< \frac{(d+1)^d d! \text{part}(d+1)}{(1-w)^d} w p^d, \end{aligned}$$

where $\text{part}(d+1)$ denotes the number of set partitions of the set $[d+1]$. Combining with (3.11) and setting $c := (d+1)^d d! \text{part}(d+1)$ yield

$$\text{Prob}(\beta_d(Y_{n,p,d}; \mathbb{Z}/2\mathbb{Z}) > 0) < w n^d \frac{c}{(1-w)^d} w p^d = c \frac{w^{d+2}}{(1-w)^d}.$$

We conclude that $\lim_{n \rightarrow \infty} \text{Prob}(\beta_d(Y_{n,p,d}; \mathbb{Z}/2\mathbb{Z}) > 0) = 0$ if $\lim_{n \rightarrow \infty} w(n) = 0$, also for all $d \geq 3$. $\square$

4. PROOF OF THE SECOND PART OF THEOREM 2.1

Before we present the proof of the second part of Theorem 2.1, we need to recall some standard tools of combinatorial probability from [1]. More specifically, a certain application of Chebyshev inequality has come to be known as the Second Moment Method. We need the symmetric version of that method, which we now proceed to describe.

Consider an infinite sequence of probability spaces $\mathcal{P}^n$, where n is a natural number. Let us fix n for now, and assume that we have random events $A_1^n, \dots, A_m^n$ in $\mathcal{P}^n$. For $i \in [m]$, let X_i^n denote indicator random variable of A_i^n , and set $X^n = \sum_{i=1}^m X_i^n$. Assume furthermore that the events A_i^n are *symmetric* in the following sense: for every $i, j \in [m]$, $i \neq j$, there exists an automorphism of the underlying probability space sending event A_i^n to event A_j^n . An example of such symmetric events in $Y_{n,p,d}$ can be found by setting $m := \binom{n}{d+1}$, indexing the d -simplices with the set $[m]$, and letting A_i^n denote the event that the d -simplex indexed with i lies in the chosen simplicial complex.

For distinct indices $i, j \in [m]$, we write $i \sim j$ in case the events A_i^n and A_j^n are not independent. Furthermore, we set

$$(4.1) \quad \xi := \sum_{i \sim j} \text{Prob}(A_i^n \wedge A_j^n).$$

We mention explicitly that the sum in (4.1) is taken over all ordered pairs (i, j) ; that is, if the summand $\text{Prob}(A_i^n \wedge A_j^n)$ occurs in the sum, then the summand $\text{Prob}(A_j^n \wedge A_i^n)$ occurs in the sum as well, since $i \sim j$ if and only if $j \sim i$. Since for all $i, j \in [m]$ we have $\text{Prob}(A_i^n \wedge A_j^n) = \text{Prob}(A_i^n) \text{Prob}(A_j^n | A_i^n)$, the equation (4.1)

now yields

$$(4.2) \quad \xi = \sum_{i=1}^m \text{Prob}(A_i^n) \sum_{j:i \sim j} \text{Prob}(A_j^n | A_i^n).$$

We set

$$(4.3) \quad \xi^* := \sum_{j:i \sim j} \text{Prob}(A_j^n | A_i^n),$$

which is well-defined, since that sum does not depend on the choice of i .

The following result can be found in [1].

Lemma 4.1 ([1, Corollary 3.5]). *With the notation above, if $\lim_{n \rightarrow \infty} E(X^n) = \infty$ and $\xi^* = o(E(X^n))$, then*

$$(4.4) \quad \lim_{n \rightarrow \infty} \text{Prob}(X^n > 0) = 1,$$

and, furthermore,

$$(4.5) \quad \lim_{n \rightarrow \infty} X^n / E(X^n) = 1.$$

We now have all the necessary tools to proceed with the proof of Theorem 2.1(2).

Proof of Theorem 2.1(2). Our argument is a direct application of the second moment method. For fixed d and p , we set $\mathcal{P}^n := Y_{n,p,d}$. We let $\{\tau_1^n, \dots, \tau_{\binom{n}{d+2}}^n\}$ be the set of all $(d+1)$ -simplices of Δ_n . For all $i = 1, \dots, \binom{n}{d+2}$, let A_i^n denote the event that $\Delta \in Y_{n,p,d}$ contains the boundary of τ_i^n , i.e., $\Delta(d+1) \supseteq \text{supp}(\partial\tau_i^n)$.

As above, let X_i^n denote the corresponding indicator random variables, and set again $X^n := X_1^n + \dots + X_{\binom{n}{d+2}}^n$. Clearly, $E(X_i^n) = \text{Prob}(A_i^n) = p^{d+2}$, for all i . Hence

$$\begin{aligned} E(X^n) &= \sum_{i=1}^{\binom{n}{d+2}} E(X_i^n) = \binom{n}{d+2} p^{d+2} > \frac{(n-d-1)^{d+2} p^{d+2}}{(d+2)!} \\ &= \frac{1}{(d+2)!} \left(1 - \frac{d+1}{n}\right)^{d+2} w^{d+2}, \end{aligned}$$

and so we see that $E(X^n) = \Omega(w^{d+2})$, and, in particular, $\lim_{n \rightarrow \infty} E(X^n) = \infty$.

Furthermore, we have $i \sim j$ if and only if the $(d+1)$ -simplices τ_i and τ_j share precisely one boundary d -simplex. Thus, in this case, the dependency graph has $\binom{n}{d+2}$ vertices and is regular of valency $(d+2)(n-d-1)$.

Given $i, j \in \{1, \dots, \binom{n}{d+2}\}$, such that $i \sim j$, we get $\text{Prob}(A_j | A_i) = p^{d+1}$, since $|\text{supp}(\partial\tau_j) \setminus \text{supp}(\partial\tau_i)| = d+1$. Plugging this data into the definition (4.3), we get

$$\xi^* = \sum_{j:i \sim i} p^{d+1} = (d+2)(n-d-1)p^{d+1}.$$

Since

$$E(X^n) = \binom{n}{d+2} p^{d+2} > \frac{n(n-d-1)^{d+1}}{(d+2)!} p^{d+2},$$

we get

$$(4.6) \quad \xi^* / E(X^n) < \frac{(d+2)(d+2)!}{np(n-d-1)^d} = \frac{(d+2)(d+2)!}{w(n-d-1)^d}.$$

Since we assumed that $\lim_{n \rightarrow \infty} w(n) = \infty$, the inequality (4.6) yields the equality $\lim_{n \rightarrow \infty} \xi^*/E(X^n) = 0$, i.e., $\xi^* = o(E(X^n))$. It then follows from Lemma 4.1 that $\lim_{n \rightarrow \infty} \text{Prob}(X^n > 0) = 1$.

Since $X^n > 0$ implies that $\beta_d(\Delta; \mathbb{Z}/2\mathbb{Z}) > 0$, we get $\text{Prob}(\beta_d(\Delta; \mathbb{Z}/2\mathbb{Z}) > 0) \geq \text{Prob}(X^n > 0)$; hence $\lim_{n \rightarrow \infty} \text{Prob}(\beta_d(Y_{n,p,d}; \mathbb{Z}/2\mathbb{Z}) > 0) = 1$. $\square$

5. THRESHOLD PROBABILITY FOR TOP HOMOLOGY GROUP WITH COEFFICIENTS IN AN ARBITRARY ABELIAN GROUP

In this short final section we shall indicate how to adjust our proofs in order to deal with the case of homology with coefficients in an arbitrary abelian group. The exact statement which we get is the following.

Theorem 5.1. *Assume that $p = p(n) = w(n)/n$, $d \geq 1$, and F is an arbitrary nontrivial abelian group. Then we have*

- (1) *if $\lim_{n \rightarrow \infty} w(n) = 0$, then $\lim_{n \rightarrow \infty} \text{Prob}(H_d(Y_{n,p,d}; F) = 0) = 1$;*
- (2) *if $\lim_{n \rightarrow \infty} w(n) = \infty$, then $\lim_{n \rightarrow \infty} \text{Prob}(H_d(Y_{n,p,d}; F) \neq 0) = 1$.*

To start with, we need a new piece of notation: for a subset $T \subseteq \Delta_n(d)$ we let $r(T)$ denote the number of $(d-1)$ -simplices σ for which there exists a unique $\tau \in T$ such that $\sigma \in \text{supp } \partial\tau$. One may intuitively think of such $(d-1)$ -simplices as the “rim” of the set T .

Next, the set Σ_d should be replaced with $\tilde{\Sigma}_d \subseteq 2^{\Delta_n(d)} \times 2^{\Delta_n(d)} \times \mathbb{Z}_{\geq 0}$ defined by the following: $(T, S, \lambda) \in \tilde{\Sigma}_d$ if and only if $r(T) > (\lambda - 1)(d + 1)$.

Accordingly, Definition 2.3 should be altered. For $(T, S, \lambda) \in \tilde{\Sigma}_d$, we define $\tilde{\rho}(T, S, \lambda)$ to be the probability that $\Delta \in Y_{n,p,d}$ satisfies the following two conditions:

- (1) $\Delta(d) \cap S = \emptyset$;
- (2) there exists $\sigma \in Z_d(\Delta_n)$ such that $T \subseteq \text{supp } \sigma \subseteq T \cup \Delta(d)$.

With this notation the inequality (3.1) gets replaced with

$$(5.1) \quad \tilde{\rho}(T, S, \lambda) \leq c(d, \lambda) p^\lambda / (1 - w)^\lambda,$$

with the proof holding almost verbatim. Essentially $|\text{supp } \sigma|$ should be replaced with $r(\text{supp } \sigma)$, and $\rho(\partial\tau, S, \lambda)$ should be replaced with $\tilde{\rho}(\text{supp } \tau, S, \lambda)$. For example, in the definition of A and B the expression $|\text{supp } (\sigma + \partial\tau)|$ should be replaced with $r(T \cup \{\tau\})$, the inequality (3.2) becomes

$$(5.2) \quad \tilde{\rho}(T, S, \lambda) \leq \sum_{\tau \in A \cup B} \text{Prob}(\tau) \tilde{\rho}(T \cup \{\tau\}, S \cup \{\tau\}, \lambda_\tau),$$

and the chain of inequalities (3.5) becomes

$$(5.3) \quad (\lambda - 1)(d + 1) \geq r(T \cup \{\tau\}) \geq r(T) - |\text{supp } \partial\tau| \\ = r(T) - (d + 1) > (\lambda - 2)(d + 1).$$

Also the proof of Theorem 2.1(1) holds verbatim with similar changes. For example, the inequality (3.12) becomes

$$(5.4) \quad \text{Prob}(A_{t_0}) \leq \sum_{i=4}^n \text{Prob}(s_i) \tilde{\rho}(\{t_0, s_i\}, \{s_i\}, 2),$$

the inequality (3.13) in the claim becomes

$$(5.5) \quad r(\{t_0, \dots, t_{d+1}\}) > (m - 2)(d + 1),$$

and the inequality (3.16) becomes

$$(5.6) \quad \text{Prob}(A_{t_0}) \leq \sum_{\pi} (n-d-1)^m p^{d+1} \tilde{\rho}(\{t_0, \dots, t_{d+1}\}, \{t_0, \dots, t_{d+1}\}, m-1).$$

Finally, the proof of Theorem 2.1(2) holds without any changes at all since the $\mathbb{Z}/2\mathbb{Z}$ -cycles $\partial\tau_i^n$ presented are in fact cycles for arbitrary coefficients.

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