NONCOMMUTATIVE RESIDUES AND A CHARACTERISATION OF THE NONCOMMUTATIVE INTEGRAL

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ABSTRACT. We continue the study of the relationship between Dixmier traces and noncommutative residues initiated by A. Connes. The utility of the residue approach to Dixmier traces is shown by a characterisation of the noncommutative integral in Connes' noncommutative geometry (for a wide class of Dixmier traces) as a generalised limit of vector states associated to the eigenvectors of a compact operator (or an unbounded operator with compact resolvent). Using the characterisation, a criteria involving the eigenvectors of a compact operator and the projections of a von Neumann subalgebra of bounded operators is given so that the noncommutative integral associated to the compact operator is *normal*, i.e. satisfies a monotone convergence theorem, for the von Neumann subalgebra. Flat tori, noncommutative tori, and a link with the QUE property of manifolds are given as examples.

1. INTRODUCTION

For a separable complex Hilbert space H , denote by $\mu_n(T)$, $n \in \mathbb{N}$, the singular values of a positive compact operator T , [17]. A. Connes introduced the association between a generalised zeta function,

$$\zeta_T(s) := \operatorname{Tr}(T^s) = \sum_{n=1}^{\infty} \mu_n(T)^s,$$

and the logarithmic divergence of the partial sums,

$$\left\{ \sum_{n=1}^N \mu_n(T) \right\}_{N=1}^{\infty},$$

with the result that

$$(1.1) \quad \lim_{s \rightarrow 1^+} (s-1)\zeta_T(s) = \lim_{N \rightarrow \infty} \frac{1}{\log(1+N)} \sum_{n=1}^N \mu_n(T)$$

if either limit exists ([6], p. 306). In [13], with coauthor A. Sedaev, we showed that the right hand side of equation (1.1) is the Dixmier trace for Connes' notion of

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a measurable operator; i.e., an operator $T \in \mathcal{L}^{1,\infty}$ where $\mathcal{L}^{1,\infty} := \{T \mid \|T\|_{1,\infty} := \sup_{n \in \mathbb{N}} \log(1+n)^{-1} \sum_{j=1}^n \mu_j(T) < \infty\}$ is called measurable if the value of a Dixmier trace $\text{Tr}_\omega(T)$, [8], ([5], p. 674), is independent of the ‘invariant mean’ (dilation invariant state) ω on ℓ^∞ ([6], Def. 7, p. 308). As a result,

$$(1.2) \quad \text{Tr}_\omega(T) = \lim_{s \rightarrow 1^+} (s-1)\zeta_T(s)$$

enables the calculation of the Dixmier trace of any measurable operator $0 < T \in \mathcal{L}^{1,\infty}$ as the residue at $s = 1$ of the zeta function ζ_T . Connes, in ([6], pp. 303-308), showed that a dilation-invariant state on ℓ^∞ was sufficient to define a Dixmier trace. In practice Connes used a Dixmier trace defined by a more restricted class of states involving Cesàro means. In [13] we showed that the (weaker) notion of measurable from Connes’ smaller class of states involving Cesàro means, the notion of measurable from dilation-invariant states, or the notion of measurable from any larger class of generalised limits, were all equivalent; see ([13], §5.3) or ([13], Thm. 6.6) in particular.

A. Carey, J. Phillips and the second author, by the content of [1], extended the formula (1.1) to nonmeasurable operators (in the sense of Connes). The results were a generalisation to τ -compact operators for von Neumann algebras with faithful normal semifinite trace τ . In this setting the s -numbers $\mu_s(T)$ of a τ -compact operator, the generalisation of singular values, are continuous instead of discrete and one considers the Dixmier trace as an expression $\text{tr}_v(T) := v(\frac{1}{\log(1+t)} \int_1^t \mu_s(T) ds)$ for a dilation invariant state v on $L^\infty([1, \infty))$. We phrase the extension of (1.1) in the language of $B(H)$ (the bounded linear operator on H). For $A \in B(H)$ and $0 < T \in \mathcal{L}^{1,\infty}$, define

$$\zeta_{A,T}(s) := \text{Tr}(AT^s).$$

Then ([1], Thm. 3.8) states

$$(1.3) \quad \text{tr}_v(AT) = \tilde{v}\text{-}\lim_{s \rightarrow 1^+} (s-1)\zeta_{A,T}(s)$$

for a ‘maximally invariant mean’ (a dilation, power and Cesàro invariant state) v of $L^\infty([1, \infty))$. The notation $\tilde{v}\text{-}\lim_{s \rightarrow 1^+} f(s)$ stands for $\tilde{v}(f(1+t^{-1}))$, where $\tilde{v}(f(t)) := v(f(\log t))$ for $f \in L^\infty([0, \infty))$, and $\text{tr}_v(AT)$ stands for the linear extension of the weight $\text{tr}_v(\sqrt{AT}\sqrt{A})$, $A > 0$. The conditions on v were reduced to dilation and power invariance in a later text of Carey, A. Rennie, Sedaev and the second author; see ([2], Thm. 4.11). However, ([2], Thm. 4.11) achieved (1.3) *only* for $A = 1$.

In this note we show the utility of the noncommutative residue to the study of the noncommutative integral (taken in most texts on noncommutative geometry to be given by the left hand side of (1.2) or (1.3)).

Our first task will be to prove that ([2], Thm. 4.11) can achieve the formula (1.3) for $A \neq 1$ with the same weakened conditions on the generalised limit v . This is shown in Corollary 3.3. We also adapt the formula (1.3) to the class $\mathcal{L}(BL \cap DL)$ of ‘dilation and power invariant’ states on ℓ^∞ . The preliminaries will make the notation $\mathcal{L}(BL \cap DL)$ apparent. This is done in Corollary 3.5. The adaptation shows that the generalisations in [1] and [2] apply to the ‘original’ type I construction of Dixmier (used originally by Connes in [5]). This step is not entirely trivial. There are subtle distinctions between ‘continuous’ and ‘discrete’ Dixmier traces, even though they provide equivalent sets of traces; see ([13], Thm. 6.2).

With the correspondence between noncommutative residues and the noncommutative integral (the left hand side of (1.2) or (1.3)) firmly in hand, we use residues to show two analytic results.

The first result is a structure result for the noncommutative integral. Assume that the situation is nontrivial, i.e. $0 < T \in \mathcal{L}^{1,\infty}$ with $\text{Tr}_\omega(T) > 0$. For $A \in B(H)$ set

$$(1.4) \quad \phi_\omega(A) := \frac{\text{Tr}_\omega(AT)}{\text{Tr}_\omega(T)}.$$

Note that ϕ_ω is a state of $B(H)$. Then Theorem 3.6 says that when $\omega \in \mathcal{L}(BL \cap DL)$,

$$(1.5) \quad \phi_\omega(A) = L_\omega(\langle h_m, Ah_m \rangle),$$

where $\{h_m\}_{m=1}^\infty$ is any complete orthonormal system of eigenvectors for T ([17], §1) and L_ω is a generalised limit. The characterisation (1.5) shows, when the sequence $\{\langle h_m, Ah_m \rangle\}_{m=1}^\infty$ is convergent at infinity, that the state ϕ_ω is *uniquely* and completely characterised by the *eigenvectors* of T . The *eigenvalues* of T are linked solely to the scale factor $\text{Tr}_\omega(T)$. The flat 1-torus and the noncommutative torus provide examples in the text. The characterisation (1.5) also provides an interesting link between Connes' noncommutative integral and quantum ergodicity on compact Riemannian manifolds [18, 4, 19, 16]; see Example 3.10.

The second result we obtain is normality criteria for the noncommutative integral. While ϕ_ω is a state of $B(H)$, it is not a normal state. This is problematic for an 'integral', since monotone and dominated convergence cannot be applied. The normality of ϕ_ω on proper weakly closed *-subalgebras of $B(H)$ is an open question. The characterisation (1.5) of ϕ_ω gives us a valuable purchase. Let $\mathcal{M} \subset B(H)$ be a von Neumann algebra. We say that a positive compact operator T is $(\mathcal{M}, h)$ -dominated if there exists $h \in H$ such that $\|Ph_m\| \leq \|Ph\|$ for all projections $P \in \mathcal{M}$. Think here of $H = L^2(F, \mathcal{B}, \mu)$ for a σ -finite measure space $(F, \mathcal{B}, \mu)$ and $\mathcal{M}$ as multiplication operators of L^∞ -functions. Then T being $(\mathcal{M}, h)$ -dominated is the same as $\int_J |h_m(x)|^2 d\mu(x) \leq \int_J |h(x)|^2 d\mu(x)$ for all $J \subset \mathcal{B}$, which is equivalent to the statement that the $|h_m|^2$ are dominated by $|h|^2 \in L^1(F, \mathcal{B}, \mu)$ μ -a.e.. In Theorem 3.12 we show that if T is $(\mathcal{M}, h)$ -dominated, then $\text{Tr}_\omega(\cdot T) \in \mathcal{M}_*$.

2. PRELIMINARIES

2.1. Preliminaries on (discrete) Dixmier traces. Let $\lceil x \rceil$, $x \geq 0$, denote the ceiling function. Define the maps $\ell^\infty \rightarrow \ell^\infty$ for $j \in \mathbb{N}$ by

$$\begin{aligned} T_j(\{a_k\}_{k=1}^\infty) &= \{a_{k+j}\}_{k=1}^\infty, \\ D_j(\{a_k\}_{k=1}^\infty) &= \{a_{\lceil j^{-1}k \rceil}\}_{k=1}^\infty. \end{aligned}$$

Set $BL := \{0 < \omega \in (\ell^\infty)^* \mid \omega(1) = 1, \omega \circ T_j = \omega \ \forall j \in \mathbb{N}\}$ (the set of Banach Limits) and $DL := \{0 < \omega \in (\ell^\infty)^* \mid \omega(1) = 1, \omega \circ D_j = \omega \ \forall j \in \mathbb{N}\}$. Both sets of states on ℓ^∞ satisfy

$$(2.1) \quad \liminf_k a_k \leq \omega(\{a_k\}_{k=1}^\infty) \leq \limsup_k a_k, \ a_k \geq 0.$$

Any state of ℓ^∞ satisfying (2.1) is termed a generalised limit since it extends $\lim$ on c to ℓ^∞ . Let $0 < T \in \mathcal{L}^{1,\infty}$. Set $\gamma(T) := \left\{ \log(1+k)^{-1} \sum_{j=1}^k \mu_j(T) \right\}_{k=1}^\infty \in \ell^\infty$

and define $DL_2 := \{0 < \omega \in (\ell^\infty)^* \mid \omega(1) = 1, \omega \text{ satisfies (2.1), and } \omega(D_2(\gamma(T))) = \omega(\gamma(T)) \forall 0 < T \in \mathcal{L}^{1,\infty}\}$. For any $\omega \in DL_2$,

$$\text{Tr}_\omega(T) := \omega(\gamma(T))$$

defines a tracial weight [8], ([6], pp. 304-305), ([3], Prop 5.2). Denote by Tr_ω as well the linear extension. Then Tr_ω is a finite trace on $\mathcal{L}^{1,\infty}$ that vanishes on the separable part $\mathcal{L}_0^{1,\infty}$. The separable part $\mathcal{L}_0^{1,\infty}$ is the closure of finite rank operators in the norm $\|\cdot\|_{1,\infty}$.

2.2. Preliminaries on (continuous) Dixmier traces. For $a > 0$ define the maps $L^\infty([0, \infty)) \rightarrow L^\infty([0, \infty))$ by

$$\begin{aligned} T_a(f)(t) &= f(t+a), \\ D_a(f)(t) &= f(a^{-1}t). \end{aligned}$$

Let ϕ be a state on $L^\infty([0, \infty))$ satisfying:

- (i) $\text{ess.-lim inf}_{t \rightarrow \infty} g(t) \leq \phi(g) \leq \text{ess.-lim sup}_{t \rightarrow \infty} g(t)$ for $0 < g \in L^\infty([0, \infty))$;
- (ii) $\phi(g) = \phi(T_a(g))$, $a > 0$, for $g \in L^\infty([0, \infty))$.

Then ϕ is called a (continuous) Banach limit, $\phi \in BL[0, \infty)$, [1, 13]. If ϕ satisfies

- (ii)' $\phi(g) = \phi(D_a(g))$, $a > 0$, for $g \in L^\infty([0, \infty))$

instead of (ii), we denote this by $\phi \in DL[0, \infty)$. Define $L^{-1} : L^\infty([1, \infty)) \rightarrow L^\infty([0, \infty))$ by

$$L^{-1}(g)(t) = g(e^t)$$

and set

$$L(\phi) := \phi \circ L^{-1}.$$

It is known that $T_a L^{-1} = L^{-1} D_{e^{-a}}$ and $D_{a^{-1}} L^{-1} = L^{-1} P^a$, $a \geq 1$, where

$$P^a(f)(t) = f(t^a);$$

see ([1], §1.1). Thus L provides isometries $BL[0, \infty) \rightarrow DL[1, \infty)$ and $DL[0, \infty) \rightarrow P[1, \infty)$, where the notation should be evident.

Let $0 < T \in \mathcal{L}^{1,\infty}$. Set $\Gamma(T)(t) := \log(1+t)^{-1} \int_1^t \mu_s(T) ds$, where $\mu_s(T)$ are the s -numbers of T relative to the canonical trace Tr on $B(H)$. Denote $L(\phi) \in DL_2[1, \infty)$ if ϕ satisfies (i) above and $L(\phi)(D_2(\Gamma(T))) = L(\phi)(\Gamma(T)) \forall 0 < T \in \mathcal{L}^{1,\infty}$. From ([13], §6) and ([3], §5), for any $L(\phi) \in DL_2[1, \infty)$,

$$\text{tr}_{L(\phi)}(T) := L(\phi)(\Gamma(T))$$

defines a tracial weight. Denote by $\text{tr}_{L(\phi)}$ as well the linear extension. Then $\text{tr}_{L(\phi)}$ is a finite trace on $\mathcal{L}^{1,\infty}$ that vanishes on the separable part $\mathcal{L}_0^{1,\infty}$. It is evident that if $\phi \in BL[0, \infty)$, then $L(\phi) \in DL_2[1, \infty)$.

3. THE MAIN RESULTS

We state the extension of ([2], Thm. 4.11). For brevity we state the result only for $B(H)$. The statement and proof for a general semifinite von Neumann algebra is apparent. The proofs of Theorems 3.1 and 3.4 and Corollaries 3.3 and 3.5 are reserved for the technical section, Section 4.

Theorem 3.1. *Let P be a projection and $0 < T \in \mathcal{L}^{1,\infty}$. For any $\phi \in BL[0, \infty) \cap DL[0, \infty)$,*

$$\mathrm{tr}_{L(\phi)}(PTP) = \phi \left(\frac{1}{r} \mathrm{Tr}(PT^{1+\frac{1}{r}}P) \right).$$

Moreover, $\lim_{s \rightarrow 1^+} (s-1) \mathrm{Tr}(PT^sP)$ exists iff PTP is measurable, and in either case

$$\mathrm{tr}_v(PTP) = \lim_{s \rightarrow 1^+} (s-1) \mathrm{Tr}(PT^sP)$$

for all $v \in DL_2[1, \infty)$.

The condition that ϕ be dilation-invariant as well as translation-invariant (hence $L(\phi)$ is power-invariant in addition to being dilation-invariant) is needed to apply the weak*-Karamata theorem [1, 2] in the proof of Theorem 3.1 (see Lemma 4.7 below). Part of the rationale for the introduction of continuous Dixmier traces in [1] was the ability to apply the weak*-Karamata theorem.

Definition 3.2. Let $0 < T \in \mathcal{L}^{1,\infty}$. We say that T is *spectrally measurable* w.r.t. $A \in B(H)$ (in the sense of Connes) if PTP is measurable for all projections P in the von Neumann algebra generated by A and A^* .

Corollary 3.3. *Let $A \in B(H)$ and $0 < T \in \mathcal{L}^{1,\infty}$. For any $\phi \in BL[0, \infty) \cap DL[0, \infty)$,*

$$\mathrm{tr}_{L(\phi)}(AT) = \phi \left(\frac{1}{r} \zeta_{A,T} \left(1 + \frac{1}{r} \right) \right).$$

Moreover, AT is measurable if T is spectrally measurable w.r.t. A and

$$\mathrm{tr}_v(AT) = \lim_{s \rightarrow 1^+} (s-1) \zeta_{A,T}(s)$$

for all $v \in DL_2[1, \infty)$.

We now state the adaptation to ‘original’ type I (discrete) Dixmier traces. Define the averaging sequence $E : L^\infty([0, \infty)) \rightarrow \ell^\infty$ by

$$E_k(f) := \int_{k-1}^k f(t) dt.$$

Define the floor mapping $p : \ell^\infty \rightarrow L^\infty([1, \infty))$ by

$$p(\{a_k\}_{k=1}^\infty)(t) := \sum_{k=1}^\infty a_k \chi_{[k, k+1)}(t).$$

Define, finally, the isometry $\mathcal{L} : (\ell^\infty)^* \rightarrow (L^\infty)^*$ by

$$\mathcal{L}(\xi) := \xi \circ E \circ L^{-1} \circ p.$$

Denote by $\mathcal{L}(BL \cap DL)$ the image of translation-invariant and dilation-invariant states on ℓ^∞ under $\mathcal{L}$. Unlike the continuous case, it is not evident that $\mathcal{L}(\xi) \in DL_2$ if $\xi \in BL \cap DL$.

Theorem 3.4. *Let P be a projection and $0 < T \in \mathcal{L}^{1,\infty}$. For any $\xi \in BL \cap DL$, $\mathcal{L}(\xi) \in DL_2$ and*

$$\mathrm{Tr}_{\mathcal{L}(\xi)}(PTP) = \xi \left(\frac{1}{k} \mathrm{Tr}(PT^{1+\frac{1}{k}}P) \right).$$

Moreover, $\lim_{k \rightarrow \infty} \frac{1}{k} \operatorname{Tr}(PT^{1+\frac{1}{k}}P)$ exists iff PTP is measurable, and in either case

$$\operatorname{Tr}_\omega(PTP) = \lim_{k \rightarrow \infty} \frac{1}{k} \operatorname{Tr}(PT^{1+\frac{1}{k}}P)$$

for all $\omega \in DL_2$.

Corollary 3.5. *Let $A \in B(H)$ and $0 < T \in \mathcal{L}^{1,\infty}$. For any $\xi \in BL \cap DL$,*

$$\operatorname{Tr}_{\mathcal{L}(\xi)}(AT) = \xi \left(\frac{1}{k} \zeta_{A,T} \left(1 + \frac{1}{k} \right) \right).$$

Moreover, AT is measurable if T is spectrally measurable w.r.t. A and

$$\operatorname{Tr}_\omega(AT) = \lim_{k \rightarrow \infty} \frac{1}{k} \zeta_{A,T} \left(1 + \frac{1}{k} \right) = \lim_{s \rightarrow 1^+} (s-1) \zeta_{A,T}(s)$$

for all $\omega \in DL_2$.

Here AT measurable means that $\operatorname{Tr}_\omega(AT)$ is independent of $\omega \in DL_2$. In Corollary 3.3 AT measurable meant that $\operatorname{tr}_v(AT)$ was independent of $v \in DL_2[1, \infty)$. Spectral measurability is sufficient for equivalence of the two notions when $A \neq P$ (equivalence when $A = P$ was shown in ([13], Cor. 3.9)).

We show two applications of residues.

3.1. Structure of the noncommutative integral. Let $0 < T \in \mathcal{L}^{1,\infty}$ be non-trivial; i.e. $\operatorname{Tr}_\omega(T) > 0$ for all $\omega \in DL_2$. Following (1.4) in the introduction,

$$\phi_\omega(A) := \frac{\operatorname{Tr}_\omega(AT)}{\operatorname{Tr}_\omega(T)}, \quad A \in B(H)$$

is a state of $B(H)$.

Theorem 3.6. *Let $\omega \in \mathcal{L}(BL \cap DL)$. For any orthonormal basis $\{h_m\}_{m=1}^\infty$ of eigenvectors of T , $\phi_\omega(A) = L_\omega(\langle h_m, Ah_m \rangle)$ for a generalised limit L_ω .*

The proof is not technical. We provide it here. Take an orthonormal basis of eigenvectors $\{h_m\}_{m=1}^\infty$ for T , where $Th_m = \lambda_m h_m$. Let P_m , $m \in \mathbb{N}$, denote the one-dimensional projections onto h_m . Define the map $\theta : \ell^\infty \rightarrow B(H)$ by

$$(3.1) \quad \theta(\{a_k\}_{k=1}^\infty) = \sum_{k=1}^\infty a_k P_k,$$

where convergence of partial sums is in the strong operator topology. The proof of the following two lemmas is routine and therefore omitted.

Lemma 3.7. *The map $\theta : \ell^\infty \rightarrow B(H)$ is an isometric injection such that $\theta(\mathbf{1}) = I$. Here I is the identity of $B(H)$.*

Lemma 3.8. *A state on ℓ^∞ vanishes on finite sequences if and only if it is a generalised limit.*

Let $\omega \in DL_2$. Define the linear functional $L_\omega : \ell^\infty \rightarrow \mathbb{C}$ by

$$(3.2) \quad L_\omega(\{a_k\}_{k=1}^\infty) = \phi_\omega(\theta(\{a_k\}_{k=1}^\infty)).$$

Proposition 3.9. *The map $L_\omega : \ell^\infty \rightarrow \mathbb{C}$ is a generalised limit.*

Proof. Without loss $\operatorname{Tr}_\omega(T) = 1$. It is evident that $L := L_\omega$ is positive and $L(\mathbf{1}) = 1$. Hence L is a state of ℓ^∞ . Suppose $\{a_k\}_{k=1}^\infty \geq 0$. For $N \geq 1$, $\theta(\{a_k\}_{k=1}^N)T$ is of finite rank and $L(\{a_k\}_{k=1}^N) = \operatorname{Tr}_\omega(\theta(\{a_k\}_{k=1}^N)T) = 0$. Apply the previous lemma. $\square$

3.1.1. *Proof of Theorem 3.6.* Without loss $\text{Tr}_\omega(T) = 1$. Let $\xi \in BL \cap DL$. Using Corollary 3.5,

$$\begin{aligned}
 \phi_{\mathcal{L}(\xi)}(A) &= \xi(p^{-1} \text{Tr}(AT^{1+p^{-1}})) \\
 &= \xi(p^{-1} \sum_{m=1}^{\infty} \langle h_m, AT^{1+p^{-1}} h_m \rangle) \\
 (3.3) \quad &= \xi(p^{-1} \sum_{m=1}^{\infty} \lambda_m^{1+p^{-1}} \langle h_m, Ah_m \rangle).
 \end{aligned}$$

Conversely, from (3.2), $L_{\mathcal{L}(\xi)}(\{\langle h_k, Ah_k \rangle\}_{k=1}^{\infty}) = \phi_{\mathcal{L}(\xi)}(\theta(\{\langle h_k, Ah_k \rangle\}_{k=1}^{\infty}))$ and

$$\begin{aligned}
 \phi_{\mathcal{L}(\xi)}(\theta(\{\langle h_k, Ah_k \rangle\}_{k=1}^{\infty})) &= \xi(p^{-1} \text{Tr}(\sum_{k=1}^{\infty} \langle h_k, Ah_k \rangle P_k T^{1+p^{-1}})) \\
 &= \xi(p^{-1} \sum_{m=1}^{\infty} \langle h_m, \sum_{k=1}^{\infty} \langle h_k, Ah_k \rangle P_k T^{1+p^{-1}} h_m \rangle) \\
 (3.4) \quad &= \xi(p^{-1} \sum_{m=1}^{\infty} \lambda_m^{1+p^{-1}} \langle h_m, Ah_m \rangle).
 \end{aligned}$$

Comparing (3.3) and (3.4) yields the result. $\square$

Example 3.10. 1. Consider the Laplacian $\Delta = -d^2/d\theta^2$ on the flat 1-torus $\mathbb{T}$. From Theorem 3.6, $\phi_\omega(A) = L_\omega(\langle f_m, Af_m \rangle)$, where $f_m(\theta) = e^{im\theta}$, $m \in \mathbb{Z} \cong \{0, 1, -1, 2, -2, \dots\}$, for any $A \in B(L^2(\mathbb{T}))$. If M_f is the multiplier of $f \in L^\infty(\mathbb{T})$ on $L^2(\mathbb{T})$, $\langle f_m, M_f f_m \rangle = \int_{-\pi}^{\pi} f(\theta) d\theta$, and $\text{Tr}_\omega(M_f \Delta^{-1/2}) = 2\phi_\omega(M_f) = 2 \int_{-\pi}^{\pi} f(\theta) d\theta$.

2. Let M be a compact Riemannian manifold without boundary with Hodge Laplacian Δ . Consider the Friedrichs quantification $\text{Op}^F(a) \in B(L^2(M))$ (the extension of a zeroth-order pseudodifferential operator) associated to a function $a \in C^\infty(S^1M)$. Here S^1M is the unit sphere bundle of the cotangent bundle of M . We have used the notation of [9]. From Connes' Trace Theorem [5, 10] and Theorem 3.6, for any orthonormal basis $\{h_m\}_{m=1}^{\infty}$ of eigenvectors of the Laplacian

$$(3.5) \quad \int_{S^1M} a d\omega = L(\langle h_m, \text{Op}^F(a) h_m \rangle)$$

for a generalised limit L . Here $d\omega$ denotes the normalised Liouville measure on S^1M . If $a \in C^\infty(M) \hookrightarrow C^\infty(S^1M)$, then $d\omega$ reverts to the Lebesgue measure on M and $\text{Op}^F(a)$ to the multiplier by a .

Equation (3.5) is an interesting link to the notion of quantum ergodicity (QE). QE is the property that $\int_{S^1M} a d\omega = \lim_{j \rightarrow \infty} \langle h_{m_j}, \text{Op}^F(a) h_{m_j} \rangle$ for a 'density one' subsequence of eigenfunctions $\{h_{m_j}\}_{j=1}^{\infty}$; see [18, 4, 19, 9]. It is known to occur for compact Riemannian manifolds M with ergodic geodesic flow on S^1M . That $\int_{S^1M} a d\omega = \lim_{m \rightarrow \infty} \langle h_m, \text{Op}^F(a) h_m \rangle$ for all orthonormal bases of eigenfunctions of the Laplacian is called quantum unique ergodicity (QUE), [9, 16]. It is not well known which manifolds have the QUE property.¹ The characterisation in Theorem 3.6 shows though, if one goes in a certain direction of looking at generalised limits as opposed to convergence of subsequences, that *any* basis of eigenfunctions

¹Private communication from A. Hassell. We also thank A. Hassell for discussions about the QE and QUE properties and references.

of the Hodge Laplacian on a compact Riemannian manifold displays a weakened quantum unique ergodicity in the sense of (3.5).

3. Consider two unitaries u, v such that $uv = \lambda vu$, for $\lambda := e^{2\pi i\theta} \in \mathbb{S}$ (the unit circle). Denote by $F_\theta(u, v)$ the $*$ -algebra of linear combinations $\sum_{(m,n) \in J} a_{m,n} u^m v^n$, where $J \subset \mathbb{Z}^2$ is a finite set, with product $ab = \sum_{r,s} (\sum_{m,n} a_{r-m,n} \lambda^{mn} b_{m,s-n}) u^r v^s$ and involution $a^* = \sum_{r,s} (\lambda^{rs} \bar{a}_{-r,-s}) u^r v^s$, $a, b \in F_\theta(u, v)$. The assignment $\tau_0(a) = a_{0,0}$ is a faithful trace on $F_\theta(u, v)$. Let (H_θ, π_θ) denote the cyclic representation associated to τ_0 . The closure, $C_\theta(u, v)$, of $\pi_\theta(F_\theta(u, v))$ in the operator norm is called a rotation C^* -algebra, [15], or the noncommutative torus ($\lambda \neq 1$), [6, 7, 10]. Canonically, finite linear combinations of $u^m v^n \hookrightarrow H_\theta$ are dense in H_θ . Define $\Delta_\theta(u^m v^n) = (m^2 + n^2)u^m v^n$. It can be shown that the ‘noncommutative Laplacian’ Δ_θ has a unique positive extension (also denoted Δ_θ) $\Delta_\theta : \text{Dom}(\Delta_\theta) \rightarrow H_\theta$ with compact resolvent; see [10]. The eigenvectors $h_{m,n} = u^m v^n \in H_\theta$ form a complete orthonormal system. Note that

$$\langle h_{m,n}, \pi_\theta(a) h_{m,n} \rangle =: \tau_0((u^m v^n)^* a u^m v^n) = \tau_0(a)$$

for any $(m, n) \in \mathbb{Z}^2$. Using the Cantor enumeration of $\mathbb{Z}^2$, from Theorem 3.6 we obtain $\text{Tr}_\omega(\pi_\theta(a) \Delta_\theta^{-1}) = \pi \langle h_{0,0}, \pi_\theta(a) h_{0,0} \rangle = \pi \tau_0(a)$, $\forall a \in F_\theta(u, v)$ (since $\text{Tr}_\omega(\Delta_\theta^{-1}) = \pi$; see [10]). By uniform continuity the same result follows for $C_\theta(u, v)$. Thus Theorem 3.6 provides a short proof of the known facts that $A \Delta_\theta^{-1}$, $A \in C_\theta(u, v)$, is measurable (in the sense of Connes) and $\text{Tr}_\omega(\cdot \Delta_\theta^{-1})$ is a faithful trace on $C_\theta(u, v)$.

3.2. Conditions for normality of the noncommutative integral. Let $\mathcal{M}$ be a weakly closed $*$ -subalgebra of $B(H)$.

Definition 3.11. A positive compact operator T is $(\mathcal{M}, h)$ -dominated if, for some orthonormal basis $\{h_m\}_{m=1}^\infty$ of eigenvectors of T , there exists $h \in H$ such that $\|Ph_m\| \leq \|Ph\|$ for all projections $P \in \mathcal{M}$.

Theorem 3.12. Let $0 < T \in \mathcal{L}^{1,\infty}$ be $(\mathcal{M}, h)$ -dominated. Then $\text{Tr}_\omega(\cdot T) \in \mathcal{M}_*$ for all $\omega \in \mathcal{L}(BL \cap DL)$.

Proof. By hypothesis $\langle h_m, Ph_m \rangle \leq \langle h, Ph \rangle$ for all projections $P \in \mathcal{M}$. Then $\langle h_m, Ah_m \rangle \leq \langle h, Ah \rangle$, $0 < A \in \mathcal{M}$, as A is a uniform limit of finite linear positive spans of projections ([14], p. 23). For any generalised limit L and $0 < A \in \mathcal{M}$,

$$(3.6) \quad L(\langle h_m, Ah_m \rangle) \leq \limsup_{m \rightarrow \infty} \langle h_m, Ah_m \rangle \leq \langle h, Ah \rangle.$$

Let $\{A_\alpha\}$ be a net of monotonically increasing positive elements of $\mathcal{M}$ with upper bound. It follows that $\{A_\alpha\}$ converges strongly to a l.u.b. $A \in \mathcal{M}$ ([14], p. 22). From (3.6) $L(\langle h_m, (A - A_\alpha)h_m \rangle) \leq \langle h, (A - A_\alpha)h \rangle$. Since $\langle h, (A - A_\alpha)h \rangle \xrightarrow{\alpha} 0$, $L(\langle h_m, Ah_m \rangle) = \sup_\alpha L(\langle h_m, A_\alpha h_m \rangle)$. From Theorem 3.6 $\phi_\omega(A) = \sup_\alpha \phi_\omega(A_\alpha)$ and ϕ_ω is normal on $\mathcal{M}$ ([14], §3.6.1). Thus $\text{Tr}_\omega(\cdot T) = \phi_\omega(\cdot) \text{Tr}_\omega(T)$ is normal. $\square$

Example 3.13. 1. Let (F, μ) be a σ -finite measure space. Take $H = L^2(F, \mu)$ and $\mathcal{M} = L^\infty(F, \mu)$ acting by multipliers on H . Let T be any positive compact operator with eigenfunctions f_m satisfying $|f_m|^2 \leq g \in L^1(F, \mu)$ μ -a.e.. Then T is $(\mathcal{M}, g)$ -dominated. For example, the eigenfunctions $f_m(\theta) = e^{im\theta}$ of the Laplacian Δ on the 1-torus $\mathbb{T}$ satisfy $|f_m|^2 = 1 \in L^1(\mathbb{T})$. Normality of the noncommutative integral for more general compact Riemannian manifolds is discussed in [12].

2. Let Δ_θ be the ‘noncommutative Laplacian’ from Example 3.10.3. From the example, $\langle h_{m,n}, Sh_{m,n} \rangle = \langle h_{0,0}, Sh_{0,0} \rangle$ for any $S \in C_\theta(u, v)''$. Hence $\|Ph_{m,n}\| = \|Ph_{0,0}\|$ for all projections $P \in C_\theta(u, v)''$. By Theorem 3.12, $\text{Tr}_\omega(\cdot \Delta_\theta^{-1})$ is a faithful normal trace on $C_\theta(u, v)''$.

4. TECHNICAL RESULTS

4.1. Proof of Theorem 3.1 and Corollary 3.3.

Lemma 4.1. *Let P be a projection and $0 < T \in \mathcal{L}^{1,\infty}$. For any $\phi \in BL[0, \infty)$,*

$$\phi\left(\frac{1}{r} \text{Tr}(PT^{1+\frac{1}{r}}P)\right) = \phi\left(\frac{1}{r} \text{Tr}((PTP)^{1+\frac{1}{r}})\right).$$

If either function is convergent at infinity, ϕ can be replaced by $\lim$.

Proof. By ([1], Prop. 3.6), $\phi\left(\frac{1}{r} \text{Tr}(PT^{1+1/r})\right) = \phi\left(\frac{1}{r} \text{Tr}((\sqrt{P}T\sqrt{P})^{1+1/r})\right)$ for $\phi \in BL[0, \infty)$, and, if either function is convergent at infinity, ϕ can be replaced by $\lim$. Clearly $\frac{1}{r} \text{Tr}(PT^{1+1/r}P) = \frac{1}{r} \text{Tr}(PT^{1+1/r})$ and $\frac{1}{r} \text{Tr}((\sqrt{P}T\sqrt{P})^{1+1/r}) = \frac{1}{r} \text{Tr}((PTP)^{1+1/r})$. $\square$

4.1.1. *Proof of Theorem 3.1.* From ([2], Thm. 4.11) $\text{tr}_{L(\phi)}(V) = \phi\left(\frac{1}{r} \text{Tr}(V^{1+1/r})\right)$ where $0 < V = PTP \in \mathcal{L}^{1,\infty}$. An application of Lemma 4.1 yields the first display of Theorem 3.1. Note that V is measurable iff V is Tauberian by ([13], Cor. 3.9). From formula (1.1), V is Tauberian iff the residue of ζ_V exists at $s = 1$. From Lemma 4.1 the residue exists iff $\lim_{r \rightarrow \infty} \frac{1}{r} \text{Tr}(PT^{1+1/r}P)$ exists. $\square$

4.1.2. *Proof of Corollary 3.3.* Let $A \in B(H)$ and let $\mathcal{M}(A)$ denote the von Neumann algebra generated by A and A^* . Note that A is the uniform limit of finite linear spans of projections in $\mathcal{M}(A)$ ([14], p. 23). Note also that $\text{tr}_{L(\phi)}(\cdot T)$ and $\phi\left(\frac{1}{r} \text{Tr}(\cdot T^{1+1/r})\right)$ are positive linear functionals on $B(H)$ and so uniformly continuous ([14], p. 42). Hence there is a finite set of scalars $c_{j,N} \in \mathbb{C}$ and projections $P_{j,N} \in \mathcal{M}(A)$, $N \in \mathbb{N}$, such that

$$\begin{aligned} \text{tr}_{L(\phi)}(AT) &= \lim_{N \rightarrow \infty} \sum_j c_{j,N} \text{tr}_{L(\phi)}(P_{j,N}TP_{j,N}) \\ &\stackrel{\text{Thm. (3.1)}}{=} \lim_{N \rightarrow \infty} \sum_j c_{j,N} \phi\left(\frac{1}{r} \text{Tr}(P_{j,N}T^{1+\frac{1}{r}}P_{j,N})\right) \\ &= \phi\left(\frac{1}{r} \text{Tr}(AT^{1+\frac{1}{r}})\right). \end{aligned}$$

If T is spectrally measurable w.r.t. A , notice that $|\lim_{r \rightarrow \infty} \frac{1}{r} \text{Tr}(AT^{1+1/r})| = \lim_{r \rightarrow \infty} \frac{1}{r} |\text{Tr}(APT^{1+1/r}P)| \leq \|A\| \lim_{r \rightarrow \infty} \frac{1}{r} \text{Tr}(PT^{1+1/r}P)$. Here P is the maximal projection in $\mathcal{M}(A)$; see ([11], p. 309). Hence $\lim_{r \rightarrow \infty} \frac{1}{r} \text{Tr}(\cdot T^{1+1/r})$ is uniformly continuous on $\mathcal{M}(A)$. For each $v \in DL_2[1, \infty)$,

$$\begin{aligned} \text{tr}_v(AT) &= \lim_{N \rightarrow \infty} \sum_j c_{j,N} \text{tr}_v(P_{j,N}TP_{j,N}) \\ &\stackrel{\text{Thm. (3.1)}}{=} \lim_{N \rightarrow \infty} c_{j,N} \sum_j \lim_{r \rightarrow \infty} \frac{1}{r} \text{Tr}(P_{j,N}T^{1+\frac{1}{r}}P_{j,N}) \\ &= \lim_{r \rightarrow \infty} \frac{1}{r} \text{Tr}(AT^{1+\frac{1}{r}}). \end{aligned}$$

The value $\text{tr}_v(AT)$ is independent of v , so AT is measurable.

4.2. Proof of Theorem 3.4 and Corollary 3.5. Let $f \in L^\infty([0, \infty))$ be an everywhere defined function of the form $f(t) = \frac{g(t)}{t}$ where g is increasing. For $b > a > 0$ we note the trivial fact that

$$\sup_{t \in [a, b]} f(t) - \inf_{t \in [a, b]} f(t) \leq \frac{g(b)}{a} - \frac{g(a)}{b}.$$

Throughout this section ξ is a state on ℓ^∞ . For brevity $\xi(a_k)$ denotes $\xi(\{a_k\}_{k=1}^\infty)$. Also $\xi(h(ak - b))$, for $h \in L^\infty([0, \infty))$, $a, b > 0$, denotes $\xi(\{h(ak - b)\}_{k=\lceil b/a \rceil}^\infty)$.

Lemma 4.2. *Let ξ be T_1 -invariant. Then ξ is T_j -invariant for all $j \in \mathbb{N}$ and $\xi(f(k+a)) = \xi(f(k))$, $a > 0$.*

Proof. That T_1 -invariance implies T_j -invariance, $j \in \mathbb{N}$, is evident from induction. For a not a natural number choose $j \in \mathbb{N}$ such that $j - 1 < a < j$. Then,

$$\begin{aligned} \xi(|f(k+a) - f(k+j)|) &\leq \xi\left(\sup_{t \in [k+j-1, k+j]} f(t) - \inf_{t \in [k+j-1, k+j]} f(t)\right) \\ &\leq \xi\left(\frac{g(k+j)}{k+j-1}\right) - \xi\left(\frac{g(k+j-1)}{k+j}\right) \\ &\leq \|f\|_\infty \xi\left(\frac{2k}{(k-1)(k+1)}\right) = 0 \end{aligned}$$

since ξ vanishes on c_0 . Hence $\xi(f(k+a)) = \xi(f(k+j)) = \xi(f(k))$. $\square$

Lemma 4.3. *Let ξ be T_1 -invariant. For any $a > 0$, $b \geq 0$,*

$$\xi\left(\sup_{t \in [ak-b, ak+b]} f(t) - \inf_{t \in [ak-b, ak+b]} f(t)\right) = 0.$$

Proof. Let $M_k := \sup_{t \in [ak-b, ak+b]} f(t) - \inf_{t \in [ak-b, ak+b]} f(t)$ for $k \geq \lceil b/a \rceil$. Set $f_1(t) := g(at - b)/(at + b) = (g(at - b)(at + b)^{-1}t)/t$, $t \geq b/a$; 0 otherwise. Then,

$$\begin{aligned} \xi(M_k) &\leq \xi\left(\frac{g(ak+b)}{ak-b}\right) - \xi(f_1(k)) \\ &\stackrel{(*)}{=} \xi\left(\frac{g(ak+b)}{ak-b}\right) - \xi(f_1(k + 2b/a)) \\ &= \xi\left(\frac{g(ak+b)}{ak+b} \frac{4b(ak+b)}{(ak-b)(ak+3b)}\right) \\ &\leq \|f\|_\infty \xi\left(\frac{4b(ak+b)}{(ak-b)(ak+3b)}\right) = 0. \end{aligned}$$

At (*) Lemma 4.2 was applied to f_1 . $\square$

Lemma 4.4. *Let ξ be T_1 -invariant and D_j -invariant, $j \in \mathbb{N}$. For any $a > 0$, $\xi(f(\frac{k}{a})) = \xi(f(k))$.*

Proof. Note that $\xi(|f(\frac{k}{j}) - f(\lceil \frac{k}{j} \rceil)|) \leq \xi(\sup_{t \in [\frac{k}{j}-1, \frac{k}{j}+1]} f(t) - \inf_{t \in [\frac{k}{j}-1, \frac{k}{j}+1]} f(t)) = 0$ for any $j \in \mathbb{N}$ by the previous lemma. Hence $\xi(f(\frac{k}{j})) = \xi(f(\lceil \frac{k}{j} \rceil)) = \xi(f(k))$ (*). Take $p \in \mathbb{N}$. By applying the previous lemma and (*) to the function $f_1(t) = (p^{-1}g(pt))/t = f(pt)$, it follows that $\xi(f(\frac{p}{j}k)) = \xi(f(\frac{p}{j}k)) = \xi(f(pk)) = \xi(f(\frac{p}{j}k)) = \xi(f(k))$ (†). Choose integers p, j such that $\frac{p}{j+1} \leq \frac{1}{a} \leq \frac{p}{j}$. Now $|f(\frac{k}{a}) - f(\frac{p}{j}k)| \leq \sup_{t \in [\frac{p}{j+1}k, \frac{p}{j}k]} f(t) - \inf_{t \in [\frac{p}{j+1}k, \frac{p}{j}k]} f(t)$. Hence

$$\begin{aligned} \xi(|f(\frac{k}{a}) - f(\frac{p}{j}k)|) &\leq \frac{j+1}{j} \xi(f(\frac{p}{j}k)) - \frac{j}{j+1} \xi(f(\frac{p}{j+1}k)) \\ &\stackrel{\text{by } (\dagger)}{=} (\frac{j+1}{j} - \frac{j}{j+1}) \xi(f(k)) \\ &\leq \|f\|_{\infty} \frac{2j+1}{j(j+1)}. \end{aligned}$$

Without loss, by adjusting p proportionately, j can be chosen arbitrarily large. Hence $\xi(f(\frac{k}{a})) = \lim_{j \rightarrow \infty} \xi(f(\frac{p}{j}k)) = \xi(f(k))$ by (†). $\square$

Define the averaging sequence $E : L^{\infty}([0, \infty)) \rightarrow \ell^{\infty}$ by $E_k(f) := \int_{k-1}^k f(t) dt$. For $a > 0$, $b \geq 0$, we abuse notation and write $E_{ak+b}(f) := \int_{ak+b-1}^{ak+b} f(t) dt$.

Lemma 4.5. *Let ξ be T_1 -invariant. For $a > 0$, $b \geq 0$, $\xi(E_{ak+b}(f)) = \xi(f(ak+b))$.*

Proof. Let $c = b+1$. Then $\inf_{t \in [ak-c, ak+c]} f(t) \leq E_{ak+b}(f) \leq \sup_{t \in [ak-c, ak+c]} f(t)$. Hence $|f(ak+b) - E_{ak+b}(f)| \leq \sup_{t \in [ak-c, ak+c]} f(t) - \inf_{t \in [ak-c, ak+c]} f(t)$. The result follows by Lemma 4.3. $\square$

Lemma 4.6. *Let ξ be T_1 -invariant and D_j -invariant, $j \in \mathbb{N}$. For any $a > 0$,*

$$\xi(f(k)) = \xi(E_k(T_a(f))) = \xi(E_k(D_a(f))).$$

Proof. That $\xi(E_k(T_a(f))) = \xi(E_{k+a}(f)) = \xi(f(k+a)) = \xi(f(k))$ is immediate with Lemma 4.5. It also follows from ([13], Lemma 2.10). Using the substitution $\frac{t}{a} \rightarrow t$,

$$\xi(E_k(D_a(f))) := \xi(\int_{k-1}^k f(\frac{t}{a}) dt) = \xi(a \int_{\frac{k}{a}-1}^{\frac{k}{a}} f(t) dt).$$

We have the equality $\xi(a \int_{\frac{k}{a}-1}^{\frac{k}{a}} f(t) dt) = \xi(\int_{\frac{k}{a}-1}^{\frac{k}{a}} f(t) dt)$ from Lemma 4.3 since

$$\left| \int_{\frac{k}{a}-1}^{\frac{k}{a}} f(t) dt - a \int_{\frac{k}{a}-1}^{\frac{k}{a}} f(t) dt \right| \leq 2 \left(\sup_{t \in [\frac{k}{a}-1, \frac{k}{a}]} f(t) - \inf_{t \in [\frac{k}{a}-1, \frac{k}{a}]} f(t) \right).$$

From

$$\xi(\int_{\frac{k}{a}-1}^{\frac{k}{a}} f(t) dt) = \xi(E_{\frac{k}{a}}(f)) \stackrel{\text{Lemma (4.5)}}{=} \xi(f(\frac{k}{a})) \stackrel{\text{Lemma (4.4)}}{=} \xi(f(k))$$

we obtain $\xi(E_k(D_a(f))) = \xi(f(k))$. $\square$

Let $\xi \in BL \cap DL$. It then follows, see ([13], Lemma 2.10) for example, that $\phi := \xi \circ E \in BL[0, \infty)$. With Lemma 4.6 we have, in addition, the property

- (iii) $\phi(f) = \phi(D_a(f))$, $a > 0$, where $f(t) := \frac{1}{t} \text{Tr}(QV^{1+\frac{1}{t}}Q)$, $t \geq 1$ (0 otherwise), $0 < V \in \mathcal{L}^{1,\infty}$, Q a projection.

Let $\lfloor x \rfloor$, $x > 0$, denote the floor function. Note that $L(\phi) = \phi \circ L^{-1} = \xi \circ E \circ L^{-1}$ belongs to $DL[1, \infty)$ and satisfies

$$(iii') \quad \phi \circ L^{-1}(g) = \phi \circ L^{-1}(P^a(g)), \quad a > 0, \text{ where } g(t) := \frac{1}{\ln(1+t)} \int_1^t \mu_{\lfloor s \rfloor}(QVQ)ds, \\ t \geq 1 \text{ (0 otherwise), } 0 < V \in \mathcal{L}^{1,\infty}, Q \text{ a projection.}$$

Property (iii') follows from Lemma 4.6 by noting that we have $\phi \circ L^{-1}(P^a(g)) = \xi(E_k(D_{a^{-1}}(L^{-1}(g))))$. In particular, $L^{-1}(g)$ has the form

$$g(e^t) := \frac{1}{\ln(1+e^t)} \int_1^{e^t} \mu_{\lfloor s \rfloor}(QVQ)ds, \quad t \geq 0$$

for $0 < V \in \mathcal{L}^{1,\infty}$, Q a projection. This is equivalent to using the function $g(e^t) = \frac{h(t)}{t}$, where $h(t) = \int_1^{e^t} \mu_{\lfloor s \rfloor}(QVQ)ds$ is increasing.

The conditions (iii) and (iii') are sufficient to repeat the conclusion of ([2], Thm. 4.11), as the next lemma shows.

Lemma 4.7. *Let $\phi \in BL[0, \infty)$ satisfy (iii) and (iii') above and let $0 < T \in \mathcal{L}^{1,\infty}$. Then*

$$\phi \circ L^{-1} \left(\frac{1}{\ln(1+t)} \int_1^t \mu_{\lfloor s \rfloor}(T)ds \right) = \phi \left(\frac{1}{t} \text{Tr}(T^{1+\frac{1}{t}}) \right).$$

Proof. By assumption ϕ is a state of $L^\infty([0, \infty))$ satisfying conditions (i), (ii), (iii), (iii'). From an inspection of [2] and [1], these conditions allow the same conclusion as ([2], Thm. 4.11). We explain.

The requirement for D_2 - and P^α -invariance, $\alpha > 1$, of $L(\phi)$ in the proof of ([2], Thm. 4.11) occurred in three places. Firstly the application of the weak*-Karamata theorem, then ([2], Prop. 4.3) and ([2], Cor. 4.4). Condition (iii') is exactly what is required in the last display of ([2], p. 264), which is the only place P^α -invariance is used for ([2], Prop. 4.3). Hence ([2], Prop. 4.3) is true under condition (iii'). The result ([2], Cor. 4.4) follows from ([2], Prop. 4.3). The property of D_2 -invariance is not an issue for $L(\phi)$ since it is dilation-invariant by $\phi \in BL[0, \infty)$.

What is left is weak*-Karamata, i.e. to achieve the last display on ([2], p. 271). In ([1], Thm. 2.2), take the special choice of $h_T(t) = \text{Tr}(T^{1+1/r}) = rf(r)$, where f is in (iii) ($Q = 1$, $V = T$), $0 < T \in \mathcal{L}^{1,\infty}$. Dilation invariance is used in the proof of ([1], Thm. 2.2) on the last display of ([1], p. 77). Indeed, for our special choice of h_T , using the notation of β and C from [1], $\phi(\frac{1}{r} \int_0^\infty e^{-t/(r/(n+1))} d\beta(t)) = 1/(n+1)\phi(1/(r/(n+1)) \text{Tr}(T^{1+1/(r/(n+1))})) = 1/(n+1)\phi(\frac{1}{r} \text{Tr}(T^{1+1/r})) = C/(n+1)$. The second equality is exactly (iii). So the last display of ([1], p. 77) holds. The rest of the argument of ([1], Thm. 2.2) carries through and with its result we obtain the last display on ([2], p. 271). The rest of the argument of ([2], Thm. 4.11) now carries through. $\square$

Define the floor mapping $p : \ell^\infty \rightarrow L^\infty([1, \infty))$ by the formula $p(\{a_k\}_{k=1}^\infty)(t) := \sum_{k=1}^\infty a_k \chi_{[k, k+1)}(t)$ and the restriction mapping $r : B([1, \infty)) \rightarrow \ell^\infty$ for everywhere defined bounded functions by $r(f) = \{f(k)\}_{k=1}^\infty$.

Lemma 4.8. For $\xi \in BL$ set $\phi := \xi \circ E$. Then

$$\lim_{n \rightarrow \infty} \left(\sup_{t \in [n, n+1)} g(t) - \inf_{t \in [n, n+1)} g(t) \right) = 0$$

and

$$\phi \circ L^{-1}(|g(t) - pr(g)(t)|) = 0.$$

Proof. Let $M_n = \sup_{t \in [n, n+1)} g(t) - \inf_{t \in [n, n+1)} g(t)$. Then

$$\begin{aligned} M_n &\leq \frac{1}{\ln(1+n)} \int_1^{n+1} \mu_{[s]}(QVQ) ds - \frac{1}{\ln(1+n+1)} \int_1^n \mu_{[s]}(QVQ) ds \\ &= \left(1 - \frac{\ln(1+n)}{\ln(2+n)}\right) \frac{1}{\ln(1+n)} \int_1^n \mu_{[s]}(QVQ) ds \\ &\quad + \frac{1}{\ln(1+n)} \int_n^{n+1} \mu_{[s]}(QVQ) ds \\ &\leq \left(1 - \frac{\ln(1+n)}{\ln(2+n)}\right) \|QVQ\|_{1,\infty} + \ln(1+n)^{-1} \mu_n(QVQ). \end{aligned}$$

Hence $\lim_n M_n = 0$. Now,

$$\sup_{t \in [n, n+1)} |g(t) - pr(g)(t)| \leq \sup_{t \in [n, n+1)} |g(t) - g(n)| \leq M_n.$$

Consequently $|g(t) - pr(g)(t)| \leq \sum_{n=1}^{\infty} M_n \chi_{[n, n+1)}(t)$, and we have that $\limsup_{t \rightarrow \infty} |g(t) - pr(g)(t)| \leq \limsup_{n \rightarrow \infty} M_n = 0$. It follows that $\phi \circ L^{-1}(|g(t) - pr(g)(t)|) = 0$. $\square$

Define the mapping $\mathcal{L} : (\ell^\infty)^* \rightarrow (\ell^\infty)^*$ by

$$\mathcal{L}(\xi) := \xi \circ E \circ L^{-1} \circ p.$$

4.2.1. *Proof of Theorem 3.4.*

Proposition 4.9. Let $\xi \in BL$. Then $\mathcal{L}(\xi) \in DL_2$.

Proof. It is clear that $\mathcal{L}(\xi)(1) = 1$ and $\mathcal{L}$ is positivity-preserving. Hence $\mathcal{L}(\xi)$ is a state on ℓ^∞ . If $\{b_k\}$ is a finite sequence, $E \circ L^{-1} \circ p(\{b_k\})$ is a finite sequence. Hence $\mathcal{L}(\xi)(\{b_k\}) = \xi(E \circ L^{-1} \circ p(\{b_k\})) = 0$. From Lemma 3.8, (2.1) is fulfilled. Set $\gamma := r(g) = \{\ln(1+k)^{-1} \sum_{j=1}^k \mu_j(T)\}_{k=1}^\infty$. Notice that

$$(D_2 p - p D_2)(\{a_k\}_{k=1}^\infty) = \sum_{k=1}^\infty (a_{k+1} - a_k) \chi_{[2k+1, 2k+2)}(t).$$

So

$$|(D_2 p - p D_2)(\gamma)| \leq \sum_{k=1}^\infty |\gamma_{k+1} - \gamma_k| \chi_{[2k, 2k+1)}(t) \leq \sum_{k=1}^\infty M_k \chi_{[2k, 2k+1)}(t),$$

where $M_k := \sup_{t \in [k, k+1)} g(t) - \inf_{t \in [k, k+1)} g(t)$. Hence $\phi \circ L^{-1}(|(D_2 p - p D_2)(\gamma)|) \leq \limsup_k M_k = 0$ by Lemma 4.8. Thus

$$\mathcal{L}(\xi)(D_2(\gamma)) = \phi \circ L^{-1} \circ D_2 \circ p(\gamma) = \phi \circ L^{-1} \circ p(\gamma) = \mathcal{L}(\xi)(\gamma)$$

by dilation invariance of $\phi \circ L^{-1}$. $\square$

Proof of Theorem. For $\xi \in BL \cap DL$ set $\phi := \xi \circ E$. Then ϕ satisfies the properties (i), (ii), (iii), (iii'). By Lemma 4.7 we have

$$(4.1) \quad \phi \circ L^{-1} \left(\frac{1}{\ln(1+t)} \int_1^t \mu_{[s]}(PTP) ds \right) = \phi \left(\frac{1}{t} \operatorname{Tr}((PTP)^{1+\frac{1}{t}}) \right).$$

By Lemma 4.1 the right hand side is equal to $\phi(\frac{1}{t} \operatorname{Tr}(PT^{1+1/t}P))$. By Lemma 4.5 we have $\phi(\frac{1}{t} \operatorname{Tr}(PT^{1+1/t}P)) = \xi(\frac{1}{k} \operatorname{Tr}(PT^{1+1/k}P))$ (*). As before, define $g(t) := \ln(1+t)^{-1} \int_1^t \mu_{[s]}(PTP) ds$, $t \geq 1$ (0 otherwise). By Lemma 4.8, which is similar to ([13], Prop. 2.12), $\phi \circ L^{-1}(g(t) - pr(g)(t)) = 0$. Hence $\phi \circ L^{-1}(g(t)) = \mathcal{L}(\xi)(\{g(n)\}_{n=1}^\infty) = \operatorname{Tr}_{\mathcal{L}(\xi)}(PTP)$ (**). From (4.1), (*), and (**), we have shown that $\operatorname{Tr}_{\mathcal{L}(\xi)}(PTP) = \xi(\frac{1}{k} \operatorname{Tr}(PT^{1+1/k}P))$.

Set $h(t) = \frac{1}{t} \operatorname{Tr}(PT^{1+1/t}P)$. Suppose PTP is measurable. Then $\lim_{t \rightarrow \infty} h(t)$ exists by Theorem 3.1. Hence $\lim_{k \rightarrow \infty} h(k)$ exists. Note that $\xi(\frac{1}{k} \operatorname{Tr}(PT^{1+1/k}P))$ equals this limit as ξ is a generalised limit.

Conversely, suppose that $\lim_{k \rightarrow \infty} h(k)$ exists. Note that

$$\lim_{n \rightarrow \infty} \sup_{t \in [n, n+1]} |h(t) - h(n)| \leq \lim_{n \rightarrow \infty} \left(\sup_{t \in [n, n+1]} h(t) - \inf_{t \in [n, n+1]} h(t) \right) = 0$$

by the proof of Lemma 4.3. Hence $\lim_{t \rightarrow \infty} h(t)$ exists and the limits are equal. By Theorem 3.1 PTP is measurable. $\square$

4.2.2. Proof of Corollary 3.5. The proof is identical to that of Corollary 3.3. The equality of $\lim_{k \rightarrow \infty} k^{-1} \zeta_{A,T}(1+k^{-1})$ with $\lim_{s \rightarrow 1^+} (s-1) \zeta_{A,T}(s)$ is contained in the last paragraphs.

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