CENTRAL VALUES OF L -FUNCTIONS AND HALF-INTEGRAL WEIGHT FORMS

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ABSTRACT. We prove a relation between the Fourier coefficients of certain Hilbert modular forms of half-integral weight and central values of the corresponding Rankin L -functions. The result generalizes the classical theorem by Waldspurger. The approach is geometric and generalizes that of Gross and Hatcher.

1. INTRODUCTION

Let F be a totally real field of degree d . For the purpose of this paper the class number $h(F)$ of F is always assumed to be odd. Let f be a holomorphic Hilbert newform over F of weight $2\mathbf{k} = (2k_1, 2k_2, \dots, 2k_d)$ ($k_i \geq 1$) and level $N = \mathfrak{p}^e$ for a prime ideal $\mathfrak{p}$ of F and $e \geq 1$. Assume further that if d is odd, then so is e . Let B be the quaternion algebra over F such that:

- (1) If d is even, then B is only ramified at all infinite places.
- (2) If d and e are both odd, then B is only ramified at infinite places and $\mathfrak{p}$.

For $D \gg 0$ in F , let χ_D be the quadratic character of the idèle class group $\mathbb{A}_F^\times / F^\times$ associated to the quadratic extension $K_D = F(\sqrt{-D})$. Suppose that the conductor of χ_D is relatively prime to $\mathfrak{p}$ and that K_D can be embedded in B , that is, $\chi_D(\mathfrak{p}) = -1$. Write $L(s, f_D) = L(s, f)L(s, f \otimes \chi_D)$ for the (complete) L -function of f over K_D . It is well known that the sign of the functional equation of $L(s, f_D)$ is given by $(-1)^d \chi_D(\mathfrak{p}^e)$, which is equal to 1 by the above assumptions.

Let us give a brief account of the paper. In Section 2, among other things, we introduce a Gross curve X and a vector bundle V associated to the quaternion algebra B . In Section 3 we construct a Hilbert modular form g of half-integral weight with coefficients in $\text{Pic}(V)$. Let g_f be the f -isotypical component of g , and denote the D -th Fourier coefficient of g_f by m_D . Then Section 4, Theorem 2 gives the main result of this paper, which is that under certain conditions on the odd fundamental discriminants $-D$ the central value of $L(s, f_D)$ is given by

$$(1.1) \quad L(1/2, f_D) = C_D \frac{(f, f)}{\langle \nu_f, \nu_f \rangle} |m_D|^2,$$

where C_D is an explicit positive constant and $\langle \nu_f, \nu_f \rangle$ is some height pairing; see Section 4 for the precise statement.

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Formula (1.1) can be viewed as a geometric generalization of the well-known result due to Waldspurger [12] (for $F = \mathbb{Q}$). A more general but probably less explicit generalization can be found in Shimura [10]. When $F = \mathbb{Q}$, similar results have been obtained by Gross [2] (weight 2) and Hatcher [6] (higher weight). Our approach and results are extensions of theirs.

Now we make a few comments on the assumptions stated above. The oddness of the class number $h(F)$ of F is used to guarantee the passage from the original central value formula, which is stated on $B^\times/F^\times$ in [13], to the current setting on $B^\times$. This assumption may be dropped if one can prove a central value formula directly on $B^\times$. The assumption on the level of f is used to eliminate the dependence of the central value formula on the choice of the $\text{Pic}(\mathcal{O}_{K_D})$ orbit of special points. This assumption can be lifted if either a central value formula involving all special points is obtained, or if the Fourier coefficients of automorphic forms of more general type, such as Jacobi forms or vector-valued forms, are considered (see [4] for a similar situation). The assumption on the parity of d and e is to ensure that f comes from a form on $B^\times$ under the Jacquet-Langlands correspondence. It might be more natural to start with a form on $B^\times$ directly.

2. GROSS CURVES AND VECTOR BUNDLES

Let B be the quaternion algebra over F defined as in Section 1 (1) or (2). Let R be a fixed order of discriminant $\mathfrak{p}^e$ over $\mathcal{O}_F$ in B , whose construction can be found for example in [3, Section 3] or [13, Section 3.2]. Notice that R is not unique even up to local conjugation. Let $\widehat{F} = F \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$, $\widehat{B} = B \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ and $\widehat{R} = R \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$.

2.1. Gross curve X . Let Y be the genus zero curve over F associated to B . The points of Y over any F -algebra E are given by $Y(E) = \{\alpha \in B \otimes_F E \mid \alpha \neq 0, \text{Tr}(\alpha) = \mathbf{N}(\alpha) = 0\}/E^\times$.

The Gross curve X is defined by the double coset

$$(2.1) \quad X = B^\times \backslash Y \times \widehat{B}^\times / \widehat{R}^\times.$$

Let $\widehat{B}^\times = \bigcup_{i=1}^n B^\times g_i \widehat{R}^\times$ be the double coset decomposition. Then there is an isomorphism

$$(2.2) \quad X \cong \prod_{i=1}^n \Gamma_i \backslash Y = \prod_{i=1}^n X_i,$$

where each $\Gamma_i = (B^\times \cap g_i \widehat{R}^\times g_i^{-1}) / \mathcal{O}_F^\times$ is a finite group with order denoted by w_i .

Suppose $D \gg 0$ is such that $K_D = F(\sqrt{-D})$ can be embedded into B . Then, there is a canonical identification $Y(K_D) \cong \text{Hom}_F(K_D, B)$ as follows: for any $f \in \text{Hom}_F(K_D, B)$, let y be the image of the unique K_D -line on the quadric $\{\alpha \in B \otimes_F K_D \mid \alpha \neq 0, \text{Tr}(\alpha) = \mathbf{N}(\alpha) = 0\}$ on which conjugation by $f(K_D^\times)$ acts by multiplication by the character $k \mapsto k/\bar{k}$. Notice that $f(K_D^\times)$ has two fixed points on $Y(K_D)$, one of which is y and the other one is $\bar{y}$, the image of y under the complex conjugate of $\text{Gal}(K_D/F)$. Special points on X over K_D are the images of $Y(K_D) \times \widehat{B}^\times / \widehat{R}^\times$ in $X(K_D)$.

Now suppose further that $-D \in F$ is a fundamental discriminant (Definition 3.2) and $\mathfrak{p} \nmid D$. A special point $x = (y, g) \in X(K_D)$ is said to have discriminant $-D$ if $f(K_D) \cap g \widehat{R} g^{-1} = f(\mathcal{O}_{K_D})$, where $f : K \rightarrow B$ is the embedding corresponding to y . If the component g of x is congruent to g_i in $B^\times \backslash \widehat{B}^\times / \widehat{R}^\times$, then the special

point x lies on the component X_i . Let $h_i(D)$ be the number of embeddings of $\mathcal{O}_{K_D}$ into R_i , modulo conjugation by $R_i^\times$. Then there are exactly $h_i(D)$ special points of discriminant $-D$ on the component X_i . There is a free action of the group $\text{Pic}(\mathcal{O}_{K_D}) \cong K_D^\times \backslash \widehat{K}_D^\times / \widehat{\mathcal{O}}_{K_D}^\times$ on the set of special points of discriminant $-D$ as follows. Let $x = (y, g)$ be a special point of discriminant $-D$ and let $a \in \widehat{K}_D^\times$. Let $\widehat{f} : \widehat{K}_D^\times \rightarrow \widehat{B}^\times$ be the homomorphism induced by the embedding $f : K_D \rightarrow B$ that corresponds to y . The action of a on x is then given by

$$(2.3) \quad x_a = (y, \widehat{f}(a)g).$$

This action is well-defined and defines a free action of $\text{Pic}(\mathcal{O}_{K_D})$ on the set of special points of discriminant $-D$; see [2, p. 133]. The argument of [2, p. 133] shows that the orbit set classifies embeddings of $\mathcal{O}_{K_D, \mathfrak{p}}$ into $R_{\mathfrak{p}}$ modulo conjugation by $R_{\mathfrak{p}}^\times$. The latter set is known [7, Theorem 5.12] or [1] to have cardinality 2. On the other hand, the complex conjugate of $\text{Gal}(K_D/F)$ on $Y(K_D)$ induces an action on special points of discriminant $-D$. Thus the product group $\text{Gal}(K_D/F) \times \text{Pic}(\mathcal{O}_{K_D})$ acts simply transitively on the set of all special points of discriminant $-D$ on X by counting the number of points.

The group $\text{Pic}(X)$ of line bundles (or linearly equivalent divisor classes) on X is isomorphic to $\mathbb{Z}^n$, and is generated by the classes e_i of degree 1 on each component X_i . We define two divisor classes associated to a fundamental discriminant $-D$ (and $\mathfrak{p} \nmid D$). Let $u(D) = |\mathcal{O}_{K_D}^\times / \mathcal{O}_F^\times|$. The first one is given by

$$(2.4) \quad c_D = \frac{1}{2u(D)} \sum_{\text{disc}(x)=-D} (x),$$

where the sum is over all special points of discriminant $-D$. The second one is given by the $\text{Pic}(\mathcal{O}_{K_D})$ orbit

$$(2.5) \quad y = \frac{1}{u(D)} \sum_{a \in \text{Pic}(\mathcal{O}_{K_D})} (x_a) \in \text{Pic}(X),$$

where $x \in X$ is any fixed special point of discriminant $-D$. Since $\bar{x}$ and x lie on the same component of X , we obtain the following equality as divisor classes:

$$(2.6) \quad c_D = y = \frac{1}{2u(D)} \sum_{i=1}^n h_i(D) e_i \in \text{Pic}(X).$$

If $a = \sum_{i=1}^n a_i e_i$ and $b = \sum_{i=1}^n b_i e_i$ are two divisor classes in $\text{Pic}(X)$, the height pairing between them is given by

$$(2.7) \quad \langle a, b \rangle = \sum_{i=1}^n w_i a_i b_i.$$

2.2. Vector bundle V over X . In the following, let $v = 1, \dots, d$ be the subscripts corresponding to infinite places of F . Each local component $B_v = B \otimes_F F_v$ with the standard Hamiltonian quaternion algebra over $\mathbb{R}$.

Let $W = \mathbb{C}x \oplus \mathbb{C}y$ be the standard two-dimensional representation of $\text{SU}(2)$ with an inner product given by $[x, x] = [y, y] = 1$ and $[x, y] = 0$. For each infinite place $v = 1, \dots, d$, the space $\text{Sym}^{2k_v-2}(W)$ is also an inner product space with basis $\{x^{2k_v-2}, x^{2k_v-1}y, \dots, y^{2k_v-2}\}$. The induced inner product on $\text{Sym}^{2k_v-2}(W)$ is such that $[x^i y^j, x^i y^j] = i!j!$; see [5, Section 3] for more details. Notice that $\text{Sym}^{2k_v-2}(W)$

is also a representation of $\mathrm{SO}(3)$, thus can be regarded as a representation of $B_v^\times$. Let U_v be the vector space of trace-free elements of B_v . Then $B_v^\times$ has a natural representation on U_v by conjugation. An inner product on U_v is given by $[u_1, u_2] = \frac{1}{2}\mathrm{Tr}(u_1 \bar{u}_2)$. The space $\mathrm{Sym}^{k_v-1}(U_v)$, as an inner product space, has an orthogonal decomposition [5, Section 3]

$$(2.8) \quad \mathrm{Sym}^{k_v-1}(U_v) = \mathrm{Sym}^{2k_v-2}(W) \oplus M_v.$$

Let $W_{2\mathbf{k}-1} = \prod_v \mathrm{Sym}^{2k_v-2}(W)$. The space $W_{2\mathbf{k}-1}$ has an action of $B^\times$ through the diagonal embedding $B^\times \rightarrow (B \otimes_{\mathbb{Q}} \mathbb{R})^\times = \prod_v B_v^\times$. The vector bundle V is then defined by

$$(2.9) \quad V = B^\times \backslash Y \times W_{2\mathbf{k}-1} \times \widehat{B}^\times / \widehat{R}^\times.$$

If we write out the double coset quotient, then

$$(2.10) \quad V \cong \prod_{i=1}^n \Gamma_i \backslash (Y \times W_{2\mathbf{k}-1}) = \prod_{i=1}^n V_i.$$

For each i , let $W_{2\mathbf{k}-1}^{\Gamma_i}$ be the elements invariant under the action of Γ_i . Then $\mathrm{Pic}(V)$ is defined by

$$(2.11) \quad \mathrm{Pic}(V) = \bigoplus_{i=1}^n \mathbb{C}e_i \otimes W_{2\mathbf{k}-1}^{\Gamma_i}.$$

Notice that $\mathrm{Pic}(V)$ is already a vector space over $\mathbb{C}$. Letting $\nu = (y, g_i, w) \in V$, define the class of ν to be

$$(2.12) \quad (\nu) = \frac{1}{|\Gamma_i|} \sum_{\gamma \in \Gamma_i} e_i \otimes \gamma(w) \in \mathrm{Pic}(V).$$

The height pairing on $V \times V$ is defined as follows. Let $\nu_1 = (y_1, g_i, w_1)$ and $\nu_2 = (y_2, g_j, w_2)$. Then

$$(2.13) \quad \langle \nu_1, \nu_2 \rangle = \begin{cases} 0, & i \neq j, \\ \sum_{\gamma \in \Gamma_i} [w_1, \gamma(w_2)]_{W_{2\mathbf{k}-1}}, & i = j. \end{cases}$$

Formula (2.13) induces a height pairing on $\mathrm{Pic}(V) \times \mathrm{Pic}(V)$, which coincides with (2.7) if $k_v = 2$ for all v .

Let $D \gg 0$ be such that there is an embedding $f : K_D \rightarrow B$. At each infinite place v , let $u_v^D = \sqrt{-D_v} \in U_v$, and let w_v^D be the component of $(u_v^D)^{k_v-1}$ in $\mathrm{Sym}^{2k_v-2}(W)$ through (2.8). Write $w_0^D = \prod_v w_v^D \in W_{2\mathbf{k}-1}$. A straight computation [5, p. 556] gives

$$(2.14) \quad [w_0^D, w_0^D] = \prod_v \frac{2^{2k_v-2} D_v^{k_v-1} (k_v-1)!^3}{(2k_v-2)!}.$$

Suppose further that $-D$ is a fundamental discriminant with $\mathfrak{p} \nmid D$. A special point on V of discriminant $-D$ is a point of the form $\nu = (y, g, w_0^D)$, where $x = (y, g)$ is a special point on X of discriminant $-D$. The action of $\mathrm{Pic}(\mathcal{O}_{K_D})$ on special points of X extends naturally to special points of V through $\nu_a = (x_a, w_0^D)$; see (2.3). Similar to (2.6) define the special cycle of V by

$$(2.15) \quad \nu_D = \frac{1}{u(D)} \sum_{a \in \mathrm{Pic}(\mathcal{O}_{K_D})} (\nu_a) = \frac{1}{2u(D)} \sum_{i=1}^n h_i(D) (e_i \otimes w_0^D) \in \mathrm{Pic}(V).$$

2.3. Curve X' and vector bundle V' . To utilize the central value formula of [13] we need to define a new curve X' and a vector bundle V' on it. The construction is parallel to that in Section 2.2. After the construction the relationship between them is studied.

The curve X' is defined by

$$(2.16) \quad X' = B^\times \backslash Y \times \widehat{B}^\times / \widehat{F}^\times \widehat{R}^\times.$$

If we write $\widehat{B}^\times = \bigcup_{j=1}^m B^\times g'_j \widehat{F}^\times \widehat{R}^\times$ for the double coset decomposition, then

$$(2.17) \quad X' \cong \coprod_{j=1}^m \Gamma'_j \backslash Y,$$

where $\Gamma'_j \cong (B^\times \cap g'_j \widehat{F}^\times \widehat{R}^\times g_j'^{-1}) / F^\times$.

Thus the group of line bundles $\text{Pic}(X')$ of X' is isomorphic to $\mathbb{Z}^m$. If $a' = \sum_{j=1}^m a_j e'_j$ and $b' = \sum_{j=1}^m b_j e'_j$, the height pairing between them is given by

$$(2.18) \quad \langle a', b' \rangle' = \sum_{j=1}^m w'_j a_j b_j.$$

Assumption. From now on we will assume that the class number $h(F)$ of F is odd.

Lemma 2.1. *The (multiplication) action of $\text{Pic}(\mathcal{O}_F) = F^\times \backslash \widehat{F}^\times / \widehat{\mathcal{O}}_F^\times$ on $S = B^\times \backslash \widehat{B}^\times / \widehat{R}^\times$ is free. The quotient space is given by*

$$B^\times \backslash \widehat{B}^\times / \widehat{F}^\times \widehat{R}^\times.$$

Proof. The second claim is obvious, and we only need to prove the first one. Let $t \in \widehat{F}^\times$ be a stabilizer of $g \in \widehat{B}^\times$ such that $gt = agr$ with $a \in B^\times$ and $r \in \widehat{R}^\times$. Then $\mathbf{N}(g)\mathbf{N}(t) = \mathbf{N}(a)\mathbf{N}(g)\mathbf{N}(r)$, which implies that $t^2 = \mathbf{N}(t) = \mathbf{N}(a)\mathbf{N}(r) \in F^\times \widehat{\mathcal{O}}_F^\times$. Since $h(F)$ is assumed to be odd, we conclude that t itself has to define a trivial class in $\text{Pic}(\mathcal{O}_F)$. $\square$

By Lemma 2.1 we can and will take $g'_j = g_j$ for $j = 1, \dots, m$. Also, $n = mh(F)$.

Remark 2.1. The same argument shows that the natural homomorphism $\text{Pic}(\mathcal{O}_F) \rightarrow \text{Pic}(\mathcal{O}_K)$ is injective for any quadratic extension K/F .

Lemma 2.2. *For each $j = 1, \dots, m$, there is a natural isomorphism from Γ_j to Γ'_j . More precisely, the natural homomorphism*

$$(2.19) \quad \left(B^\times \cap g_j \widehat{R}^\times g_j^{-1} \right) / \mathcal{O}_F^\times \cong \left(B^\times \cap g_j \widehat{F}^\times \widehat{R}^\times g_j^{-1} \right) / F^\times$$

is an isomorphism. Thus $w'_j = w_j$ for $j = 1, \dots, m$.

Proof. The homomorphism in (2.19) is induced by the identity map. To see that it is surjective, suppose $b \in B^\times \cap g_j \widehat{F}^\times \widehat{R}^\times g_j^{-1}$. Then we need to find $b_1 \in B^\times \cap g_j \widehat{R}^\times g_j^{-1}$ such that $b_1 = bf$ for some $f \in F^\times$. Since $h(F)$ is odd, and taking the norms of $b = g_j t r g_j^{-1} \in B^\times$ we get $\mathbf{N}(b) = t^2 \mathbf{N}(r) \in F^\times$, which implies that $t \in F^\times \widehat{\mathcal{O}}_F^\times$, so $b_1 = bf$ for some $f \in F$. For injectivity, suppose $b_1, b_2 \in B^\times \cap g_j \widehat{R}^\times g_j^{-1}$ such that $b_1 = fb_2$. Since $b_1 b_2^{-1} \in g_j \widehat{R}^\times g_j^{-1}$, we get that $f \in F^\times \cap g_j \widehat{R}^\times g_j^{-1} = \mathcal{O}^\times$. $\square$

Corollary 2.3. *The natural projection*

$$\pi : X = B^\times \backslash Y \times \widehat{B}^\times / \widehat{R}^\times \rightarrow X' = B^\times \backslash Y \times \widehat{B}^\times / \widehat{F}^\times \widehat{R}^\times$$

is an étale covering of equal degree $h(F)$.

Corollary 2.4. *Let $c'_D = \pi(c_D) \in \text{Pic}(X')$. Then*

$$(2.20) \quad \pi^*(c'_D) = c_D.$$

Thus $\langle c_D, c_D \rangle = \deg(\pi) \langle c'_D, c'_D \rangle' = h(F) \langle c'_D, c'_D \rangle'$ by the projection formula.

Notice that the centers of $(B \otimes \mathbb{R})^\times$ and $B_v^\times$ act trivially on $W_{2\mathbf{k}-1}$ and U_v (Section 2.2) respectively. Thus the vector bundle V descends to a vector bundle on X' and is denoted by V' . The group $\text{Pic}(V')$ can also be defined similarly. $\text{Pic}(V)$ is equipped with a height pairing, denoted again by $\langle \cdot, \cdot \rangle'$. The projection of a special point ν of fundamental discriminant $-D$ on V defines a special point ν' on V' . Hence the special cycle ν_D (2.15) descends to a special cycle on V :

$$(2.21) \quad \nu'_D = \sum_{a \in \text{Pic}(\mathcal{O}_{K_D}) / \text{Pic}(\mathcal{O}_F)} (\nu'_a) = \frac{1}{2u(D)} \sum_{j=1}^m h_j(D) (e'_j \otimes w_0^D) \in \text{Pic}(V').$$

Similar to Corollary 2.4, we have the following.

Proposition 2.5. *Let ν_D and ν'_D be defined in (2.15) and (2.21) respectively. Then*

$$(2.22) \quad \langle \nu_D, \nu_D \rangle = h(F) \langle \nu'_D, \nu'_D \rangle'.$$

3. HILBERT MODULAR FORMS OF HALF-INTEGRAL WEIGHT

Let B^0 be the subspace of trace-free elements of B , and let $U = \prod_v B^0 \otimes_v \mathbb{R}$. For each infinite place v , let $\mu_v^1, \mu_v^2, \mu_v^3$ be a fixed basis of $B^0 \otimes_v \mathbb{R}$. A homogeneous polynomial $P = \prod_v P_v$ of degree $\mathbf{k} - 1$ on U is said to be spherical harmonic if for each v ,

$$\sum_{i=1}^3 \frac{\partial^2}{\partial x_{i,v}^2} P_v = 0.$$

Lemma 3.1. *For each $\nu \in \text{Pic}(V)$, there is a spherical harmonic polynomial $P_i^\nu = \prod_v P_{i,v}^\nu$ of degree $\mathbf{k} - 1$ on U such that for every $D \gg 0$, one has*

$$\langle \nu, (e_i \otimes w_0^D) \rangle = \prod_v P_{i,v}^\nu(\sqrt{-D_v}) = \prod_v P_{i,v}^\nu(x_{1,v}, x_{2,v}, x_{3,v}),$$

where $\sqrt{-D_v} = x_{1,v}\mu_v^1 + x_{2,v}\mu_v^2 + x_{3,v}\mu_v^3$.

Proof. Here w_0^D is defined in Section 2.2. See [9, Proposition 2.10] or [5, Lemma 1] for a proof. $\square$

Let R_i , $i = 1, \dots, n$ be the order associated to $g_i \in B^\times \backslash \widehat{B}^\times / \widehat{R}^\times$, that is, $R_i = B \cap g_i \widehat{R} g_i^{-1}$. Let

$$S_i = \mathcal{O}_F + 2R_i$$

be a suborder of R_i . Write S_i^0 for the subset of trace-free elements of S_i . For each $D \gg 0$ in $\mathcal{O}_F$, define $A_i(D) = \{b \in S_i^0 \mid \mathbf{N}(b) = -b^2 = D\}$ and $a_i(D) = |A_i(D)|$.

Definition 3.2. Let $D \gg 0$ be in $\mathcal{O}_F$ and $K_D = F(\sqrt{-D})$. Then $-D$ is called a fundamental discriminant if $\mathcal{O}_{K_D}$ has relative discriminant (D) over $\mathcal{O}_F$, and

$$(3.1) \quad \mathcal{O}_K = \mathcal{O}_F + \frac{a + \sqrt{-D}}{2} \mathcal{O}_F$$

for some $a \in \mathcal{O}_F$.

Proposition 3.3. Suppose the relative discriminant of K_D over F is (D) , and suppose K_D splits over all primes of F dividing 2. Then $-D$ is a fundamental discriminant.

Proof. First, notice that $x = \frac{a + \sqrt{-D}}{2}$ for $a \in \mathcal{O}_F$ is in $\mathcal{O}_{K_D}$ if and only if $4|(a^2 + D)$.

Let $(2) = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_g^{e_g}$ be the prime decomposition of (2) in $\mathcal{O}_F$. Since each $\mathfrak{p}_i$ splits in K_D , the equation $y_i^2 = -D$ has solutions in $\mathcal{O}_{F, \mathfrak{p}_i}$, which modulo $\mathfrak{p}_i^{2e_i}$ implies that $y_i^2 \equiv -D \pmod{\mathfrak{p}_i^{2e_i}}$ has solutions for all i . Now the Chinese Remainder Theorem tells us that there is an $a \in \mathcal{O}_F$ such that $a \equiv y_i$ for all i . Such an a satisfies the required property $a^2 \equiv -D \pmod{4}$. Hence $x = \frac{a + \sqrt{-D}}{2} \in \mathcal{O}_{K_D}$.

Let $\mathcal{O} = \mathcal{O}_F + \frac{a + \sqrt{-D}}{2} \mathcal{O}_F$. As $\left(\frac{a + \sqrt{-D}}{2}\right)^2 = \frac{a^2 - D}{4} + \frac{a\sqrt{-D}}{2} = \frac{-a^2 - D}{4} + a \frac{a + \sqrt{-D}}{2} \in \mathcal{O}$, so $\mathcal{O}$ is an order in K . Moreover, the discriminant of $\mathcal{O}$ is (D) , which is the same as that of $\mathcal{O}_K$. Therefore $\mathcal{O}_K = \mathcal{O} = \mathcal{O}_F + \frac{a + \sqrt{-D}}{2} \mathcal{O}_F$ and $-D$ is a fundamental discriminant. $\square$

Proposition 3.3 implies that there are plenty of fundamental discriminants. For instance, let D be square-free such that $(D, d_F) = 1$ and $-D \equiv 1 \pmod{8}$. Then $-D$ is a fundamental discriminant.

Proposition 3.4. Suppose $-D$ is a fundamental discriminant. Then, for each $i = 1, \dots, n$,

$$(3.2) \quad \frac{a_i(D)}{w_i} = \frac{h_i(D)}{u(D)},$$

where $h_i(D)$ is the number of (optimal) embeddings of $\mathcal{O}_K$ into R_i , modulo conjugation by $R_i^\times$.

Proof. First let $f : \mathcal{O}_K \rightarrow R_i$ be an embedding. Write $\mathcal{O}_K = \mathcal{O}_F + \frac{a + \sqrt{-D}}{2} \mathcal{O}_F$. The element $b = f(\sqrt{-D}) \in R_i$ then satisfies $\text{Tr}(b) = 0$ and $\mathbf{N}(b) = -D$. Since $\frac{a+b}{2} = f\left(\frac{a + \sqrt{-D}}{2}\right) \in R_i$, we have $b \in A_i(D)$.

Conversely let $b \in A_i(D)$. Thus $a' + b \in 2R_i$ for some $a' \in \mathcal{O}_F$. In particular, $\frac{a' + b}{2}$ is integral over $\mathcal{O}_F$, which implies that $\frac{a' + \sqrt{-D}}{2} \in K_D$ is integral over K_D , that is, $\frac{a' + \sqrt{-D}}{2} \in \mathcal{O}_K$. By comparing discriminants we get

$$\mathcal{O}_K = \mathcal{O}_F + \frac{a' + \sqrt{-D}}{2} \mathcal{O}_F.$$

Thus we obtain an embedding $f : \mathcal{O}_K \rightarrow R_i$ by letting $f\left(\frac{a' + \sqrt{-D}}{2}\right) = \frac{a' + b}{2}$.

The group $\Gamma_i = R_i^\times / \mathcal{O}_F^\times$ acts on $A_i(D)$ and the set of embeddings by conjugation. Hence we have proved that

$$(3.3) \quad |A_i(D)/\Gamma_i| = h_i(D).$$

Now (3.2) follows from (3.3) because the order of the stabilizer of an element $b \in A_i(D)$ under the action of Γ_i is equal to $u(D)$. $\square$

Proposition 3.5. *The series*

$$(3.4) \quad g = \sum_{D \gg 0, \text{ or } D=0} \left(\sum_{i=1}^n \frac{a_i(D)}{2w_i} (e_i \otimes w_0^D) \right) q^D$$

defines a Hilbert modular form of weight $\mathbf{k} + \frac{1}{2}$ with coefficients in $\text{Pic}(V)$. Here $z = (z_1, z_2, \dots, z_d) \in \mathbb{H}^d$, and $q^D = \exp(2\pi i \text{Tr} D z)$. Moreover, if $-D$ with $\mathfrak{p} \nmid D$ is a fundamental discriminant, then the D -th Fourier coefficient of g equals ν_D .

Proof. By Lemma 3.1, the i -th series in (3.4) is a theta series with spherical harmonic coefficients attached to the quadratic space S_i^0 . The modularity and weight of such a series is well known when $F = \mathbb{Q}$; see for instance [8]. For a general field F , the proof can be found in [11, Section 5].

If $-D$ is a fundamental discriminant, the equality between the D -th Fourier coefficient of g and ν_D follows by comparing (2.15) and (3.2). $\square$

The term for $D = 0$ in (3.4) is nonzero only if $k_v = 2$ for all v .

4. CENTRAL VALUES AND FOURIER COEFFICIENTS OF HALF-INTEGRAL WEIGHT FORMS

In this section we first recall a formula for the central value $L(1/2, f_D)$ which is expressed in terms of the height pairing between special cycles of $\text{Pic}(V)$. Then we relate these central values to the Fourier coefficients of the Hilbert modular form of half-integral weight constructed in Proposition 3.5.

From now on let $-D$ with $\mathfrak{p} \nmid D$ be a fundamental discriminant, and assume there exists a special point of discriminant $-D$ on X .

Theorem 1. *The central value of $L(s, f_D)$ is given by*

$$(4.1) \quad L(1/2, f_D) = C_D(f, f) \langle \nu_{D,f}, \nu_{D,f} \rangle,$$

where C_D is given by

$$(4.2) \quad C_D = C_1 h(F) d_F^{3/2} d_{K_D}^{-1/2} \prod_v \frac{(k_v - 1)!}{2D_v^{1-k_v}}$$

with C_1 defined in [13, (1.4)], (f, f) is the Petersson inner product, and $\nu_{D,f}$ is the f -isotypical component of ν_D .

Proof. By [13, Theorem 1.2],

$$(4.3) \quad L(1/2, f_D) = C_1 d_F^{3/2} d_{K_D}^{-1/2} (f, f) \langle \nu'_{D,f}, \nu'_{D,f} \rangle_G,$$

where C_1 is the rational number given by [13, (1.4)] and $\langle \cdot, \cdot \rangle_G$ is the geometric pairing defined in [13, Section 3.1] through a multiplicity function $M_\infty = \prod_v M_v$, which is defined as follows. Choose a decomposition $B = K_D + K_{Dj}$, and let $\xi(a + bj) = \frac{\mathbf{N}(bj)}{\mathbf{N}(a+bj)}$. Then

$$M_v(\gamma_v) = \frac{2^{2k_v-1} (k_v - 1)!^2}{(2k_v - 2)!} P_{k_v-1}(1 - 2\xi(\gamma_v)),$$

where $P_n(t) = \sum_{j=0}^n \binom{n}{j} \binom{n+j}{n} \left(\frac{t-1}{2} \right)^j$ is the standard n -th Legendre polynomial.

By [5, (7.6)] one has

$$\begin{aligned}
 (4.4) \quad [w_0^D, \gamma(w_0^D)] &= \left(\prod_v P_{k_v-1}(1 - 2\xi(\gamma_v)) \right) [w_0^D, w_0^D] \\
 &= \prod_v P_{k_v-1}(1 - 2\xi(\gamma_v)) \cdot \prod_v \frac{2^{2k_v-2} D_v^{k_v-1} (k_v-1)!^3}{(2k_v-2)!} \\
 &= \left(\prod_v \frac{(k_v-1)!}{2D_v^{1-k_v}} \right) M_\infty(\gamma).
 \end{aligned}$$

Comparing the height pairing (2.13) and the geometric pairing in [13, Section 3.1], we obtain the following relation between them:

$$(4.5) \quad \langle \nu'_{D,f}, \nu'_{D,f} \rangle' = \left(\prod_v \frac{(k_v-1)!}{2D_v^{1-k_v}} \right) \langle \nu'_{D,f}, \nu'_{D,f} \rangle_G.$$

By Proposition 2.5,

$$(4.6) \quad \langle \nu_{D,f}, \nu_{D,f} \rangle = h(F) \left(\prod_v \frac{(k_v-1)!}{2D_v^{1-k_v}} \right) \langle \nu'_{D,f}, \nu'_{D,f} \rangle_G,$$

which combined with (4.3) completes the proof of (4.1). $\square$

Remark 4.1. (1) Strictly speaking, the special cycle in [13] is defined over a curve associated to the group $G = B^\times / F^\times$, but there is an isomorphism

$$(4.7) \quad B^\times \backslash \widehat{B}^\times / \widehat{F}^\times \widehat{R}^\times \cong G(F) \backslash G(\mathbb{A}_f) / \Delta,$$

with $\Delta = \widehat{R}^\times / \widehat{\mathcal{O}}_F^\times$. This isomorphism is induced by $\widehat{B}^\times / \widehat{F}^\times \cong G(\mathbb{A}_f)$ and $\widehat{R}^\times \cap \widehat{F}^\times = \widehat{\mathcal{O}}_F^\times$.

(2) The existence of a special point of discriminant $-D$ implies that the functional equation of $L(s, f_D)$ has sign $+1$ (due to the assumption on the parity of d and e). Hence the result of [13] can be applied.

(3) There are naturally defined Hecke operators acting on all the curves and line bundles defined in Section 2. By the Jacquet-Langlands correspondence, f determines an eigenform on the group $B^\times \backslash B^\times(\mathbb{A}_F)$, which has the same Hecke eigenvalues as f . Hence it is legitimate to speak of f -isotypical components on $\text{Pic}(V)$.

Let $\nu_f \in \text{Pic}(V) \otimes \mathbb{C}$ be a nonzero element in the f -isotypical component. By the strong multiplicity-one theorem, such a ν_f is unique up to a scalar multiple. Let

$$(4.8) \quad g(\nu_f) = \sum_D m_D q^D = \langle g, \nu_f \rangle.$$

Now we state the main result of this paper.

Theorem 2. *Let $-D$ with $\mathfrak{p} \nmid D$ be an odd fundamental discriminant such that there is a special point on X of discriminant $-D$. Then*

$$(4.9) \quad L(1/2, f_D) = C_D \frac{(f, f)}{\langle \nu_f, \nu_f \rangle} |m_D|^2,$$

where C_D is given by (4.2).

Proof. By Proposition 3.5 and (4.8),

$$(4.10) \quad m_D = \langle \nu_D, \nu_f \rangle = \langle \nu_{D,f}, \nu_f \rangle.$$

Thus

$$(4.11) \quad \nu_{D,f} = \frac{m_D}{\langle \nu_f, \nu_f \rangle} \nu_f \in \text{Pic}(V).$$

So

$$\langle \nu_{D,f}, \nu_{D,f} \rangle = \frac{|m_D|^2}{\langle \nu_f, \nu_f \rangle}.$$

Now (4.9) follows immediately from (4.1). $\square$

The form $g(\nu_f)$ is a Niwa-Shintani lifting of f . Hence (4.9) can be regarded as an explicit geometric generalization of the result of [12].

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