THE LANDAU THEOREM AND BLOCH THEOREM FOR PLANAR HARMONIC AND PLURIHARMONIC MAPPINGS

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ABSTRACT. For a normalized quasiregular pluriharmonic mapping f of the unit ball B^n of $\mathbb{C}^n$ into $\mathbb{C}^n$, we estimate the supremum of numbers R such that some subdomain Ω of the ball is mapped by f diffeomorphically onto some ball of radius R . Our estimates significantly improve earlier estimates, even in the case of harmonic functions in the disc.

1. INTRODUCTION

The classical Bloch theorem asserts the existence of a positive absolute constant b such that for any holomorphic mapping f of the unit disk D , with the normalization $f'(0) = 1$, the image $f(D)$, as a covering surface, contains a univalent disk of radius b . Bloch's constant is defined as the supremum of such constants b .

In 1990, Mario Bonk [3] improved Ahlfors' lower estimate of Bloch's constant which had prevailed for more than fifty years. It has been said that finding the precise value of the Bloch constant replaced the Bieberbach conjecture as the most important open problem in function theory, after the latter was confirmed by Louis de Branges in 1985. At the present time the best lower bound of Bloch's constant is $\frac{\sqrt{3}}{4} + 0.0002$, which is due to the authors of the present paper [5].

For planar harmonic mappings, in order to obtain a Bloch theorem, some extra restriction, other than the normalization at the origin, must be added. For K -quasiregular harmonic mappings (even in higher dimensions), Bochner [2] had already proved the existence of a Bloch constant, but gave no estimate. In [7], a concrete lower bound of the Bloch constant for planar K -quasiregular harmonic mappings was given by us together with Hengartner; moreover, it is proved there that the Bloch theorem is valid for open harmonic mappings. Related to the Bloch theorem, the Landau theorem for planar harmonic mappings was established in the same paper. In the case of several variables, we proved a Bloch theorem for quasiregular holomorphic mappings in another paper [6].

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In the present paper, we continue this investigation. First, we give significant improvements of the lower bounds of the Bloch constants for K -quasiregular harmonic mappings and for open harmonic mappings. Then, we consider pluriharmonic mappings in higher dimensions and establish the Schwarz-Pick lemma and Landau theorem for pluriharmonic mappings as well as the Bloch theorem for pluriharmonic K -mappings.

2. PLANAR HARMONIC MAPPINGS

For a continuously differentiable complex-valued function $f(z) = u(z) + iv(z)$, $z = x + iy$, we use the common notation for its formal derivatives:

$$f_z = \frac{1}{2}(f_x - if_y), \quad f_{\bar{z}} = \frac{1}{2}(f_x + if_y).$$

Then f is a harmonic mapping if and only if f is twice continuously differentiable and $\Delta f = 4f_{z\bar{z}} = 0$.

Let f be a harmonic mapping of a domain G . f is said to be univalent or locally univalent if f is one-to-one or locally one-to-one on G . It is known [12] that a harmonic mapping is locally univalent if and only if the determinant $\det J_f$ of its Jacobian matrix J_f does not vanish anywhere. Such a result does not hold in higher dimensions [17]. Locally, f can be written as $f = \bar{g} + h$, where g and h are holomorphic. Since $f_z = h'$ and $f_{\bar{z}} = \bar{g}'$, we see that f_z and $\overline{f_{\bar{z}}}$ are holomorphic and that

$$\det J_f = u_x v_y - u_y v_x = |f_z|^2 - |f_{\bar{z}}|^2 = |h'|^2 - |g'|^2.$$

A mapping of a domain $G \subset \mathbb{C}$ is said to be open if it maps any open subset of G to an open set in $\mathbb{C}$. A mapping is said to be light if no continuum (connected closed set containing more than one point) is mapped to a single point. A continuous mapping of a domain in the plane is called an interior transformation in the sense of Stoilow if f is both open and light. The famous Stoilow theorem [16] says that an interior transformation f is topologically holomorphic, that is, f can be written in the form $f = F \circ \phi$, where F is a non-constant holomorphic mapping and ϕ is a homeomorphism. An interior transformation f is said to be sense-preserving if the homeomorphism ϕ is sense-preserving. It is proved in [7] that for harmonic mappings, the openness implies the lightness; i.e., *an open harmonic mapping is an interior transformation*.

For a continuously differentiable mapping f , denote

$$\begin{aligned} \Lambda_f &= \max_{0 \leq \theta \leq 2\pi} |e^{i\theta} f_z + e^{-i\theta} f_{\bar{z}}| = |f_z| + |f_{\bar{z}}|, \\ \lambda_f &= \min_{0 \leq \theta \leq 2\pi} |e^{i\theta} f_z + e^{-i\theta} f_{\bar{z}}| = ||f_z| - |f_{\bar{z}}||. \end{aligned}$$

Λ_f and λ_f are the maximum dilation and minimum dilation of the mapping f respectively. Note that $|\det J_f| = \Lambda_f \lambda_f$. f is said to be K -quasiregular ($K \geq 1$) on a domain G if $\Lambda_f \leq K \lambda_f$ holds everywhere on G .

Theorem 1. *A non-constant K -quasiregular harmonic mapping is an interior transformation in the sense of Stoilow.*

Proof. Let f be a K -quasiregular harmonic mapping of a domain G , which is not a constant. For $z_0 \in G$, f has an expansion around z_0 :

$$f(z) = f(z_0) + \sum_{n=k}^{\infty} a_n(z - z_0)^n + b_n(\bar{z} - \bar{z}_0)^n,$$

where $k \geq 1$ and one of a_k and b_k is not zero. $|a_k|$ and $|b_k|$ cannot be equal. Otherwise, we may assume that neither f_z nor $f_{\bar{z}}$ vanishes identically, so there is a sequence $z_j \rightarrow z$ with $|f_z(z_j)/f_{\bar{z}}(z_j)| \rightarrow 1$ and $\lambda_f(z_j)/\Lambda_f(z_j) \rightarrow 0$ as $z_j \rightarrow z_0$, which contradicts the quasiregularity.

Without loss of generality, assume that $|a_k| > |b_k|$. Write $f(z) - f(z_0) = h(z) + g(z)$ with

$$h(z) = a_k(z - z_0)^k, \quad g(z) = \sum_{n=k+1}^{\infty} a_n(z - z_0)^n + \sum_{n=k}^{\infty} b_n(\bar{z} - \bar{z}_0)^n.$$

For each r sufficiently small, there is a small $\delta > 0$ so that $|h(z)| > |g(z)| + \delta$ for $z \in \gamma = \{z : |z - z_0| = r\}$. Then, for w' with $|w' - f(z_0)| < \delta$, we have $|g(z) + f(z_0) - w'| < |g(z)| + \delta < |h(z)|$ for $z \in \gamma$ and, consequently,

$$\Delta_\gamma \arg(f(z) - w') = \Delta_\gamma \arg(h(z) + g(z) + f(z_0) - w') = \Delta_\gamma \arg h(z) = 2k\pi.$$

Thus, there is at least one zero of $f(z) - w'$ in the disk bounded by γ . This shows that $f(D)$ contains the disk of center $f(z_0)$ and radius δ . Thus, f is open and, consequently, f is an interior transformation in the sense of Stoilow. The theorem is proved. $\square$

3. THE HARMONIC BLOCH CONSTANTS

For a non-constant harmonic mapping f of the unit disk D , a univalent disk contained in $f(D)$ is defined as a disk which is the homeomorphic image of some region in D . The supremum b_f of the radii of all univalent disks is called the inradius of $f(D)$ and denoted by b_f . The class of all K -quasiregular harmonic mappings of D is denoted by $\text{Har}_K(D)$, and that of all open harmonic mappings of D is denoted by $\text{Har}_O(D)$. The Bloch constants for K -quasiregular harmonic mappings and open harmonic mappings are defined by

$$\beta_{har}^K = \inf\{b_f : f \in \text{Har}_K(D), \lambda_f(0) = 1\},$$

$$B_{har}^K = \inf\{b_f : f \in \text{Har}_K(D), f_z(0) = 1, |f_{\bar{z}}(0)| \leq (K-1)/(K+1)\},$$

and

$$B_{har}^O = \inf\{b_f : f \in \text{Har}_O(D), f_z(0) = 1, f_{\bar{z}}(0) = 0\}.$$

It is obvious that $\beta_{har}^K \geq B_{har}^K$. With this notation the results in [7] are formulated by

$$\beta_{har}^K \geq \frac{\pi}{8\sqrt{2}(1+2K)}, \quad B_{har}^O \geq 0.02.$$

These results have been improved slightly by A. Grigoryan [9] and M. Liu [13]. Liu's lower bound for B_{har}^O is 0.0277.

Now, by a different approach we obtain new lower bounds for B_{har}^K and B_{har}^O . Denote by B_{hol} the classical Bloch constant for holomorphic functions.

Theorem 2. $B_{har}^K \geq \frac{2}{K+1}B_{hol}$, and $B_{har}^O \geq \frac{1}{4}B_{hol}$.

Proof. Let $f \in \text{Har}_K(D)$, $f_z(0) = 1$ and $|f_{\bar{z}}(0)| \leq (K-1)/(K+1)$. Since f_z is holomorphic, the integral

$$\zeta = F(z) = \int_0^z f_z(z) dz$$

defines a function, holomorphic on D , such that $F'(0) = 1$. Thus, for any $r < B_{hol}$, there is a disk Δ of center ζ_0 and radius r which is the biholomorphic image of a domain $G \subset D$ under the mapping F . The composition $g = f \circ F^{-1}$ is a K -quasiregular harmonic mapping. Since f is an interior transformation as indicated in Section 2, f is locally univalent outside of a discrete set $E \subset D$ of branch points. As indicated in Section 2, $\det J_f \neq 0$ on $D \setminus E$ and, consequently, $\det J_f > 0$ on $D \setminus E$ since $\det J_f(0) > 0$. Thus, $|f_z| \geq |f_{\bar{z}}|$ and

$$|f_{\bar{z}}| \leq \frac{K-1}{K+1} |f_z|$$

holds for $z \in D$.

We have, for $\zeta \in \Delta$,

$$g_\zeta = f_z/F'(z) = 1, \quad g_{\bar{\zeta}} = f_{\bar{z}}/\overline{f_z}, \quad |g_{\bar{\zeta}}| \leq \frac{K-1}{K+1}.$$

Thus, for $\zeta_1, \zeta_2 \in \Delta$,

$$|g(\zeta_1) - g(\zeta_2)| \geq \left| \int_{\zeta_1, \zeta_2} g_\zeta d\zeta \right| - \left| \int_{\zeta_1, \zeta_2} g_{\bar{\zeta}} d\bar{\zeta} \right| \geq \frac{2|\zeta_1 - \zeta_2|}{K+1}.$$

This shows that g is univalent on Δ and, consequently, f is univalent on G . Further, for $\zeta \in \Delta$,

$$|g(\zeta) - g(\zeta_0)| \geq \frac{2|\zeta - \zeta_0|}{K+1}.$$

The right side of the above estimate tends to $2r/(K+1)$ as $\zeta \rightarrow \partial\Delta$. This shows that $g(\Delta) = f(G)$ contains a disk of radius $2r/(K+1)$. Since r can be close to B_{hol} arbitrarily, this shows that $b_f \geq 2B_{hol}/(K+1)$. The first estimate is proved.

Now, let $f \in \text{Har}_O(D)$, $f_z(0) = 1$ and $f_{\bar{z}}(0) = 0$. f is also an interior transformation as proved in Section 2. As above, letting $E \subset D$ be the discrete set of branch points, we have $\det J_f > 0$ and $|f_z| > |f_{\bar{z}}|$ for $z \in D \setminus E$. Thus, the holomorphic function $h = \overline{f_{\bar{z}}}/f_z$ has its modulus less than 1 on $D \setminus E$. Note that E is removable and $h(0) = 0$. Using the Schwarz lemma, we have $|h(z)| \leq |z|$ and, consequently, $|f_{\bar{z}}(z)| \leq |z||f_z(z)|$ for $z \in D$. This shows that f is a K -quasiregular harmonic mapping, with $K = (1+\rho)/(1-\rho)$, of the disk with center origin and radius ρ , and $g(z) = \rho^{-1}f(\rho z)$, $0 < \rho < 1$, is a K -quasiregular harmonic mapping of the unit disk. Note that $g_z(0) = 1$. Taking $\rho = 1/2$ and using the first estimate, we obtain that $b_f \geq B_{hol}/4$. The second estimate is proved and the proof is complete. $\square$

The first lower bound estimate in the above theorem is quite satisfactory, since the lower bound tends to B_{hol} as $K \rightarrow 1$. Using Ahlfors' [1] lower estimate $B_{hol} \geq \sqrt{3}/4$, from the above theorem, we have

$$B_{har}^K \geq \frac{1}{K+1} \frac{\sqrt{3}}{2}, \quad B_{har}^O \geq \frac{\sqrt{3}}{16} \approx 0.108.$$

4. LANDAU'S THEOREM FOR HARMONIC MAPPINGS

Let f be a holomorphic function on the unit disk D . If $f(0) = 0$, $f'(0) = \alpha > 0$ and $|f(z)| < 1$ for $z \in D$, the classical Landau's theorem asserts that f is univalent in the disk D_{r_0} of center 0 and radius

$$r_0 = \frac{\alpha}{1 + \sqrt{1 - \alpha^2}} > \frac{\alpha}{2},$$

and $f(D_{r_0})$ contains a disk D_{R_0} of center 0 and radius $R_0 = r_0^2$. In [7], it is proved that Landau's theorem holds also for harmonic mappings with the constants r_0 and R_0 replaced by $r_0 = \pi^2 \alpha / (16m)$ and $R_0 = r_0 \alpha / 2 \approx \alpha^2 / 22.2$, where $m \approx 6.85$, under the normalization $f_z(0) = \alpha$ and $f_{\bar{z}}(0) = 0$. Now we improve this result for open harmonic mappings. To do this, we need the Schwarz-Pick lemma for harmonic mappings: for a harmonic mapping f on the unit disk such that $f(0) = 0$ and $|f(z)| < 1$ for $z \in D$, it is known [11] that

$$(4.1) \quad |f(z)| \leq \frac{4}{\pi} \arctan |z|, \quad z \in D,$$

and

$$(4.2) \quad \Lambda_f(0) \leq \frac{4}{\pi}.$$

Since the composition $f \circ \phi$ of f and a biholomorphic mapping ϕ of D is harmonic, as a consequence of (4.2), if the condition $f(0) = 0$ is replaced by $f(z) = 0$ for some $z \in D$, we have

$$(4.3) \quad \Lambda_f(z) \leq \frac{4}{\pi} \frac{1}{1 - |z|^2}.$$

Moreover, (4.3) is always true [8] for any harmonic self-mapping f of D and any point $z \in D$ even without the assumption $f(z) = 0$.

Theorem 3. *Let f be a harmonic mapping of D such that $f(0) = 0$, $f_z(0) = \alpha > 0$ and $f_{\bar{z}}(0) = 0$, and $|f(z)| < 1$ for $z \in D$. In addition, suppose that $|f_{\bar{z}}| \leq |f_z|$ on D . Then, f is univalent on a domain which contains the origin and whose image is a disk of center 0 and radius*

$$R_0 = kr'^2(1 - r'/2),$$

where

$$k = \frac{2 \log 3}{\pi}, \quad r' = \frac{\alpha}{2k + \sqrt{4k^2 - \alpha^2}}.$$

Proof. By E , denote the set of zeros of the holomorphic function f_z . The function $h(z) = \overline{f_z}/f_z$ is holomorphic and has its modulus not exceeding 1 on $D \setminus E$. Since E is removable, using the Schwarz lemma, we have

$$(4.4) \quad |f_{\bar{z}}(z)| \leq |z| |f_z(z)|, \quad z \in D.$$

Define $\zeta = F(z) = \int_0^z f_z(z) dz$ for $z \in D$. For $|z'| < 1/2$, by (4.3), we have

$$\begin{aligned} |F(z')| &\leq \int_{0, z'} |f_z(z)| |dz| \leq \int_{0, z'} \Lambda_f(z) |dz| \\ &\leq \frac{4}{\pi} \int_0^{|z'|} \frac{dr}{1 - r^2} = \frac{2}{\pi} \log \frac{1 + |z'|}{1 - |z'|} < k = \frac{2 \log 3}{\pi}. \end{aligned}$$

Using Landau's theorem for $g(z) = k^{-1}F(z/2)$, we see that g is univalent on the disk $D_{r'}$ of center the origin and radius r' and $g(D_{r'})$ contains the disk $D_{R'}$ with $R' = r'^2$. Consequently, F is univalent on $D_{r'/2}$, and $F(D_{r'/2})$ contains the disk $D_{kR'}$.

Let $G = F_{D_{r'/2}}^{-1}(D_{kR'}) \subset D_{r'/2}$. For $z_1, z_2 \in G$, let $\gamma = F^{-1}(\overline{F(z_1)}, F(z_2))$. Then,

$$|F(z_2) - F(z_1)| = \left| \int_{\gamma} f_z(z) dz \right| = \int_{\gamma} |f_z(z)| |dz|.$$

We have, by (4.4),

$$\begin{aligned} |f(z_1) - f(z_2)| &\geq \left| \int_{\gamma} f_z(z) dz \right| - \left| \int_{\gamma} f_{\bar{z}}(z) d\bar{z} \right| \\ &\geq |F(z_2) - F(z_1)| - \int_{\gamma} |z| |f_z(z)| |dz| \geq \left(1 - \frac{r'}{2}\right) |F(z_2) - F(z_1)|, \end{aligned}$$

from which we see that f is univalent on G since $r' < 1$. Further, $f(0) = 0$ and $|f(z)| \geq (1 - r'/2)|F(z)| \rightarrow kR'(1 - r'/2)$ as $z \rightarrow \partial G$. This shows that $f(G)$ contains the disk of center the origin and radius $kR'(1 - r'/2)$. The theorem is proved. $\square$

We notice that $R_0 \approx \alpha^2/(16k) \approx \alpha^2/11.2$ as $\alpha \rightarrow 0$. As in the proof of Theorem 2, under the normalization in Theorem 3, the openness of f implies that $|f_{\bar{z}}| \leq |f_z|$ on D . So, Theorem 3 holds if the additional assumption is replaced by the openness of f .

5. THE SCHWARZ-PICK LEMMA FOR PLURIHARMONIC MAPPINGS

In this section, we prove that (4.3), without the assumption $f(z) = 0$, holds also for any pluriharmonic mapping f of the unit ball B^n into the complex space $\mathbb{C}^n$ of dimension n . A continuous complex-valued function ϕ defined on a domain $\Omega \subset \mathbb{C}^n$ is said to be pluriharmonic if, for each fixed $z' \in \Omega$ and $\theta \in \partial B^n$, the function $\phi(z' + \theta\zeta)$ is harmonic in the complex variable ζ , for $|\zeta|$ smaller than the distance of z' from $\partial\Omega$. A mapping f of Ω into $\mathbb{C}^n$ is said to be pluriharmonic if every component of f is pluriharmonic. A mapping f of B^n into $\mathbb{C}^n$ is pluriharmonic if and only if f has a representation $f = g + \bar{h}$, where g and h are holomorphic mappings.

For a continuously differentiable mapping $w = f(z) = (f_1(z), \dots, f_m(z))$, $z = (z_1, \dots, z_n)$, of a domain $\Omega \subset \mathbb{C}^n$ into $\mathbb{C}^m$, by f_z and $f_{\bar{z}}$ denote the matrices $(\partial f_j / \partial z_k)_{m \times n}$ and $(\partial f_j / \partial \bar{z}_k)_{m \times n}$ respectively. Denote the maximum dilation and minimum dilation by

$$\Lambda_f = \max_{\theta \in \partial \mathbb{B}^n} |f_z \theta + f_{\bar{z}} \bar{\theta}|, \quad \lambda_f = \min_{\theta \in \partial \mathbb{B}^n} |f_z \theta + f_{\bar{z}} \bar{\theta}|.$$

Here, θ is regarded as a column vector. From now on, we identify a point in $\mathbb{C}^n$ or $\mathbb{R}^n$ (real space of dimension n) with a complex or real column vector.

Theorem 4. *Let f be a pluriharmonic mapping of B^n into B^m . Then*

$$(5.1) \quad \Lambda_f(z) \leq \frac{4}{\pi} \frac{1}{1 - |z|^2} \quad \text{for } z \in B^n.$$

If $f(0) = 0$, then

$$(5.2) \quad |f(z)| \leq \frac{4}{\pi} \arctan |z| \quad \text{for } z \in B^n.$$

Proof. To prove (5.1), consider $z = 0$ first. Let $\Lambda_f(0) = |f_z(0)\theta' + f_{\bar{z}}(0)\overline{\theta'}|$ and $\omega = f_z(0)\theta' + f_{\bar{z}}(0)\overline{\theta'} \neq 0$. Define the harmonic function $\phi(\zeta) = \langle f(\zeta)\theta', \omega/|\omega| \rangle$ for $\zeta \in D$. Then, $|\phi(\zeta)| \leq 1$ for $\zeta \in D$ and $\phi_\zeta(0) + \phi_{\bar{\zeta}}(0) = \langle \omega, \omega/|\omega| \rangle = |\omega|$. Since ϕ is harmonic, (4.3) holds for ϕ . Thus,

$$\Lambda_f(0) = |\omega| = |\phi_\zeta(0) + \phi_{\bar{\zeta}}(0)| \leq \Lambda_\phi(0) \leq \frac{4}{\pi}.$$

This shows (5.1) for $z = 0$.

Let $z' \in B^n$ be fixed. By $\varphi_{z'}$ we denote the Möbius transformation of B^n such that $\varphi(z') = 0$ and $\varphi_{z'} = \varphi_{z'}^{-1}$; see [15]. It is known that

$$\Lambda_{\varphi_{z'}}(z') = \frac{1}{1 - |z'|^2}.$$

By what was proved above, we have $\Lambda_{f \circ \varphi_{z'}}(0) \leq 4/\pi$ and, consequently,

$$\Lambda_f(z') \leq \Lambda_{\varphi_{z'}}(z') \Lambda_{f \circ \varphi_{z'}}(0) \leq \frac{4}{\pi} \frac{1}{1 - |z'|^2}.$$

This shows (5.1) completely.

For any fixed $z' \in B^n \setminus \{0\}$, letting $\phi(\zeta) = \langle f(\zeta z'/|z'|), f(z')/|f(z')| \rangle$ for $\zeta \in D$ and using (4.1), we obtain $|f(z')| = |\phi(|z'|)| \leq (4/\pi) \arctan |z'|$. This shows (5.2), and the lemma is proved. $\square$

6. THE LANDAU THEOREM FOR PLURIHARMONIC MAPPINGS

The following Landau theorem for holomorphic mappings of B^n into $\mathbb{C}^n$ was proved in [6]: *Let f be a holomorphic mapping of B^n into $\mathbb{C}^n$ such that $f(0) = 0$, $\det f'(0) = \alpha > 0$, and $|f(z)| < M$ for $z \in B^n$. Let*

$$\rho_0 = \frac{\alpha}{mM^n}, \quad \rho_1 = \frac{\alpha}{2M^n}, \quad R_0 = M\rho_0\rho_1 = \frac{\alpha^2}{2mM^{2n-1}},$$

where $m \approx 4.2$ is the minimum of the function $(2 - r^2)/(r(1 - r^2))$ for $0 \leq r \leq 1$. Then f maps $B^n(0, \rho_0)$ injectively onto a domain which contains $B^n(0, R_0)$. Here, by $B^n(z, r)$ we denote the ball in $\mathbb{C}^n$ of center z and radius r . In this section, we establish the Landau theorem for pluriharmonic mappings.

For a continuously differentiable mapping $w = f(z) = (f_1(z), \dots, f_n(z))$, $z = (z_1, \dots, z_n)$, of a domain $\Omega \subset \mathbb{C}^n$ into $\mathbb{C}^n$, let $z_k = x_k + iy_k$ and $f_j = u_j + iv_j$. f can be regarded as a mapping of a domain in $\mathbb{R}^{2n}$ into $\mathbb{R}^{2n}$. We denote the real Jacobian matrix of this mapping by

$$J_f = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial y_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial y_2} & \dots & \frac{\partial u_1}{\partial x_n} & \frac{\partial u_1}{\partial y_n} \\ \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial y_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial y_2} & \dots & \frac{\partial v_1}{\partial x_n} & \frac{\partial v_1}{\partial y_n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial y_1} & \frac{\partial u_n}{\partial x_2} & \frac{\partial u_n}{\partial y_2} & \dots & \frac{\partial u_n}{\partial x_n} & \frac{\partial u_n}{\partial y_n} \\ \frac{\partial v_n}{\partial x_1} & \frac{\partial v_n}{\partial y_1} & \frac{\partial v_n}{\partial x_2} & \frac{\partial v_n}{\partial y_2} & \dots & \frac{\partial v_n}{\partial x_n} & \frac{\partial v_n}{\partial y_n} \end{pmatrix}.$$

Let $\mathbb{B}^{2n}$ denote the unit ball in $\mathbb{R}^{2n}$. With this notation, the maximum dilation and minimum dilation Λ_f and λ_f have another expression:

$$\Lambda_f = \max_{\theta \in \partial \mathbb{B}^{2n}} |J_f \theta|, \quad \lambda_f = \min_{\theta \in \partial \mathbb{B}^{2n}} |J_f \theta|.$$

For a complex or real $n \times n$ matrix A , the operator norm of A is defined by

$$|A| = \max_{\theta \in \partial B^n} |A\theta| \quad \text{or} \quad |A| = \max_{\theta \in \partial \mathbb{B}^n} |A\theta|.$$

Note that $\Lambda_f = |J_f|$.

Lemma 1. *Let A be an $n \times n$ complex (real) matrix. Then, for any unit vector $\theta \in \partial B^n$ ($\partial \mathbb{B}^n$), the following inequality holds:*

$$(6.1) \quad |A| \geq |\det(A)|^{1/n}, \quad |A\theta| \geq \frac{|\det(A)|}{|A|^{n-1}}.$$

If A satisfies $|A| \leq K|\det(A)|^{1/n}$, $K \geq 1$, then, for any $\theta \in \partial B^n$ ($\partial \mathbb{B}^n$), we have

$$(6.2) \quad |A\theta| \geq \frac{|\det(A)|^{1/n}}{K^{n-1}}.$$

Proof. First assume that A is a complex matrix. We may assume that $\det(A) \neq 0$. There exist two unitary matrices U_1 and U_2 such that $\Lambda = U_1 A U_2$ is a diagonal matrix with positive elements $\lambda_1, \lambda_2, \dots, \lambda_n$. Then

$$|\det(A)| = \lambda_1 \lambda_2 \cdots \lambda_n.$$

Assume that $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. For any $\theta \in \partial B^n$, since U_1 and U_2 are unitary, we have

$$|A\theta| = |U_1 A U_2 U_2^{-1} \theta| = |\Lambda \omega| \geq \lambda_1 = \frac{|\det(A)|}{\lambda_2 \cdots \lambda_n},$$

where $\omega = U_2^{-1} \theta \in \partial B^n$. Let ω be the unit vector, of which only the k -th component is not zero. Then, since U_1 is unitary,

$$\lambda_k = |\Lambda \omega| = |U_1 A U_2 \omega| = |A U_2 \omega|.$$

Also, since U_2 is unitary, $\omega' = U_2 \omega$ is also a unit vector since U_2 is unitary. Thus,

$$\lambda_k = |A \omega'| \leq |A|,$$

and, if $|A| \leq K|\det(A)|^{1/n}$, we have

$$\lambda_k \leq K|\det(A)|^{1/n}.$$

This shows the lemma for a complex matrix. The proof is completely the same in the real case. $\square$

As a consequence of (6.1), for a continuously differentiable mapping f of a domain $\Omega \subset \mathbb{C}^n$ into $\mathbb{C}^n$, we have

$$(6.3) \quad \lambda_f \geq |\det J_f| / \Lambda_f^{2n-1}.$$

Theorem 5. *Let f be a pluriharmonic mapping of B^n into $\mathbb{C}^n$ such that $f(0) = 0$, $|f(z)| < M$ for $z \in B^n$ and $(4M/\pi)^{2n} \geq \det J_f(0) = \alpha > 0$. Let*

$$\rho_0 = \frac{\alpha \pi^{2n+1}}{4m(4M)^{2n}}, \quad R_0 = \frac{\alpha^2 \pi^{4n}}{8m(4M)^{4n-1}},$$

where $m \approx 4.2$ is the minimum of the function $\frac{2-r^2}{r(1-r^2)}$ on the interval $(0, 1)$. Then f is univalent on the ball $B^n(0, \rho_0)$, and $f(B^n(0, \rho_0))$ covers the ball $B^n(0, R_0)$.

Proof. The function $\frac{2-r^2}{r(1-r^2)}$ attains its minimum $m \approx 4.2$ at $r_0 \approx 0.66$. For fixed different points $z', z'' \in B^n(0, \rho_0)$, let $z'' - z' = |z'' - z'|\theta$ and define the pluriharmonic mapping

$$\phi_\theta(z) = (f_z(z) - f_z(0))\theta + (f_{\bar{z}}(z) - f_{\bar{z}}(0))\bar{\theta}.$$

Then, by (5.1),

$$|\phi_\theta(z)| \leq \Lambda_f(0) + \Lambda_f(z) \leq \frac{4M}{\pi} \left(1 + \frac{1}{1 - |z|^2}\right) < \frac{4M}{\pi} \frac{2 - r_0^2}{1 - r_0^2} \quad \text{for } |z| < r_0.$$

Consequently, using (5.2), we obtain

$$(6.4) \quad |\phi_\theta(z)| \leq \frac{16M|z|}{\pi^2} \frac{2 - r_0^2}{r_0(1 - r_0^2)} = \frac{16mM|z|}{\pi^2} \quad \text{for } |z| < r_0.$$

We have

$$\begin{aligned} |f(z'') - f(z')| &= \left| \int_{z'z''} f_z(z)dz + f_{\bar{z}}(z)d\bar{z} \right| \\ &\geq \left| \int_{z'z''} f_z(0)dz + f_{\bar{z}}(0)d\bar{z} \right| - \left| \int_{z'z''} (f_z(z) - f_z(0))dz + (f_{\bar{z}}(z) - f_{\bar{z}}(0))d\bar{z} \right|, \\ &\quad \left| \int_{z'z''} f_z(0)dz + f_{\bar{z}}(0)d\bar{z} \right| \geq |f_z(0)\theta + f_{\bar{z}}(0)\bar{\theta}| \int_{z'z''} ds \\ &\geq \lambda_f(0)|z'' - z'| \geq \frac{|\det J_f(0)|}{\Lambda_f(0)^{2n-1}}|z'' - z'| \geq \frac{\alpha\pi^{2n-1}}{(4M)^{2n-1}}|z'' - z'|, \end{aligned}$$

where (6.3) and (5.1) are used, and

$$\begin{aligned} &\left| \int_{z'z''} (f_z(z) - f_z(0))dz + (f_{\bar{z}}(z) - f_{\bar{z}}(0))d\bar{z} \right| \\ &\leq \int_{z'z''} |(f_z(z) - f_z(0))\theta + (f_{\bar{z}}(z) - f_{\bar{z}}(0))\bar{\theta}| ds \\ &< \frac{16mM\rho_0}{\pi^2}|z'' - z'| = \frac{\alpha\pi^{2n-1}}{(4M)^{2n-1}}|z'' - z'|, \end{aligned}$$

since $\rho_0 \leq \pi/(4m) < r_0$. Thus $f(z'') \neq f(z')$. This shows that f is univalent in $B^n(0, \rho_0)$.

Now, let $z' \in \partial B^n(0, \rho_0)$. We have

$$\begin{aligned} |f(z')| &= \left| \int_{0z'} f_z(z)dz + f_{\bar{z}}(z)d\bar{z} \right| \\ &\geq \left| \int_{0z'} f_z(0)dz + f_{\bar{z}}(0)d\bar{z} \right| - \left| \int_{0z'} (f_z(z) - f_z(0))dz + (f_{\bar{z}}(z) - f_{\bar{z}}(0))d\bar{z} \right| \\ &\geq \frac{\alpha^2\pi^{4n}}{4m(4M)^{4n-1}} - \int_0^{\rho_0} \frac{16mMr}{\pi^2} dr = \frac{\alpha^2\pi^{4n}}{8m(4M)^{4n-1}}. \end{aligned}$$

This shows that $f(B^n(0, \rho_0))$ covers the ball $B^n(0, R_0)$. The theorem is proved. $\square$

7. THE BLOCH THEOREM FOR PLURIHARMONIC MAPPINGS

In the case of several variables, the notions of univalent ball and inradius b_f are defined in a similar way as that in one variable. For general holomorphic mappings of more than one complex variable, there is no (positive) Bloch constant ([10], [18]); in order to obtain higher dimensional analogs of Bloch's theorem, it is necessary to

restrict the class of mappings considered. In [18], H. Wu defined the quasiregularity of a holomorphic mapping in a way which is a little different from that in Section 3 of this paper. A holomorphic mapping f of B^n into $\mathbb{C}^n$ is said to be a Wu K -mapping, $K \geq 1$, if

$$|f'(z)| \leq K |\det f'(z)|^{1/n} \quad \text{for } z \in B^n,$$

where $f'(z)$ is the Jacobian matrix of the mapping f at z , and $|f'(z)|$ is the operator norm which is just the maximum dilation $\Lambda_f(z)$. For a Wu K -mapping of the unit ball B^n into $\mathbb{C}^n$, $n > 1$, such that $\det f'(0) = 1$, the authors of the present paper [6] gave two lower bounds of the inradius b_f :

$$b_f \geq \frac{1}{10K^{2n-1}}, \quad b_f \geq \frac{1}{12K^{n-1}}.$$

Now, we generalize the notion of Wu K -mapping to pluriharmonic mappings and establish a lower bound estimate of the Bloch constant for such mappings.

A pluriharmonic mapping f of B^n into $\mathbb{C}^n$ is said to be a K -mapping if

$$|J_f(z)| \leq K |\det J_f(z)|^{1/(2n)} \quad \text{for } z \in B^n,$$

where J_f is defined in Section 8. For pluriharmonic K -mappings, we have the following Bloch theorem.

Theorem 6. *Let f be a pluriharmonic K -mapping of the unit ball B^n into $\mathbb{C}^n$, $n > 1$, such that $\det J_f(0) = 1$. Then*

$$b_f \geq R_n = \frac{k_n \pi}{8m} \left(\frac{k_n \pi}{4K \log(1/(1 - k_n))} \right)^{4n-1},$$

where $0 < k_n < 1$ is the unique number such that

$$(7.1) \quad 4n \log \frac{1}{1 - k_n} = (4n - 1) \cdot \frac{k_n}{1 - k_n}.$$

Proof. Without loss of generality, we may assume that f is pluriharmonic on $\overline{B}^n$. Let $z' \in B^n$ be a point such that

$$(7.2) \quad (1 - |z'|)^{2n} |\det J_f(z')| = 1$$

and

$$(7.3) \quad (1 - |z|)^{2n} |\det J_f(z)| \leq 1 \quad \text{for } |z'| = r \leq |z| \leq 1.$$

In particular,

$$(7.4) \quad |\det J_f(z)| \leq |\det J_f(z')| \quad \text{for } |z| = r.$$

Let $|z_0| \leq r$. Assume that $\Lambda_f(z_0) = |f_z(z_0)\theta + f_{\bar{z}}(z_0)\bar{\theta}|$ with $\theta \in \partial B^n$. Define $\phi(\zeta) = f_z(\zeta z_0/|z_0|)\theta + f_{\bar{z}}(\zeta z_0/|z_0|)\bar{\theta}$ for $\zeta \in D$. Since ϕ is harmonic, by the maximum principle,

$$\Lambda_f(z_0) = |\phi(|z_0|)| \leq |f_z(\zeta' z_0/|z_0|)\theta + f_{\bar{z}}(\zeta' z_0/|z_0|)\bar{\theta}|$$

with $|\zeta'| = r$. Let $z'' = \zeta' z_0/|z_0|$. Note that $|z''| = r$. Thus, by (7.4), and the definition of K -mappings,

$$\begin{aligned} \Lambda_f(z_0) &\leq |f_z(z'')\theta + f_{\bar{z}}(z'')\bar{\theta}| \\ &\leq \Lambda_f(z'') = |J_f(z'')| \leq K |\det J_f(z'')|^{1/(2n)} \leq K |\det J_f(z')|^{1/(2n)}. \end{aligned}$$

This shows that

$$(7.5) \quad \Lambda_f(z) \leq K |\det J_f(z')|^{1/(2n)} \quad \text{for } |z| \leq r.$$

For $\zeta \in B^n$, define

$$g(\zeta) = z' + k(1-r)\zeta, \quad F(\zeta) = \frac{1}{k} (f(g(\zeta)) - f(z')),$$

where $k = k_n$. Then,

$$F(0) = 0, \quad |\det J_F(0)| = (1-r)^{2n} |\det J_f(z')| = 1.$$

If $|g(\zeta)| \leq r$, by (7.5) and (7.2),

$$\begin{aligned} \Lambda_F(\zeta) &= (1-r)\Lambda_f(g(\zeta)) \\ &\leq K(1-r)|\det J_f(z')|^{1/(2n)} = K, \end{aligned}$$

and if $|g(\zeta)| \geq r$,

$$\begin{aligned} \Lambda_F(\zeta) &= (1-r)\Lambda_f(g(\zeta)) \leq K(1-r)|\det J_f(g(\zeta))|^{1/(2n)} \\ &= K \left(\frac{1-r}{1-|g(\zeta)|} \right) (1-|g(\zeta)|) |\det J_f(g(\zeta))|^{1/(2n)} \leq K \left(\frac{1-r}{1-|g(\zeta)|} \right) \\ &\leq K \cdot \frac{1-r}{1-r-k(1-r)|\zeta|} = \frac{K}{1-k|\zeta|}. \end{aligned}$$

We conclude that

$$\Lambda_F(\zeta) \leq \frac{K}{1-k|\zeta|} \quad \text{for } \zeta \in B^n.$$

Thus, for $\zeta' \in B^n$, we have

$$|F(\zeta')| \leq \int_{0\zeta'} |dF(\zeta)| = \int_{0\zeta'} \Lambda_F(\zeta) ds \leq K \int_0^{|\zeta'|} \frac{ds}{1-ks} < \frac{K}{k} \log \frac{1}{1-k}.$$

Now, applying Theorem 5 to the function F , we see that $F(B^n)$ contains a schlicht ball with center 0 and radius

$$\frac{\pi}{8m} \left(\frac{k\pi}{4K \log(1/(1-k))} \right)^{4n-1}.$$

Consequently, $f(B^n)$ contains the schlicht ball with center $f(z')$ and radius R_n . This proves the theorem. $\square$

Now, we give an approximative estimate for R_n . Because of (7.1), we have

$$1 = (2n-1)k_n + (4n-1-4n/3)k_n^2 + \dots,$$

from which it follows that $k_n < \frac{1}{2n-1}$ and

$$k_n \geq \frac{1}{2n-1} \left(1 - \frac{4n-1}{2n-1} \frac{k_n}{1-k_n} \right) = \frac{1}{2n-1} (1 - o(1)).$$

Replacing $\log \frac{1}{1-k_n}$ by $\frac{4n-1}{4n} \frac{k_n}{1-k_n}$ in the formula of R_n gives

$$R_n = \frac{\pi}{8m} \left(\frac{\pi}{4} \right)^{4n-1} \frac{1}{K^{4n-1}} \left(\frac{4n}{4n-1} \right)^{4n-1} k_n (1-k_n)^{4n-1}.$$

Since

$$\left(\frac{4n}{4n-1} \right)^{4n-1} = \left(1 + \frac{1}{4n-1} \right)^{4n-1} \rightarrow e, \quad k_n = \frac{1}{2n-1} (1 + o(1)),$$

and

$$(1-k_n)^{4n-1} = \left[(1-k_n)^{1/k_n} \right]^{(4n-1)k_n} \rightarrow \frac{1}{e^2},$$

we have

$$R_n = \frac{\pi}{8me} \frac{1}{2n-1} \left(\frac{\pi}{4}\right)^{4n-1} \frac{1}{K^{4n-1}} (1 + o(1)).$$

For locally univalent pluriharmonic K -mappings, Theorem 5 has another version and Theorem 6 can be improved. It is known [14] that for $n > 1$, a quasiregular holomorphic mapping is locally biholomorphic. However, it is not true that a pluriharmonic K -mapping is always locally univalent.

Theorem 7. *Let f be a locally univalent pluriharmonic K -mapping of the unit ball B^n into $\mathbb{C}^n$, $n > 1$, such that $f(0) = 0$, $(4M/\pi)^{2n} \geq \det J_f(0) = \alpha > 0$, and $|f(z)| < M$ for $z \in B^n$. Then there exists a domain $\Omega \subset B^n(0, \rho_0)$ such that $0 \in \Omega$ and f maps Ω onto a ball $B^n(0, R_0)$ injectively, where*

$$\rho_0 = \frac{\pi^2 \alpha^{1/(2n)}}{16mMK^{2n-1}}, \quad R_0 = \frac{\pi^2 \alpha^{1/n}}{32mMK^{4n-2}},$$

and m is the same number as in Theorem 5.

Proof. For any fixed z' with $|z'| = \rho_0$, letting $\theta = z'/|z'|$ and using (6.2) and (6.4), we have

$$\begin{aligned} |f(z')| &= \left| \int_{0z'} f_z(z) dz + f_{\bar{z}}(z) d\bar{z} \right| \\ &\geq \left| \int_{0z'} f_z(0) dz + f_{\bar{z}}(0) d\bar{z} \right| - \left| \int_{0z'} (f_z(z) - f_z(0)) dz + (f_{\bar{z}}(z) - f_{\bar{z}}(0)) d\bar{z} \right| \\ &\geq \lambda_f(0) \rho_0 - \left| \int_0^{\rho_0} \phi_\theta(z) dr \right| \geq \frac{\alpha^{1/2n} \rho_0}{K^{2n-1}} - \int_0^{\rho_0} \frac{16mMr}{\pi^2} dr \\ &= \frac{\alpha^{1/(2n)} \rho_0}{K^{2n-1}} - \frac{8mMr \rho_0^2}{\pi^2} = R_0. \end{aligned}$$

Denote $B = B^n(0, R_0)$, $B' = B^n(0, \rho_0)$, and let Ω be a component of $f_{B'}^{-1}(B)$ which contains 0. It follows from what we proved above that f makes Ω a regular covering space of B . We recall that a covering is said to be regular if the mapping is locally topological and the lifting is always possible. Thus, by the monodromy theorem in the theory of covering spaces, f maps Ω onto B topologically. The theorem is proved. $\square$

If replacing Theorem 5 we use Theorem 7 at the last step of the proof of Theorem 6 and let $k = 0.7$, then we obtain the following Bloch theorem.

Theorem 8. *Let f be a locally univalent pluriharmonic K -mapping of the unit ball B^n into $\mathbb{C}^n$, $n > 1$, such that $\det J_f(0) = 1$. Then*

$$b_f \geq \frac{1}{134K^{4n-1}}.$$

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