REMARKS ON THE AREA THEOREM IN THE THEORY OF UNIVALENT FUNCTIONS

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ABSTRACT. We prove an integral extension of the classical area theorem for univalent functions. We give an application finding geometric conditions on the image domain of a univalent function f which imply that f belongs to the Hardy space H^p , $0 < p < \infty$.

1. INTRODUCTION AND MAIN RESULTS

Let $\mathbb{D}$ denote the open unit disk of the complex plane $\mathbb{C}$. A complex-valued function defined in $\mathbb{D}$ is said to be *univalent* if it is analytic and one-to-one there. We refer to [5] and [8] for the theory of these functions. Throughout the paper, $\mathcal{U}$ will stand for the class of all univalent functions in $\mathbb{D}$. The classical area theorem [5, p. 29], which is a key in the proof of a good number of results in the theory of univalent functions, can be stated as follows:

Theorem 1. *If $\varphi(z) = \sum_{n=0}^{\infty} b_n z^n$, $b_0 \neq 0$, is an analytic function in $\mathbb{D}$ such that the meromorphic function $\varphi(z)/z$ is one-to-one in $\mathbb{D}$, then*

$$\sum_{n=1}^{\infty} \frac{n^2 |b_n|^2}{n+1} \leq \sum_{n=0}^{\infty} \frac{|b_n|^2}{n+1}$$

or, equivalently,

$$\int_{\mathbb{D}} |z\varphi'(z)|^2 dm(z) \leq \int_{\mathbb{D}} |\varphi(z)|^2 dm(z).$$

Here, $dm(z) = dxdy$ denotes the usual Lebesgue area measure.

In this paper we generalize Theorem 1 in the following way.

Theorem 2. *Let $p > 0$. If φ is a function as in Theorem 1 and $\varphi(z) \neq 0$ for all $z \in \mathbb{D}$, then*

$$(1.1) \quad \int_{\mathbb{D}} |z|^p |\varphi(z)|^{p-2} |\varphi'(z)|^2 dm(z) \leq \int_{\mathbb{D}} |z|^{p-2} |\varphi(z)|^p dm(z).$$

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Consequently, we deduce the following theorem on univalent functions.

Theorem 3. *If f is univalent in $\mathbb{D}$, $f(0) = 0$ and $p > 0$, then*

$$(1.2) \quad \int_{\mathbb{D}} |z|^{2p-2} |f(z)|^{-p} \left| 1 - \frac{zf'(z)}{f(z)} \right|^2 dm(z) \leq \int_{\mathbb{D}} |z|^{2p-2} |f(z)|^{-p} dm(z),$$

or, equivalently,

$$(1.3) \quad \int_{\mathbb{D}} |z|^{2p-2} |f(z)|^{-p} \left| \frac{zf'(z)}{f(z)} \right|^2 dm(z) \leq 2 \int_{\mathbb{D}} |z|^{2p-2} |f(z)|^{-p} \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) dm(z).$$

Given a space X of analytic functions, one of the most interesting problems in the theory of univalent functions is finding geometric conditions on a domain Ω which imply that Ω is an X -domain; that is, any analytic function f defined on $\mathbb{D}$ with $f(\mathbb{D}) \subset \Omega$ belongs to X . This problem has been solved for a good number of spaces, such as the Bloch space, Besov spaces B^p , $1 < p < \infty, \dots$ (see [1], [3]). However, this is an open problem for H^p spaces (see [4]). We shall use Theorem 3 to find geometric conditions on the image domain of a function $f \in \mathcal{U}$ which imply its membership in H^p . For simplicity, we shall assume that $0 \in f(\mathbb{D})$.

Given a domain $\Omega \subset \mathbb{C}$ and a point w in Ω , we shall write $d_{\Omega}(w)$ to denote the (Euclidean) distance from w to the boundary $\partial\Omega$. The following inequalities play an essential role in the proof of our results (see, e.g., [9], Corollary 1.4).

If Ω is a simply connected proper subdomain of $\mathbb{C}$ and if F is a conformal mapping from Δ onto Ω , then we have

$$(1.4) \quad d_{\Omega}(F(z)) \leq |F'(z)|(1 - |z|^2) \leq 4d_{\Omega}(F(z)), \quad z \in \mathbb{D}.$$

The following result is proved in [2, Corollary 7].

Corollary 1. *Suppose that $1/2 \leq p < \infty$ and that $f \in A^{2p} \cap \mathcal{U}$. Set $\Omega = f(\mathbb{D})$ and suppose that $0 \in \Omega$. For $\varepsilon > 0$, set $\Omega_{\varepsilon} = \{w \in \Omega : |w| > \varepsilon\}$. If*

$$\int_{\Omega_{\varepsilon}} \frac{d_{\Omega}(w)^{2p-2}}{|w|^{2p}} dm(w) < \infty,$$

for all sufficiently small $\varepsilon > 0$, then $f \in H^p$.

Here, we shall prove the following extension of this result.

Theorem 4. *Suppose that $0 < \beta < 1$, $1 - \frac{\beta}{2} < p < \infty$, and $f \in A^{2p} \cap \mathcal{U}$. Set $\Omega = f(\mathbb{D})$ and suppose that $0 \in \Omega$. If*

$$\int_{\Omega_{\varepsilon}} \frac{d_{\Omega}(w)^{\frac{2p-2}{\beta}}}{|w|^{\frac{2p-2}{\beta} + 2\frac{\beta-\delta}{\beta}}} dm(w) < \infty,$$

for some δ , $0 < \delta < \frac{1+p}{2}$ and all sufficiently small $\varepsilon > 0$, then $f \in H^p$.

Moreover, we shall prove that this result is sharp in a certain sense.

The paper is organized as follows. Section 2 is devoted to proving Theorem 2 and Theorem 3. Corollary 4 and some other results are proved in section 3.

2. PROOF OF THE MAIN RESULTS

The proof of Theorem 2 is based on the following fact due to Prawitz [10]. The proof is borrowed from [7] (see also [6]).

Theorem 5. Let $f: \mathbb{D} \mapsto \mathbb{C}$ be a univalent function and $f(0) = 0$, and let

$$J_p(r) = J_p(r, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^{-p} d\theta, \quad p > 0, \quad 0 < r < 1.$$

Then

$$\begin{aligned} 2\pi J'_p(r) &= -(p/r) \operatorname{Im} \int_{\Gamma_r} |w|^{-p-2} \bar{w} dw & (w = u + iv) \\ &= -(p/r) \int_{\Gamma_r} |w|^{-p-2} (u dv - v du) \\ &< 0, \end{aligned}$$

for all $r \in (0, 1)$, where Γ_r is the image under f of the circle $\{\zeta \in \mathbb{C} : |\zeta| = r\}$ and Γ_r is positively oriented.

Proof. We have

$$\begin{aligned} 2\pi J'_p(r) &= -p \int_0^{2\pi} |f(re^{i\theta})|^{-p-2} \operatorname{Re}\{\overline{f(re^{i\theta})} f'(re^{i\theta}) e^{i\theta}\} d\theta \\ &= -(p/r) \operatorname{Im} \int_{|\zeta|=r} |f(\zeta)|^{-p-2} \overline{f(\zeta)} f'(\zeta) d\zeta \\ &= -(p/r) \operatorname{Im} \int_{\Gamma_r} |w|^{-p-2} \bar{w} dw & (w = u + iv) \\ &= -(p/r) \int_{\Gamma_r} |w|^{-p-2} (u dv - v du), \end{aligned}$$

where Γ_r is the image under f of the circle $|\zeta| = r$ and the curve Γ_r is positively oriented. Now we apply Green's formula to the domain $\Omega_{r,R}$ bounded by Γ_r and the circle $|w| = R$, where $R > \max_{|z|=r} |f(z)|$. Since

$$\partial(|w|^{-p-2}u)/\partial u - \partial(-|w|^{-p-2}v)/\partial v = -p|w|^{-p-2},$$

we have

$$\begin{aligned} &\int_{|w|=R} |w|^{-p-2} (u dv - v du) - \int_{\Gamma_r} |w|^{-p-2} (u dv - v du) \\ &= -p \iint_{\Omega_{r,R}} |w|^{-p-2} du dv. \end{aligned}$$

The first integral is equal to $2\pi R^{-p}$, and therefore

$$J'_p(r) = -(p/r) R^{-p} - (p^2/2\pi r) \iint_{\Omega_{r,R}} |w|^{-p-2} du dv.$$

Letting R tend to ∞ we get

$$J'_p(r) = -(p^2/2\pi r) \iint_{\Omega_r} |w|^{-p-2} du dv,$$

where Ω_r is the exterior of the curve Γ_r . This concludes the proof. $\square$

Proof of Theorem 2. With the hypotheses of Theorem 2, let $g(z) = \varphi(z)/z$ and $f(z) = 1/g(z) = z/\varphi(z)$. Let

$$I_p(r) = I_p(r, g) = \frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\theta})|^p d\theta = J_p(r, f), \quad p > 0.$$

The function f satisfies the hypotheses of Theorem 5, and therefore

$$2\pi I'_p(r) = 2\pi J'_p(r) = -(p/r) \int_{\Gamma_r} |w|^{-p-2} (u dv - v du) < 0.$$

By the change $w \mapsto 1/w$ we get

$$(2.1) \quad 2\pi I'_p(r) = (p/r) \int_{\gamma_r} |w|^{p-2} (u dv - v du) < 0,$$

where γ_r denotes the curve $w = g(re^{it})$, $0 \leq t \leq 2\pi$, which is a curve of negative orientation. Now we try to parameterize γ_r by $w = F(e^{it})$, where $F(e^{it}) \equiv g(re^{it})$; indeed we choose

$$F(z) = \frac{\bar{z}}{r} \varphi(rz).$$

Then we have

$$(2.2) \quad \int_{\gamma_r} |w|^{p-2} (u dv - v du) = \operatorname{Im} \int_{|\zeta|=1} |F(\zeta)|^{p-2} \overline{F(\zeta)} dF(\zeta).$$

Now, we choose a circle T_ρ of radius ρ centered at 0 ($0 < \rho < \frac{1}{2}$) and apply Green's formula to the annulus $A_\rho := \{z : \rho \leq |z| \leq 1\}$ to get

$$\operatorname{Im} \int_{|\zeta|=1} |F(\zeta)|^{p-2} \overline{F(\zeta)} dF(\zeta) = p \int_{A_\rho} |F|^{p-2} J_F dm + \operatorname{Im} \int_{T_\rho} |F|^{p-2} \overline{F} dF,$$

where $J_F(z)$ is the Jacobian of F ,

$$\begin{aligned} J_F(z) &= |\partial F / \partial z|^2 - |\partial F / \partial \bar{z}|^2 \\ &= |z\varphi'(rz)|^2 - |\varphi(rz)|^2 / r^2; \end{aligned}$$

here the circles are positively oriented. From the properties of φ it follows that

$$\left| \operatorname{Im} \int_{T_\rho} |F|^{p-2} \overline{F} dF \right| \leq C\rho^p$$

and

$$\int_{\mathbb{D}} |F|^{p-2} |J_F| dm \leq C \int_{\mathbb{D}} |z|^{p-2} dm(z) < \infty.$$

Hence, by passing to the limit as $\rho \rightarrow 0$, we get

$$\operatorname{Im} \int_{|\zeta|=1} |F(\zeta)|^{p-2} \overline{F(\zeta)} dF(\zeta) = p \int_{\mathbb{D}} |F|^{p-2} J_F dm.$$

From this, (2.1) and (2.2), it follows that

$$\int_{\mathbb{D}} |F(z)|^{p-2} J_F(z) dm(z) < 0;$$

that is,

$$\begin{aligned} (2.3) \quad & \int_{\mathbb{D}} |z\varphi(rz)|^{p-2} |z\varphi'(rz)|^2 dm(z) < r^{-2} \int_{\mathbb{D}} |z|^{p-2} |\varphi(rz)|^p dm(z) \\ & = 2\pi r^{-2} \int_0^1 t^{p-1} I_p(rt, \varphi) dt, \end{aligned}$$

where, as above,

$$I_p(s, \varphi) = \frac{1}{2\pi} \int_0^{2\pi} |\varphi(se^{i\theta})|^p d\theta.$$

As is well known, $I_p(s, \varphi)$ increases with s and so we can apply the monotone convergence theorem to get

$$\begin{aligned} \lim_{r \rightarrow 1^-} \int_0^1 t^{p-1} I_p(rt, \varphi) dt &= \int_0^1 t^{p-1} \lim_{r \rightarrow 1^-} I_p(rt, \varphi) dt \\ &= \int_0^1 t^{p-1} I_p(t, \varphi) dt. \end{aligned}$$

From this and (2.3), via Fatou's lemma, we obtain (1.1). This completes the proof of Theorem 2. $\square$

Proof of Theorem 3. If f is univalent in $\mathbb{D}$, $f(0) = 0$, then let $\varphi(z) = z/f(z)$. Since

$$\varphi'(z) = \frac{f(z) - zf'(z)}{f(z)^2},$$

we see, by Theorem 2, that

$$\int_{\mathbb{D}} |z|^{2p-2} |f(z)|^{2-p} \left| \frac{f(z) - zf'(z)}{f(z)^2} \right|^2 dm(z) \leq \int_{\mathbb{D}} |z|^{p-2} \left| \frac{z}{f(z)} \right|^p dm(z).$$

The result follows.

Remark 1. Equality is possible in (1.1), (1.2) and (1.3). In the case of (1.1) we take

$$\varphi(z) = (1 - e^{-i\theta} z)^2, \quad \theta \in [0, 2\pi].$$

Then

$$\operatorname{Im} \int_{|\zeta|=1} |F(\zeta)|^{p-2} \overline{F(\zeta)} dF(\zeta) = 0,$$

where

$$F(z) = \bar{z}\varphi(z) = \bar{z}(1 - e^{-i\theta} z)^2.$$

Now an application of Green's formula as above shows that

$$\int_{\mathbb{D}} |F|^{p-2} J_F dm = 0,$$

which implies (1.1).

The above deduction of Theorem 3 from Theorem 2 shows that equality in (1.2) and (1.3) is attained if f_θ is any rotation of Koebe's function:

$$f_\theta(z) = \frac{z}{(1 - e^{-i\theta} z)^2}. \quad \square$$

3. APPLICATIONS

First, we present the following simple but useful lemma.

Lemma 1. *If f is univalent in $\mathbb{D}$, $f(0) = 0$ and $p > 0$, then*

$$\int_{\mathbb{D}} |z|^{2p} |f(z)|^{-p} \left| \frac{f'(z)}{f(z)} \right|^2 dm(z) < \infty.$$

Proof. Observe that if $f \in \mathcal{U}$ with $f(0) = 0$, then

$$\int_{\mathbb{D}} |z|^{2p-2} |f(z)|^{-p} dm(z) < \infty \quad \text{for any } p > 0,$$

which together with Theorem 3 gives that

$$\begin{aligned} & \int_{\mathbb{D}} |z|^{2p} |f(z)|^{-p} \left| \frac{f'(z)}{f(z)} \right|^2 dm(z) \\ & \leq 2 \int_{\mathbb{D}} |z|^{2p-2} |f(z)|^{-p} \left| 1 - \frac{zf'(z)}{f(z)} \right|^2 dm(z) + 2 \int_{\mathbb{D}} |z|^{2p-2} |f(z)|^{-p} dm(z) \\ & \leq 4 \int_{\mathbb{D}} |z|^{2p-2} |f(z)|^{-p} dm(z) < \infty. \end{aligned}$$

This finishes the proof. $\square$

Now, we are ready to obtain our result on H^p -univalent functions.

Proof of Theorem 4. Bearing in mind [2, Theorem 1], it is enough to prove that

$$(3.1) \quad \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-1} dm(z) < \infty.$$

Take $\eta > 0$ such that $\{|w| < \eta\} \subset \Omega$ and take ε with $0 < \varepsilon < \eta$ and set $\mathbb{D}_\varepsilon = f^{-1}(\Omega_\varepsilon)$.

By Hölder's inequality (with conjugate exponents $\frac{2}{1+\beta} > 1$ and $\frac{2}{1-\beta}$) and Lemma 1, we deduce that

$$\begin{aligned} (3.2) \quad & \int_{\mathbb{D}_\varepsilon} |f'(z)|^p (1 - |z|^2)^{p-1} dm(z) \\ & = \int_{\mathbb{D}_\varepsilon} [(1 - |z|^2)|f'(z)|]^{p-1} |f'(z)|^\beta |z|^{-2\delta} |f(z)|^{1-\beta+\delta} \left[|f(z)|^{-\delta} |z|^{2\delta} \left| \frac{f'(z)}{f(z)} \right|^{1-\beta} \right] dm(z) \\ & \leq \left(\int_{\mathbb{D}_\varepsilon} [(1 - |z|^2)|f'(z)|]^{\frac{2(p-1)}{1+\beta}} |f'(z)|^{\frac{2\beta}{1+\beta}} |z|^{-\frac{4\delta}{1+\beta}} |f(z)|^{\frac{2(1-\beta+\delta)}{1+\beta}} dm(z) \right)^{\frac{1+\beta}{2}} \\ & \quad \times \left(\int_{\mathbb{D}_\varepsilon} |f(z)|^{-\frac{2\delta}{1-\beta}} |z|^{\frac{4\delta}{1-\beta}} \left| \frac{f'(z)}{f(z)} \right|^2 dm(z) \right)^{\frac{1-\beta}{2}} \\ & \leq C \left(\int_{\mathbb{D}_\varepsilon} [(1 - |z|^2)|f'(z)|]^{\frac{2(p-1)}{1+\beta}} |f'(z)|^{\frac{2\beta}{1+\beta}} |z|^{-\frac{4\delta}{1+\beta}} |f(z)|^{\frac{2(1-\beta+\delta)}{1+\beta}} dm(z) \right)^{\frac{1+\beta}{2}}. \end{aligned}$$

On the other hand, if $0 < \delta < \frac{1+\beta}{2}$, using Hölder's inequality (with conjugate exponents $\frac{1+\beta}{\beta}$ and $1 + \beta$), making the change of variable $w = f(z)$ and bearing in

mind (1.4) and that $f \in A^{2p}$, we deduce that

$$\begin{aligned}
& \int_{\mathbb{D}_\varepsilon} [(1 - |z|^2)|f'(z)|]^{\frac{2(p-1)}{1+\beta}} |f'(z)|^{\frac{2\beta}{1+\beta}} |z|^{-\frac{4\delta}{1+\beta}} |f(z)|^{\frac{2(1-\beta+\delta)}{1+\beta}} dm(z) \\
&= \int_{\mathbb{D}_\varepsilon} \frac{[(1 - |z|^2)|f'(z)|]^{\frac{2(p-1)}{1+\beta}}}{|f(z)|^{\frac{2(p+\beta-1-\delta)}{1+\beta}}} |f'(z)|^{\frac{2\beta}{1+\beta}} |z|^{-\frac{4\delta}{1+\beta}} |f(z)|^{\frac{2p}{1+\beta}} dm(z) \\
&= \left(\int_{\mathbb{D}_\varepsilon} \frac{[(1 - |z|^2)|f'(z)|]^{\frac{2(p-1)}{\beta}}}{|f(z)|^{\frac{2(p+\beta-1-\delta)}{\beta}}} |f'(z)|^2 dm(z) \right)^{\frac{\beta}{1+\beta}} \\
&\quad \times \left(\int_{\mathbb{D}} |z|^{-4\delta} |f(z)|^{2p} dm(z) \right)^{\frac{1}{1+\beta}} \\
&\leq C \left(\int_{\Omega_\varepsilon} \frac{d\Omega(w)^{\frac{2p-2}{\beta}}}{|w|^{\frac{2p-2}{\beta} + 2 + \frac{\beta-\delta}{\beta}}} dm(w) \right)^{\frac{\beta}{1+\beta}} < \infty,
\end{aligned}$$

which, together with (3.1) and (3.2), finishes the proof. $\square$

Moreover, we are able to prove that this result is sharp in the following sense.

Theorem 6. *If $0 < \beta < 1$ and $1 - \frac{\beta}{2} < p < \infty$, then there exists a univalent function $g \in A^{2p} \setminus H^p$ with $g(0) = 0$ and such that, setting $\Omega = g(\mathbb{D})$ and $\Omega_\varepsilon = \{w \in \Omega : |w| > \varepsilon\}$,*

$$(3.3) \quad \int_{\Omega_\varepsilon} \frac{d\Omega(w)^{\frac{2p-2}{\beta}}}{|w|^{\frac{2p-2}{\beta} + 2 + \kappa}} dm(w) < \infty, \quad \varepsilon > 0,$$

for every $\kappa > 0$.

Proof of Theorem 6. We shall follow the lines of the proof of [2, Theorem 8]. Take $p \in (1/2, \infty)$ and let f be the function defined in the proof of [2, Theorem 3], that is,

$$f(z) = \left[\frac{1}{(1-z) \log \frac{2e}{1-z}} \right]^{\frac{1}{p}}, \quad z \in \mathbb{D}.$$

Set

$$g(z) = f(z) - f(0), \quad z \in \mathbb{D}.$$

Then g is univalent, $g(0) = 0$ and $g \in A^{2p} \setminus H^p$. Finally, we shall see that (3.3) holds.

Take $\varepsilon > 0$. Since $g(0) = 0$, there exists η with $0 < \eta < 1$ such that

$$g^{-1}(\Omega_\varepsilon) \subset \mathbb{D}_\eta = \{z \in \mathbb{D} : |z| > \eta\}.$$

We have that

$$g'(z) = \frac{1}{p(1-z)^{1+\frac{1}{p}}} \left[\left(\frac{1}{\log \frac{2e}{1-z}} \right)^{\frac{1}{p}} \left(1 - \frac{1}{\log \frac{2e}{1-z}} \right) \right], \quad z \in \mathbb{D},$$

and that there exists a positive constant C such that

$$|g(z)| \geq C \left| \frac{1}{(1-z) \log \frac{2e}{1-z}} \right|^{\frac{1}{p}} \quad z \in \mathbb{D}_\eta.$$

So, using (1.4) and assuming without loss of generality that $\kappa < p \left(\frac{2p-2}{\beta} + 1 \right)$, we obtain

$$\begin{aligned}
& \int_{\Omega_\varepsilon} \frac{d_\Omega(w)^{\frac{2p-2}{\beta}}}{|w|^{\frac{2p-2}{\beta}+2+\kappa}} dm(w) \\
& \leq C \int_{\mathbb{D}_\eta} (1-|z|^2)^{\frac{2p-2}{\beta}} \frac{|g'(z)|^{\frac{2p-2}{\beta}+2}}{|g(z)|^{\frac{2p-2}{\beta}+2+\kappa}} dm(z) \\
& \leq C \int_{\mathbb{D}_\eta} (1-|z|^2)^{\frac{2p-2}{\beta}} \frac{\left| (1-z) \log \frac{2e}{1-z} \right|^{\frac{2p-2}{p\beta} + \frac{2+\kappa}{p}}}{\left| 1-z \right|^{(1+\frac{1}{p})(\frac{2p-2}{\beta}+2)} \left| \log \frac{2e}{1-z} \right|^{\frac{2p-2}{p\beta} + \frac{2}{p}}} \\
& \quad \times \left| 1 - \frac{1}{\log \frac{2e}{1-z}} \right|^{\frac{2p-2}{\beta}+2} dm(z) \\
& \leq C \int_{\mathbb{D}_\eta} (1-|z|^2)^{\frac{2p-2}{\beta}} \frac{\left| \log \frac{2e}{1-z} \right|^{\frac{\kappa}{p}}}{\left| 1-z \right|^{\frac{2p-2}{\beta}+2-\frac{\kappa}{p}}} dm(z) \\
& \leq C \int_0^1 (1-r)^{-1+\frac{\kappa}{p}} \left(\log \frac{2e}{1-r} \right)^{\frac{\kappa}{p}} dr < \infty.
\end{aligned}$$

This finishes the proof. $\square$

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