ON LIOUVILLE INTEGRABILITY OF h -PROJECTIVELY EQUIVALENT KÄHLER METRICS

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ABSTRACT. Under a nondegeneracy condition we classify the compact connected Kähler manifolds admitting pairs of h -projectively equivalent metrics. Any such manifold is biholomorphically equivalent to $\mathbb{C}P^n$ and has integrable geodesic flow.

1. INTRODUCTION

In [5, 6, 8], and [9], pairs of Riemannian metrics whose Levi-Civita connections are mutually projectively equivalent were studied. It was proven, among other things, that under a nondegeneracy condition such Riemannian manifolds have integrable geodesic flows. More specifically, it was shown that these manifolds are Liouville manifolds in the sense of [1]. Also, from a single pair of projectively equivalent metrics, a hierarchy of such pairs was constructed; see e.g. [8] and the references therein. These results were generalized to metrics of arbitrary signature in [9].

The holomorphic counterpart to this problem was studied in [10]. There were considered pairs of holomorphically projectively equivalent Kähler metrics (or simply, h -projectively equivalent metrics). By definition, two Kähler metrics g and g' are h -projectively equivalent if their Levi-Civita connections ∇ and ∇' satisfy

$$(1.1) \quad \nabla'_X Y - \nabla_X Y = \phi(X)Y + \phi(Y)X - \phi(JX)JY - \phi(JY)JX,$$

where ϕ is a one-form and J stands for the complex structure on the manifold (see [4]). Equation (1.1) implies that any holomorphically planar curve with respect to one of the metrics is holomorphically planar also with respect to the other metric and vice versa; see for example [10, Introduction, Example (b)], [4, 7], for more details, as well as [11, 12] for related work. It was shown in [10] that under a nondegeneracy condition Kähler manifolds admitting pairs of h -projective equivalent metrics are Kähler-Liouville manifolds in the sense of [1]. As in the real case, there exists a hierarchy of such pairs of metrics. However, the metrics appearing in the hierarchy are no longer Kähler, but only Hermitian in general. They satisfy a modified version of (1.1):

$$(1.2) \quad \nabla'_X Y - \nabla_X Y = \phi(X)Y + \phi(Y)X - \phi(PX)QY - \phi(PY)QX,$$

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where ϕ is a one-form on M ; P is a nondegenerate, skew-symmetric with respect to both g and g' , $(1, 1)$ -type tensor field; and $Q = -P^{-1}$.

Results. The aim of this paper is twofold. First, we show that under a non-degeneracy condition Kähler manifolds admitting h -projectively equivalent metrics are Kähler-Liouville manifolds of a special type (see Theorem 2.2, Section 2). As a consequence, these manifolds have integrable geodesic flows, and, if compact and connected, are biholomorphically equivalent to the complex projective space $\mathbb{C}P^n$ (Corollary 2.3, Section 2). From this, one obtains a complete set of invariants that characterize the pairs of h -projectively equivalent Kähler metrics (see Theorem 3.3, Section 3). Second, under a non-degeneracy condition we investigate the local structure of Hermitian manifolds admitting pairs of Hermitian metrics that satisfy the more general equation (1.2). It is shown that such a Hermitian manifold M ($\dim_{\mathbb{C}} M = n$) possesses locally an n -dimensional real vector space of infinitesimal automorphisms (Theorem 4.6, Section 4).

2. h -PROJECTIVELY EQUIVALENT METRICS AND KÄHLER-LIOUVILLE MANIFOLDS

Let M be a connected complex manifold of complex dimension n , and let g and g' be two Kähler metrics on M which are mutually h -projectively equivalent. As stated in the Introduction, their Levi-Civita connections ∇ and ∇' satisfy

$$\nabla'_X Y - \nabla_X Y = \phi(X)Y + \phi(Y)X - \phi(JX)JY - \phi(JY)JX,$$

where J denotes the complex structure and ϕ is a one-form on M . Let A be the $(1, 1)$ -type tensor field defined by

$$(2.1) \quad g'(X, Y) = (\det A)^{-1/2} g(A^{-1}X, Y).$$

Then A is symmetric with respect to both g and g' , and $AJ = JA$. Denote by $\mathcal{A}$ the $\mathbb{C}$ -linear map corresponding to A when tangent spaces are regarded as complex vector spaces by putting $J = \sqrt{-1}$. Let

$$h_1(p) \geq \cdots \geq h_n(p) > 0$$

be the eigenvalues of $\mathcal{A}$ at $p \in M$. Each h_i is a continuous function on M , and it is smooth at the points $p \in M$ where $h_{i-1}(p) > h_i(p) > h_{i+1}(p)$. Note that the eigenvalues of the endomorphism A are $h_1, h_1, \dots, h_n, h_n$. Denote

$$(2.2) \quad \chi(c) := \det(\mathcal{A} - c\mathbf{1}) = \prod_{i=1}^n (h_i - c).$$

Proposition 2.1. *For any $c \in \mathbb{R}$ consider the function on the tangent bundle TM ,*

$$(2.3) \quad K_c(X) = \det(\mathcal{A} - c\mathbf{1}) g((A - c\mathbf{1})^{-1}X, X) .$$

Let K_c^ be the pullback of K_c by the Legendre transform $\sharp_g : T^*M \rightarrow TM$ corresponding to g . Then the one-parameter family of functions on T^*M ,*

$$(2.4) \quad K_c^* = \sum_{i=1}^n (-c)^{i-1} K_i,$$

*$c \in \mathbb{R}$, consists of Poisson commuting integrals of the geodesic flow of (M, g) .*¹

¹Note that the integrals $K_1, \dots, K_n$ appearing in this family are not sufficient for the integrability of the geodesic flow of g as $\dim_{\mathbb{R}} M = 2n$.

For the proof, see [10, Theorem 4, V]. Now, let us make the following assumption.

Assumption 1. *There is a point $p_0 \in M$ such that $\mathcal{A}$ has n distinct eigenvalues at p_0 , i.e., $h_1(p_0) > \cdots > h_n(p_0)$, and $dh_i \neq 0$ ($1 \leq i \leq n$) at p_0 .*

Let $\mathcal{F}$ be the n -dimensional vector space of functions on T^*M spanned by $K_1, \dots, K_n$. Note that $\frac{1}{2}K_n$ is the Hamiltonian of the geodesic flow of (M, g) .

Theorem 2.2. *$(M, g, \mathcal{F})$ is a Kähler-Liouville manifold of type (A) whose associated partially ordered set consists of one element. In particular, there exists an n -dimensional vector space $\mathfrak{h}$ of infinitesimal automorphisms of the Kähler manifold (M, g) such that*

$$(2.5) \quad [Y_1, Y_2] = 0, \quad \{F, Y\} = 0 \quad \text{for any } Y_1, Y_2, Y \in \mathfrak{h}, F \in \mathcal{F}.$$

Thus the geodesic flow of (M, g) is integrable with first integrals in $\mathcal{F}$ and $\mathfrak{h}$.

Proof. By definition, $(M, g, \mathcal{F})$ is a Kähler-Liouville manifold if the following conditions are satisfied (see [1, Part 2, Introduction]).

- (1) The Hamiltonian E of the geodesic flow belongs to $\mathcal{F}$.
- (2) $\{F, H\} = 0$ for any $F, H \in \mathcal{F}$.
- (3) $F_p = F|_{T_p^*M}$ is a Hermitian form for any $p \in M$ and $F \in \mathcal{F}$.
- (4) F_p ($F \in \mathcal{F}$) are simultaneously normalizable for every $p \in M$.
- (5) $\mathcal{F}_p = \{F_p \mid F \in \mathcal{F}\}$ is n -dimensional at some $p \in M$.

Clearly, $(M, g, \mathcal{F})$ is a Kähler-Liouville manifold. Let M^0 be the set of all points $p \in M$ where $\mathcal{A}$ has n distinct eigenvalues $h_1 > \cdots > h_n$, and let $M^1 \subseteq M^0$ consists of all points in M^0 , where $dh_i \neq 0$ for any i .

Take $p_0 \in M^1$ and an orthonormal frame $V_1, \dots, V_n, JV_1, \dots, JV_n$ of TM on a neighborhood of p_0 in M^1 such that $AV_i = h_i V_i$, $AJV_i = h_i JV_i$. The existence of such a frame follows from $AJ = JA$. Then

$$K_c^* = \sum_{i=1}^n \left(\prod_{j \neq i} (h_j - c) \right) (V_i^2 + (JV_i)^2),$$

where V_i and JV_i are regarded as functions on the cotangent bundle.² As (2.4) holds also when c is a function on M we see that

$$(2.6) \quad \sum_{j=1}^n (-h_i)^{j-1} K_j = \left(\prod_{j \neq i} (h_j - h_i) \right) (V_i^2 + (JV_i)^2).$$

Using (2.6) and the Poisson commutativity of $K_1, \dots, K_n$, we obtain by a direct computation that

$$(2.7) \quad \left\{ \left(\prod_{k \neq i} (h_k - h_i) \right) (V_i^2 + (JV_i)^2), \left(\prod_{l \neq j} (h_l - h_j) \right) (V_j^2 + (JV_j)^2) \right\} = 0, \\ dh_j(V_i) = dh_j(JV_i) = 0 \quad (j \neq i).$$

²Note that under such an identification, $\{X, Y\} = [X, Y]$ and $\{X, f\} = df(X)$ for any two vector fields X, Y and $f \in C^\infty(M)$.

Therefore, the function a_i in [1, Proposition 1.1, p. 84] can be taken as

$$(2.8) \quad a_i = \left((-1)^{n-i} \prod_{k \neq i} (h_k - h_i) \right)^{-1} \quad (1 \leq i \leq n).$$

Since $dh_i \neq 0$ ($1 \leq i \leq n$), it follows from (2.7) and (2.8) that for any $1 \leq i, j \leq n$, $da_i|_{\text{span}\{V_j, JV_j\}} \neq 0$. Hence, $(M, g, \mathcal{F})$ is of type (A) and the associated partially ordered set (see [1, Proposition 1.9]) consists of one element. See [1, pp. 84–88] for the definition of Kähler-Liouville manifolds of type (A) and the partial order.

The existence of the space $\mathfrak{h}$ of infinitesimal automorphisms now follows from the general theory of Kähler-Liouville manifolds of type (A); see [1, Proposition 1.16, Theorem 3.1].³ $\square$

Now suppose that M is compact and connected. Let G be the transformation group of M generated by $\mathfrak{g} = \mathfrak{h} + J\mathfrak{h}$. The group G is isomorphic to $(\mathbb{C}^\times)^n$, and with this action M becomes a toric variety (see [1, Section 4]). Since the associated partially ordered set consists of one element, we have the following (cf. [1, Section 7]).

Corollary 2.3. *As a toric variety, M is isomorphic to the complex projective space $\mathbb{C}P^n$.*

Remark. As stated in [3], some parts of [1] are incorrect unless an additional condition called properness is assumed. Note however that in the case where the associated partially ordered set consists of one element, the properness condition is automatically satisfied (see [3] for more details).

3. FUNDAMENTAL INVARIANTS AND CLASSIFICATION

Let (M, g, g') be a compact connected complex manifold supplied with a pair of h -projectively equivalent Kähler metrics g and g' that satisfy Assumption 1. As shown in the previous section, $(M, g, \mathcal{F})$ is a Kähler-Liouville manifold whose partially ordered set consists of one element, and therefore M is isomorphic to $\mathbb{C}P^n$ as a toric variety. In this case, $(M, g, \mathcal{F})$ is completely characterized by some invariants, several constants and a function on a circle (see [1, Section 7]). Here we assign to any triple (M, g, g') a set of invariants, called *fundamental invariants*, which is an enlargement of the set of invariants stated above, and we show that those invariants characterize the triple (M, g, g') uniquely up to isomorphisms. Let (M, g, g') and (N, s, s') be two such triples.

Definition 3.1. The triples (M, g, g') and (N, s, s') are called *isomorphic* if there is a biholomorphic diffeomorphism $\Phi : M \rightarrow N$ such that $\Phi^*s = g$ and $\Phi^*s' = g'$.

Let $\chi(c) = \det(\mathcal{A} - c\mathbf{1})$ (cf. (2.2)) be associated to the triple (M, g, g') and let $h_1, \dots, h_n$ be as above. In view of (2.8), the functions h_i here are the same as those appearing in [1, Proposition 1.12] and [1, Theorem 3.1], and where they were called *fundamental functions*.⁴ Therefore, by [1, Theorem 4.10] we have

Lemma 3.1. $\min h_k = \max h_{k+1}$ for $k = 1, \dots, n-1$.

³A local construction of these infinitesimal automorphisms in the more general case of Hermitian metrics is given in Section 4 of the present paper.

⁴Note however that the h_k 's appearing in [1, Proposition 1.12] are defined uniquely up to an affine transform $h_k \mapsto \alpha h_k + \beta$, where $\alpha, \beta \in \mathbb{R}$ are independent of k .

Define the positive numbers

$$c_0 := \max_M h_1, \quad c_k := \min_M h_k = \max_M h_{k+1} \quad (1 \leq k \leq n-1), \quad c_n := \min_M h_n.$$

It is clear that $c_0 > c_1 > \dots > c_n > 0$.

Next, consider the level sets $L_k \stackrel{\text{def}}{=} \{\chi(c_k) = 0\}$ ($k = 0, \dots, n$). By the results proved in [1] and summarized in [1, Section 7] we have

- Lemma 3.2.** (a) *The L_k are totally geodesic, complex hypersurfaces.*
 (b) *By an identification of M as a toric variety to the complex projective space $\mathbb{C}P^n$ with homogeneous coordinates $[z_0, \dots, z_n]$, each L_k identifies with the hypersurface $(\cong \mathbb{C}P^{n-1})$ given by $z_k = 0$.*
 (c) *The $Y_k = J(\text{grad } \chi(c_k))$ ($0 \leq k \leq n$) span $\mathfrak{h}$.*
 (d) *The Y_n generate an effective circle action on $S = \bigcap_{k=1}^{n-1} L_k \simeq \mathbb{C}P^1$ with two fixed points $q_1 = L_0 \cap S$ and $q_2 = L_n \cap S$.*

Denote by $\gamma(t)$ ($\|\dot{\gamma}(0)\|_g = 1$) a minimal length geodesic connecting q_1 with q_2 . Let $l > 0$ be the length of γ . (Notice that the points $t = 0$ and $t = l$ are conjugate.) Consider the function

$$(3.1) \quad h(t) \stackrel{\text{def}}{=} \chi(c_n)(\gamma(t))$$

defined on the interval $[0, l]$.

Definition 3.2. The numbers $c_0 > c_1 > \dots > c_n > 0$, the length $l > 0$, and the C^∞ function h on $[0, l]$ are called *fundamental invariants* of the triple (M, g, g') .

The main result of this section is

Theorem 3.3. *The triples (M, g, g') and (N, s, s') are isomorphic if and only if their fundamental invariants coincide.*

Proof. It follows from the definition of the fundamental invariants that isomorphic triples have the same fundamental invariants.

Let us prove the inverse statement. Suppose that (M, g, g') and (N, s, s') have the same fundamental invariants, namely $(\{c_0, \dots, c_n\}, l, h(t))$. Consider the Kähler-Liouville manifolds $(M, g, \mathcal{F})$ and $(N, s, \mathcal{H})$, where

$$\mathcal{F} := \text{span}\{K_c(g, g') | c \in \mathbb{R}\} \quad \text{and} \quad \mathcal{H} := \text{span}\{K_c(s, s') | c \in \mathbb{R}\}.$$

We have proved that $(M, g, \mathcal{F})$ and $(N, s, \mathcal{H})$ are Kähler-Liouville manifolds of type (A) whose associated partially ordered sets consist of one element.

Lemma 3.4. *The Kähler-Liouville manifolds $(M, g, \mathcal{F})$ and $(N, s, \mathcal{H})$ are mutually isomorphic.*

Proof. By [1], Section 7, they are mutually isomorphic if and only if the corresponding elements $(\{\tilde{c}_k\}, \tilde{d}, \tilde{l}, \tilde{h}) \in \mathcal{C}_n$ are mutually equivalent (see [1, p. 134] for the set $\mathcal{C}_n$ and the equivalence). For $(M, g, \mathcal{F})$ one has $\tilde{l} = 2l$, $\tilde{c}_k = (c_k - c_n)/(c_0 - c_n)$,

$$\begin{aligned} \tilde{h}(t) &= [(\tilde{h}_1(\gamma(t)) - \tilde{c}_n) \dots (\tilde{h}_n(\gamma(t)) - \tilde{c}_n)] / [(\tilde{c}_0 - \tilde{c}_n) \dots (\tilde{c}_{n-1} - \tilde{c}_n)] \\ &= h(t) / [(c_0 - c_n) \dots (c_{n-1} - c_n)], \quad t \in [0, l], \end{aligned}$$

and $\tilde{h}(t) = \tilde{h}(-t)$, $t \in [-l, 0]$, where $\tilde{h}_k := (h_k - c_n)/(c_0 - c_n)$. By Proposition 7.1 (5) in [1], $\tilde{d}$ is determined uniquely from $\tilde{l}$ and $\tilde{h}$. Hence, $(\{\tilde{c}_k\}, \tilde{d}, \tilde{l}, \tilde{h})$ are determined by the data $(\{c_k\}, l, h)$, and therefore coincide with those for $(N, s, \mathcal{H})$. $\square$

By Lemma 3.4 there exists an isomorphism $\Phi : (M, g, \mathcal{F}) \rightarrow (N, s, \mathcal{H})$ of Kähler-Liouville manifolds. In particular, $\Phi^*s = g$. For the sake of notation, we denote the symbols for (N, s, s') by attaching the hat $\hat{\cdot}$ to the corresponding symbols for (M, g, g') , e.g., $\hat{h}_k, \hat{\tilde{h}}_k, \hat{\tilde{L}}_k$, etc. Using that the invariants $(\{\tilde{c}_k\}, \tilde{d}, \tilde{l}, \tilde{h})$ and $(\{\hat{c}_k\}, \hat{d}, \hat{l}, \hat{h})$ are not only mutually equivalent as elements of $\mathcal{C}_n$ but even coincide, i.e. $\tilde{h}(t) = \hat{h}(t)$, $\tilde{l} = \hat{l}$, and $\tilde{d} = \hat{d}$, one sees from the construction of Φ in [1, Theorem 7.2] that $\Phi^*(\hat{h}_k) = \tilde{h}_k$. As $\tilde{h}_k = (h_k - c_n)/(c_0 - c_n)$ and $\hat{h}_k := (\hat{h}_k - c_n)/(c_0 - c_n)$ we obtain that

$$(3.2) \quad \Phi^*\hat{h}_k = h_k \quad (1 \leq k \leq n).$$

Recall that M^1 is the subset of M where the h_k are all distinct and $dh_k \neq 0$. M^1 is the same subset of M as the one considered in [1, p. 85]. In particular, it is open and dense in M by [1, Theorem 3.1] and, by (3.2), $\Phi(M^1) = N^1$. For each $p \in M^1$, let $D_k(p)$ be the eigenspace of the endomorphism A on T_pM corresponding to the eigenvalue $h_k(p)$. Note that the $D_k(p)$'s are mutually orthogonal J -invariant subspaces of T_pM of real dimension two. Using (2.7) one sees that $(\text{grad } h_k)_p \in D_k(p)$ and, by the J -invariance of $D_k(p)$, $(J \text{ grad } h_k)_p \in D_k(p)$ as well. Recall that Φ preserves J and $\Phi^*s = g$. This together with (3.2) and $(\text{grad } h_k)_p, (J \text{ grad } h_k)_p \in D_k(p)$ shows that Φ maps $D_k(p)$ onto $\hat{D}_k(\Phi(p))$ for any $p \in M^1$. Hence, Φ maps the tensor field A to the corresponding tensor field $\hat{A}$. Finally, in view of (2.1), $\Phi^*s' = g'$. This completes the proof of Theorem 3.3. $\square$

4. LOCAL STRUCTURE IN THE HERMITIAN CASE

Let M be an n -dimensional complex manifold, and g and g' be two Hermitian metrics on it. Suppose that the corresponding Levi-Civita connections ∇ and ∇' satisfy relation (1.2) in the Introduction; that is,

$$\nabla'_X Y - \nabla_X Y = \phi(X)Y + \phi(Y)X - \phi(PX)QY - \phi(PY)QX,$$

where ϕ is a one-form on M , P is a $(1,1)$ -type tensor field on M which is non-degenerate and skew-symmetric with respect to both metrics g and g' , and $Q = -P^{-1}$. Let J be the complex structure of M , and let A be as in the previous section, i.e., the $(1,1)$ -type tensor field defined by the formula (2.1). As P , Q , and J , are skew-symmetric with respect to both g and g' we see from (2.1) that

$$(4.1) \quad PA = AP, \quad QA = AQ, \quad JA = AJ.$$

As in the previous section, we make the following assumption.

Assumption 2. *There is a point $p_0 \in M$ such that A has n distinct eigenvalues at p_0 , $h_1(p_0) > \dots > h_n(p_0) > 0$, and $dh_i \neq 0$ ($1 \leq i \leq n$) at p_0 .*

Let U be a neighborhood of p_0 such that Assumption 2 holds for any $p \in U$. Choose an orthonormal frame $V_1, \dots, V_n, JV_1, \dots, JV_n$ on U so that D_i , the subbundle of TU spanned by V_i and JV_i , is an eigenspace of A . Then we have

$$TU = D_1 \oplus \dots \oplus D_n \quad \text{and} \quad A|_{D_i} = h_i \mathbf{1}_{D_i}$$

where $\mathbf{1}_{D_i}$ denotes the identity map on D_i . As Q commutes with A ,

$$(4.2) \quad Q|_{D_i} = b_i J|_{D_i},$$

where b_i are some nonzero functions. In particular, Q and P commute with the complex structure J .

Proposition 4.1. $dh_i|_{D_j} = 0$ if $i \neq j$.

Proof. By [10, Lemma 2, IIB],

$$(4.3) \quad \begin{aligned} -g((\nabla_X A)Y, Z) &= \phi(AY)g(X, Z) + \phi(AZ)g(X, Y) \\ &\quad - \phi(APY)g(QX, Z) - \phi(APZ)g(QX, Y). \end{aligned}$$

Using (4.2) one obtains from (4.3) that $g((\nabla_{V_j} A)V_i, V_i) = 0$ for $i \neq j$. On the other hand, by differentiating the equality $AV_i = h_i V_i$ covariantly in the direction of V_j , and by using that

$$(4.4) \quad 2g(\nabla_{V_j} V_i, V_i) = g(\nabla_{V_j} V_i, V_i) + g(V_i, \nabla_{V_j} V_i) = \nabla_{V_j}[g(V_i, V_i)] = 0,$$

one gets that $g((\nabla_{V_j} A)V_i, V_i) = V_j h_i$. Hence, $dh_i(V_j) = 0$. In a similar way one gets $dh_i(JV_j) = 0$. $\square$

Since $dh_i \neq 0$ is assumed, we have $dh_i|_{D_i} \neq 0$. Therefore there is a particular choice of V_i so that

$$(4.5) \quad dh_i(JV_i) = 0 \quad (1 \leq i \leq n).$$

From now on we assume that the V_i 's are chosen in this way. Put

$$D^+ := \text{span}\{V_1, \dots, V_n\} \quad \text{and} \quad D^- := \text{span}\{JV_1, \dots, JV_n\}.$$

Note that $d_p h_1, \dots, d_p h_n$ are linearly independent for any $p \in U$, and by Proposition 4.1 and (4.5), D^- is completely integrable.

Lemma 4.2. For $i \neq j$,

$$(4.6) \quad \begin{aligned} \nabla_{V_j} V_i &= \frac{1}{2(h_i - h_j)}(V_i h_i) V_j, \quad \nabla_{V_j}(JV_i) = \frac{1}{2(h_i - h_j)} \frac{b_j}{b_i} (V_i h_i) JV_j, \\ \nabla_{JV_j} V_i &= \frac{1}{2(h_i - h_j)} \left((V_i h_i) JV_j - \frac{b_i}{b_j} (V_j h_j) JV_i \right), \end{aligned}$$

$$(4.7) \quad \nabla_{JV_j}(JV_i) = \frac{1}{2(h_i - h_j)} \left(-\frac{b_j}{b_i} (V_i h_i) V_j + \frac{b_i}{b_j} (V_j h_j) V_i \right).$$

Proof. Using (4.5) one obtains from (4.3) that

$$(4.8) \quad (\nabla_{V_j} A)V_i = -\phi(AV_i)V_j + \phi(APV_i)b_j JV_j.$$

By formula (6) in [10, IIB], $\phi(X) = -\frac{1}{4}d(\log \det A)(X)$. This together with (4.8) shows that $(\nabla_{V_j} A)V_i = \frac{1}{2}(V_i h_i)V_j$. On the other side, a covariant differentiation of the equality $(A - h_i)V_i \equiv 0$ in the direction of V_i leads to $(\nabla_{V_j} A)V_i = -(A - h_i)\nabla_{V_j} V_i$. Hence, $-(A - h_i)\nabla_{V_j} V_i = \frac{1}{2}(V_i h_i)V_j$, and therefore

$$\nabla_{V_j} V_i \equiv \frac{1}{2(h_i - h_j)}(V_i h_i)V_j \quad \text{mod } D_i.$$

In a similar way one obtains the other formulas of the lemma modulo D_i . To prove the exact equalities, we note that $\nabla_{JV_j}(JV_i)$ is orthogonal to JV_i (cf. (4.4)). Using this and the fact that (4.7) holds modulo D_i we see that $\nabla_{JV_j}(JV_i)$ is a linear combination of V_j and V_i , and so is $[JV_j, JV_i] = \nabla_{JV_j}(JV_i) - \nabla_{JV_i}(JV_j)$. As D^- is completely integrable, the Lie bracket $[JV_j, JV_i]$ is a section of D^- . Thus we have $[JV_j, JV_i] = 0$, $\nabla_{JV_j}(JV_i) = \nabla_{JV_i}(JV_j)$, and

$$(4.9) \quad \nabla_{JV_j}(JV_i) = \frac{1}{2(h_i - h_j)} \left(-\frac{b_j}{b_i} (V_i h_i) V_j + \frac{b_i}{b_j} (V_j h_j) V_i \right).$$

As $\nabla_{JV_j} V_i$ is orthogonal to V_i and as $g(\nabla_{JV_j} V_i, JV_i) = -g(V_i, \nabla_{JV_j} JV_i)$ we see that (4.6) holds. In particular, $\nabla_{JV_i} V_j$ is a linear combination of JV_j and JV_i . Differentiating h_i in the direction of

$$(4.10) \quad [JV_i, V_j] = \nabla_{JV_i} V_j - \nabla_{V_j} (JV_i),$$

we have $0 = (JV_i)V_j h_i - V_j(JV_i)h_i = -(\nabla_{V_j} (JV_i))h_i$. This implies that the D_i -component of the derivative $\nabla_{V_j} (JV_i)$ vanishes. In particular, one has $g(\nabla_{V_j} V_i, JV_i) = -g(V_i, \nabla_{V_j} JV_i) = 0$. Hence, the D_i -component of $\nabla_{V_j} V_i$ vanishes as well. $\square$

Put

$$(4.11) \quad W_i := \sqrt{(-1)^{n-i} \prod_{k \neq i} (h_k - h_i)} V_i \quad (1 \leq i \leq n).$$

Proposition 4.3. (1) For $i \neq j$,

$$[W_i, W_j] = [JW_i, JW_j] = [W_i, JW_j] = 0,$$

(2)

$$\begin{aligned} [W_i, JW_i] &= \sum_{k \neq i} \frac{\left| \prod_{l \neq i} (h_l - h_i) \right|}{\left| \prod_{l \neq k} (h_l - h_k) \right|} \frac{1}{h_i - h_k} \frac{b_i}{b_k} (W_k h_k) JW_k \\ &\quad - \left(\sum_{l \neq i} \frac{W_l h_l}{h_l - h_i} + W_i \log \left| \frac{b_j}{b_i} \right| + \frac{W_i^2 h_i}{W_i h_i} \right) JW_i, \end{aligned}$$

where $j(\neq i)$ is arbitrary.

Proof. Item (1) is an immediate consequence of Lemma 4.2 and $\nabla_X Y - \nabla_Y X = [X, Y]$. Using Lemma 4.2 and $g([V_i, JV_i], Z) = g(\nabla_{V_i} JV_i, Z) - g(\nabla_{JV_i} V_i, Z) = g(V_i, \nabla_{JV_i} Z) - g(JV_i, \nabla_{V_i} Z)$ for any Z from the orthonormal frame, we get

$$(4.12) \quad [V_i, JV_i] \equiv \sum_{k \neq i} \frac{1}{h_i - h_k} \frac{b_i}{b_k} (V_k h_k) JV_k \quad \text{mod } D_i.$$

Hence, by (4.11), $[W_j, JW_j] = c_j W_j + \sum_{k=1}^n e_{jk} JW_k$, where $e_{jk} \neq 0$ for $j \neq k$. Using item (1) and the Jacobi identity for the Lie bracket of vector fields one obtains for $i \neq j$,

$$\begin{aligned} 0 &= [W_i, [W_j, JW_j]] = (W_i c_j) W_j + \sum_k (W_i e_{jk}) JW_k + e_{ji} [W_i, JW_i] \\ &= (W_i c_j) W_j + e_{ji} c_i W_i + \sum_k (W_i e_{jk} + e_{ji} e_{ik}) JW_k. \end{aligned}$$

By comparing both sides of the last equality we see that $c_l = 0$ and $W_i e_{jk} = -e_{ji} e_{ik}$ for $i \neq j$. In particular, $e_{ii} = -W_i \log |e_{ji}|$, $j \neq i$. This together with (4.11) and (4.12) implies (2). $\square$

Corollary 4.4. *The subbundle D^+ is completely integrable, and its integral manifolds are totally geodesic.*

Proof. Since by Lemma 4.2, $\nabla_{V_j} V_i \in D^+$ ($j \neq i$), it is enough to show that $\nabla_{V_i} V_i \in D^+$. One has $g(\nabla_{V_i} V_i, JV_j) = -g(V_i, \nabla_{V_i} (JV_j))$. If $i \neq j$, the right-hand side vanishes by Lemma 4.2. If $i = j$, $g(V_i, \nabla_{V_i} (JV_i)) = g(V_i, [V_i, JV_i])$, which vanishes by Proposition 4.3 (2). $\square$

Let $W_1^*, \dots, W_n^*, (JW_1)^*, \dots, (JW_n)^*$ be the dual frame to $W_1, \dots, W_n, JW_1, \dots, JW_n$. Then it follows from Proposition 4.3 (1) that the one-forms W_i^* are closed. Therefore there is a system of functions $(w_1, \dots, w_n)$ such that $W_i^* = dw_i$ on a neighborhood of p_0 . The function h_i is thus a function of w_i alone. More generally, a function u satisfying $du|_{D^-} = 0$ is a function of $(w_1, \dots, w_n)$. For simplicity we will write $\partial_i u$ instead of $W_i u$.

Proposition 4.5. (1) $d(b_i/b_j)|_{D^-} = 0$ for any i, j .

(2) There are nonzero functions $\hat{b}_i(w_i)$ ($1 \leq i \leq n$) such that

$$b_1/\hat{b}_1 = \dots = b_n/\hat{b}_n.$$

Proof. By Proposition 4.3 (2), $[W_i, JW_i] = \sum_k e_{ik} JW_k$. Proposition 4.3 (1) and Jacobi's identity imply that

$$(JW_j)e_{ik} = 0 \quad (j \neq i).$$

This together with the explicit formulas for the e_{ik} 's from Proposition 4.3 (2) show that $(JW_j)(b_i/b_k) = 0$ for $j \neq i$. Taking $k = j$ in the last formula and then replacing i by k and j by i , one sees that $(JW_i)(b_i/b_k) = 0$ as well. Therefore,

$$d(b_i/b_k)|_{D^-} = 0.$$

To prove (2), recall that $\partial_j e_{ik} = -e_{ij}e_{jk}$ ($j \neq i$), and hence

$$(4.13) \quad -\partial_j \log |e_{ik}| = e_{ij}e_{jk}/e_{ik}, \quad j \neq i, \quad i \neq k.$$

Using (4.13) and the explicit formulas for the e_{ik} 's we see that, if $n \geq 3$ and $j \neq i, k$,

$$\partial_j \log |b_i/b_k| = 0 \quad \text{and} \quad \partial_j \log |b_i/b_j| = \partial_j \log |b_k/b_j|.$$

The former formula shows that b_i/b_k depends only on (w_i, w_k) . Hence by the latter formula,

$$(4.14) \quad \partial_i \partial_j \log |b_i/b_j| = 0.$$

If $n = 2$, by comparing both sides of $\partial_j e_{ii} = -e_{ij}e_{ji}$, we conclude that (4.14) still holds. Hence, there are nonzero functions $\hat{b}_j(w_j)$ ($1 \leq j \leq n$) such that

$$\partial_j \log |b_i/b_j| = -\partial_j \log |\hat{b}_j|$$

for any $i \neq j$. Then

$$(4.15) \quad \mu_{ij} = \log |b_i/b_j| - \log |\hat{b}_i/\hat{b}_j|$$

are constants that satisfy $\mu_{ji} = -\mu_{ij}$ and $\mu_{ij} + \mu_{jk} = \mu_{ik}$. Putting $\nu_i = (1/n) \sum_k \mu_{ik}$ we have $\mu_{ij} = \nu_i - \nu_j$. Hence, redefining $\hat{b}_i$ as $\exp(\nu_i) \hat{b}_i$, we get from (4.15) that $\mu_{ij} = 0$. Finally, choosing appropriately the signs of the $\hat{b}_i$'s we obtain (2). $\square$

Consider the vector fields $Y_1, \dots, Y_n$ defined by

$$(4.16) \quad \sum_{k=1}^n (-h_i)^{k-1} Y_k = (-1)^{n-i} \frac{W_i h_i}{\hat{b}_i} JW_i.$$

Theorem 4.6. (1) Each Y_k is an infinitesimal automorphism of the Hermitian manifold M , i.e., $\mathcal{L}_{Y_k} g = 0$, $\mathcal{L}_{Y_k} J = 0$.

(2) $[Y_i, Y_j] = 0$ for any i, j .

(3) $\{K_i, Y_j\} = 0$ for any i, j .

Proof. We write $\sum_k c_{ik} Y_k = JW_i$. Since by (4.16), $dc_{ik} = 0$ on D^- , it follows that

$$0 = [JW_i, JW_j] = \sum_{k,l} c_{ik} c_{jl} [Y_k, Y_l].$$

Hence (2) holds. Using (2), $Y_j c_{ik} = 0$, and $JW_k = \sum_j c_{kj} Y_j$, we see that $[Y_i, JW_k] = 0$ for any i, k . Next, we show that $[Y_i, W_k] = 0$ for any i, k . Since $W_j c_{ik} = 0$ for $j \neq i$, we obtain as above

$$(4.17) \quad \sum_k c_{ik} [W_j, Y_k] = [W_j, JW_i] = 0 \quad (j \neq i).$$

We have also

$$(4.18) \quad \sum_k (W_i c_{ik}) Y_k + \sum_k c_{ik} [W_i, Y_k] = [W_i, JW_i] = \sum_j e_{ij} JW_j = \sum_{j,k} e_{ij} c_{jk} Y_k,$$

where e_{ij} are as in the proof of Proposition 4.5. On the other side, a direct computation involving Proposition 4.5(2) and the explicit expressions for the e_{ij} 's and the c_{ik} 's shows that

$$(4.19) \quad W_i c_{ik} = \sum_j e_{ij} c_{jk}.$$

Therefore, from (4.18), $\sum_k c_{ik} [W_i, Y_k] = 0$. This combined with (4.17) implies that $[W_i, Y_k] = 0$.

Hence, we have shown that $[Y_k, W_i] = [Y_k, JW_i] = 0$ for any i, k . This implies that $\mathcal{L}_{Y_k} J = 0$ and by formula (2.6), $\sum_{j=1}^n (-h_i)^{j-1} \{Y_k, K_j\} = 0$ for any i, k . Thus $\mathcal{L}_{Y_k} K_j \equiv \{Y_k, K_j\} = 0$.⁵ Since $(1/2)K_n$ is the Hamiltonian of the geodesic flow, we also have $\mathcal{L}_{Y_k} g = 0$. $\square$

Next we show that the local structure of M is completely characterized by Theorem 4.6.

Trivialization of the local structure: Let U be an open set of $\mathbb{C}^n$ with coordinates $\{(z_1, \dots, z_n)\}$. Writing $z_i = x_i + \sqrt{-1}y_i$, we put

$$X_i = \frac{\partial}{\partial x_i}, \quad Y_i = \frac{\partial}{\partial y_i}.$$

Then $JX_i = Y_i$, where J denotes the standard complex structure of $\mathbb{C}^n$. Suppose that U is of the form

$$U = \{(z_1, \dots, z_n) \mid (x_1, \dots, x_n) \in U_0, (y_1, \dots, y_n) \in V_0\} \simeq U_0 \times V_0,$$

where U_0 and V_0 are open neighborhoods of 0 in $\mathbb{R}^n$. Let U_1 be an open disk in $\mathbb{R}^n = \{(w_1, \dots, w_n)\}$, and let $h_i(w_i)$ and $\hat{b}_i(w_i)$ ($1 \leq i \leq n$) be functions of a single variable such that

$$h_1 > \dots > h_n > 0, \quad dh_i \neq 0, \quad \hat{b}_i \neq 0$$

on U_1 . Define the vector fields $\tilde{X}_i$ ($1 \leq i \leq n$) by

$$\sum_{k=1}^n (-h_i)^{k-1} \tilde{X}_k = (-1)^{n-i} \frac{1}{\hat{b}_i} \frac{\partial h_i}{\partial w_i} \frac{\partial}{\partial w_i}.$$

Note that $[\tilde{X}_i, \tilde{X}_j] = 0$ for any i, j . Therefore, taking a reference point $p_0 \in U_1$, we get a unique diffeomorphism $\Phi : U_1 \rightarrow U_0$ so that $\Phi(p_0) = 0$ and $\Phi_*(\tilde{X}_i) = X_i$

⁵We identify the symmetric $(m, 0)$ -type tensors with the corresponding functions on T^*M .

for any i . Put $\Phi_*(\partial/\partial w_i) = W_i$, and regard h_i and $\hat{b}_i$ as functions on U_0 via the diffeomorphism Φ . We define the functions K_i on the cotangent bundle T^*U by the formula

$$\sum_{j=1}^n (-h_i)^{j-1} K_j = (-1)^{n-i} (W_i^2 + (JW_i)^2) .$$

It follows from our construction that the vector fields Y_k satisfy (4.16) and commute with W_i and JW_i . Hence Y_k commutes with K_j . It is easily seen that the Lie brackets $[W_i, JW_j]$ vanish for any $i \neq j$ (cf. (4.17)). Hence the K_i are mutually Poisson commuting. Let g be the Hermitian metric on U so that $(1/2)K_n$ is the Hamiltonian of its geodesic flow. Then it follows that the Y_k are infinitesimal holomorphic isometries of the Hermitian manifold (U, g) . Define the $(1, 1)$ -type tensor fields A and Q by

$$AJ = JA, \quad QJ = JQ, \quad AW_i = h_i W_i, \quad QW_i = \hat{b}_i JW_i,$$

and put $P = -Q^{-1}$. Define the Hermitian metric g' by the relation (2.1). Then one verifies that the corresponding covariant derivatives ∇ and ∇' satisfy (1.2).

The range of the hierarchy: We refer to [10, IID] for the definition and the main properties of the hierarchy of pairs of metrics corresponding to a given pair of PQ^ϵ -projectively equivalent metrics g and g' . Starting with a pair of h -projectively equivalent Kähler metrics⁶ one gets a hierarchy of pairs of Hermitian metrics satisfying equation (1.2) with $P = J \cdot \alpha(A)$ and $Q = \alpha(A)^{-1} \cdot J$, where $\alpha : (a, b) \rightarrow \mathbb{R}$, $(a, b) \subseteq \mathbb{R}$, is a real-analytic function such that $\alpha(A)$ is well-defined and non-degenerate on M .

Following the trivialization construction discussed above we choose $h_i(w_i)$ and $\hat{b}_i(w_i)$ ($1 \leq i \leq n$) so that $h_1 > \dots > h_n > 0$, $dh_i \neq 0$, and $\hat{b}_i \neq 0$. Assume for simplicity that $U_1 = \{(w_1, \dots, w_n) \mid w_i \in (-\delta, \delta), 1 \leq i \leq n\}$, $\delta > 0$, and that the h_i 's have positive first derivatives on $(-\delta, \delta)$. Consider the functions $\alpha_j : (-\delta, \delta) \rightarrow \mathbb{R}$,

$$(4.20) \quad \alpha_i := \hat{b}_i \circ h_i^{-1}, \quad 1 \leq i \leq n,$$

and assume that there exist $1 \leq k, l \leq n$ so that $\alpha_k \not\equiv \alpha_l$. Note that the latter assumption implies that the pairs of metrics obtained from h_i and $\hat{b}_i$ via the trivialization construction do not belong to the hierarchy obtained from a pair of h -projectively equivalent Kähler metrics. One sees from (4.20) that these pairs of metrics can be parametrized by $2n$ functional parameters.

Remark. As a matter of fact, arguing similarly, one can construct global examples of such pairs of metrics on $\mathbb{C}P^n$ by modifying the examples considered in [2], [10, VI]. In particular, there are many examples of pairs of Hermitian metrics (g, g') on $\mathbb{C}P^n$ satisfying (1.2) beyond the hierarchies.

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⁶Then $P = Q = J$ and $\epsilon = -1$.

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